UNIVERSIDAD DE SANTIAGO DE CHILE
DEPARTAMENTO DE FÍSICA, FACULTAD DE CIENCIA

Dynamics of Wess-Zumino-Witten and Chern-Simons Theories

Olivera Mišković

Thesis for obtaining the degree of

Doctor of Philosophy

Supervising Professors: Dr. Ricardo Troncoso
Dr. Jorge Zanelli

SANTIAGO – CHILE
JANUARY 2004

UNIVERSIDAD DE SANTIAGO DE CHILE
DEPARTAMENTO DE FÍSICA, FACULTAD DE CIENCIA

Dinámica de las Teorías de Wess-Zumino-Witten y Chern-Simons

Olivera Mišković

Tesis para optar el grado de

Doctor en Ciencias con Mención en Física

Directores de Tesis: Dr. Ricardo Troncoso
Dr. Jorge Zanelli

SANTIAGO – CHILE
ENERO 2004

INFORME DE APROBACION
TESIS DE DOCTORADO

Se informa al Comité del Programa de Doctorado en Ciencias con mención en Física que la Tesis presentada por el candidato

Olivera Mišković

ha sido aprobada por la Comisión Informante de Tesis como requisito para la obtención del grado de Doctor en Ciencias con mención en Física.

Directores de Tesis:

Dr. Ricardo Troncoso ______________________

Dr. Jorge Zanelli ______________________

Comisión Informante de Tesis:

Dr. Máximo Bañados ______________________

Dr. Jorge Gamboa ______________________

Dr. Marc Henneaux ______________________

Dr. Mikhail Plyushchay ______________________

Dr. Lautaro Vergara (presidente) ______________________

Abstract

This thesis is devoted to the study of three problems on the Wess-Zumino-Witten (WZW) and Chern-Simons (CS) supergravity theories in the Hamiltonian framework:

1. The two-dimensional super WZW model coupled to supergravity is constructed. The canonical representation of Kac-Moody algebra is extended to the super Kac-Moody and Virasoro algebras. Then, the canonical action is constructed, invariant under local supersymmetry transformations. The metric tensor and Rarita-Schwinger fields emerge as Lagrange multipliers of the components of the super energy-momentum tensor.

2. In dimensions $D \geq 5$, CS theories are irregular systems, that is, they have constraints which are functionally dependent in some sectors of phase space. In these cases, the standard Dirac procedure is not directly applicable and must be redefined, as it is shown in the simplified case of finite number of degrees of freedom. Irregular systems fall into two classes depending on their behavior in the vicinity of the constraint surface. In one case, it is possible to regularize the system without ambiguities, while in the other, regularization is not always possible and the Hamiltonian and Lagrangian descriptions may be dynamically inequivalent. Irregularities have important consequences in the linearized approximation of nonlinear theories.

3. The dynamics of CS supergravity theory in $D = 5$, based on the supersymmetric extension of the AdS algebra, $su(2,2\,|4)$, is analyzed. The dynamical fields are the vielbein, the spin connection, 8 gravitini, as well as $SU(4)$ and $U(1)$ gauge fields. A class of backgrounds is found, providing a regular and generic effective theory. Some of these backgrounds are shown to be BPS states. The charges for the simplest choice of asymptotic conditions are obtained, and they satisfy a supersymmetric extension of the classical WZW_4 algebra, associated to $su(2,2\,|4)$.

Resumen

Esta tesis está dedicada al estudio de tres problemas en de teorías de supergravedad de Wess-Zumino-Witten (WZW) y de Chern-Simons (CS), en el formalismo hamiltoniano:

1. Se construye un modelo de súper WZW en dos dimensiones, acoplado a supergravedad. La representación canónica del álgebra de Kac-Moody es extendida a las álgebras de súper Kac-Moody y súper Virasoro. Luego, se construye la acción canónica, invariante bajo transformaciones de supersimetría locales. El tensor métrico y el campo de Rarita-Schwinger aparecen como multiplicadores de Lagrange de las componentes del súper tensor energía-momentum.

2. En dimensiones $D \geq 5$, las teorías de CS constituyen sistemas irregulares, es decir, contienen ligaduras que son funcionalmente dependientes en algunos sectores del espacio de fase. En estos casos, el procedimiento de Dirac estándar no es aplicable directamente y debe ser redefinido, como se muestra en el caso simplificado cuando el sistema tiene un número finito de grados de libertad. Los sistemas irregulares pueden pertenecer a dos clases, dependiendo de su comportamiento en la vecindad de la superficie de ligadura. En un caso, es posible regularizar el sistema sin ambigüedades, mientras que en el otro, la regularización no es siempre posible y las descripciones hamiltoniana y lagrangiana pueden no ser dinámicamente equivalentes. Estas irregularidades tienen importantes consecuencias en la aproximación linealizada de teorías no-lineales.

3. Se analiza la dinámica de la teoría de supergravedad de CS en $D = 5$, basada en la extensión supersimétrica del álgebra de AdS, $su(2,2\,|4)$. Los campos dinámicos son el vielbein, la conección de spin y 8 gravitini, además de campos de gauge para $SU(4)$ y $U(1)$. Se identifica una clase de backgrounds que da lugar a una teoría efectiva que es regular y genérica. Del mismo modo, se prueba que algunos de estos backgrounds son estados BPS. Se obtienen las cargas para la elección más simple de condiciones asintóticas. Estas cargas satisfacen una extensión supersimétrica del álgebra clásica de WZW_4, asociada a $su(2,2\,|4)$.

Резиме

Ова теза је посвећена проучавању три проблема у вези са Вес-Зумино-Витеновим (ВЗВ) и Черн-Сајмонсовим (ЧС) теоријама супергравитације у оквиру Хамилтоновог формализма:

1. Конструисан је дводимензиони супер ВЗВ-ов модел куплован са супергравитацијом. Канонска репрезентација Кац-Мудијеве алгебре је проширена до суперсиметричне Кац-Мудијеве и Виразорове алгебре, а затим је конструисано канонско дејство инваријантно под трансформацијама локалне суперсиметрије. Метрички тензор и Рарита-Швингерово поље су се појавили као Лагранжеви множитељи уз компоненте супертензора енергије-импулса.

2. ЧС-ове теорије у димензијама $D \geq 5$ су ирегуларни системи, што значи да садрже везе које су функционално зависне у појединим областима фазног простора. У тим случајевима не може директно да се примени стандардна Диракова процедура, него мора да се редефинише, што је урађено за најједноставније системе са коначним бројем степени слободе. Показано је да постоје две врсте ирегуларних система, у зависности од тога како се понашају у близини површи дефинисане везама. У једном случају систем може да се регуларише, док у другом случају регуларизација није увек могућа, пошто Хамилтонов и Лагранжев формализам могу да доведу до динамички нееквивалентних резултата. Ирегуларности имају значајне последице у линеарној апроксимацији нелинеарних теорија.

3. Анализирана је динамика ЧС-ове теорије супергравитације у $D = 5$, базиране на суперсиметричној екстензији анти-де Ситерове алгебре, $su(2,2|4)$. Динамичка поља те теорије су пентада, спинска конексија, 8 гравитина, као и градијентна поља $SU(4)$ и $U(1)$. Нађена је класа позадинских поља таквих да су ефективне теорије, дефинисане у њиховој околини, регуларне и генеричке. Такође је показано да нека од тих позадинских поља представљају тзв. BPS-стања. Избором најједноставнијих асимптотских услова су добијени очувани набоји, чија је алгебра суперсиметрична екстензија класичне WZW_4 алгебре, асоциране са $su(2,2|4)$.

Acknowledgment

There are many people who supported me from the moment I left my home in Belgrade, and came to Chile to work on my Ph.D. Thesis. First of all, I thank to my parents and my sister, without whose care, support and positive influence my trip and this work would have not been possible. I will also be in eternal debt to Rodrigo Olea, who has been sharing with me his love, his life and his knowledge in physics, and without whom everything would have been different.

The person who particularly fills me with the gratitude is my supervisor Jorge Zanelli, whose love for physics and his profound understanding of it have been irresestibly inspiring, and whose deep humanity is overwhelming. Infinite thanks to Ricardo Troncoso, my second supervisor and friend, for his help, support and cheering up in dark moments. I would like to thank to all physicists and staff of CECS, with the director Claudio Teitelboim, who accepted me from the very beginning as a member of their "second family".

For enlightening and useful discussions, as well as friendly criticism, thanks to Alexis Amézaga, Rodrigo Aros, Eloy Ayon-Beato, Máximo Bañados, Arundhati Dasgupta, Andrés Gomberoff, Mokhtar Hassaïne, Marc Henneaux, Georgios Kofinas, Cristián Martínez, Rodrigo Olea, Mikhail Plyushchay, Joel Saavedra, Claudio Teitelboim and Tatjana Vukašinac. I am particularly thankfull to Milutin Blagojević, Ivana Mišković and Branislav Sazdović, who were regularly replying my numerous questions by e-mail, and to Milovan Vasilić, for his helpful criticism. Many thanks to all physicists and staff of Institute of Physics in Belgrade, Vinča Institute and the Department of Physics USACH.

I will be forever obliged to my friends Aleksandra Ćuk, Snežana Djurdjić, Dáfni Fernanda Zenedin Marchioro, Milan Lalić, Aleksandra Lazović, Ricardo Navarro, Ljubomir Nikolić, Ivan Sajić, Jelena Tošić and Tatjana Vukašinac, for their friendship and moral support.

This work is partially funded by Chilean grants JZ 1999, MECESUP USA 9930, FONDECYT 2010017 and Millenium Project. I also want to thank to the Abdus Salam ICTP, Max Planck-Albert Einstein Institut and Université Libre de Bruxelles for their hospitality during the completition the parts of this work, as well as to Sonja and Vlado Mišković for their financial help. The generous support of Empresas CMPC to CECS is also acknowledged. CECS is a Millenium Science Institute and is funded in part by grants from Fundación Andes and the Tinker Foundation.

Contents

Chapter 1

Introduction

Wess-Zumino-Witten (WZW) and Chern-Simons (CS) field theories have been intensively studied in connection with several applications in physics and mathematics.

The two-dimensional WZW theory[1] [1, 2, 3], described by a non-linear sigma model with non-local interaction, was originally studied by Witten [3] as a theory equivalent to non-interacting massless fermions, thus providing non-Abelian bosonization rules for interacting fermionic theories. The WZW action is also known as the necessary counter-term for cancelation of quantum anomalies (the breaking down of a classical symmetry at the quantum level) [4]–[8]. This theory is exactly solvable and quantizable, and its action has two independent ("left" and "right") chiral symmetries, whose infinite-dimensional algebras are two copies of the affine, Kac-Moody (KM), algebra. The WZW theory is also conformally invariant, where the symmetry is described by the Virasoro algebra. Because of this, the WZW model is relevant in string theory, as well.

Three-dimensional CS theories have a topological origin, since they can be defined as CS forms integrated over the boundary of a compact four-dimensional manifold. These theories have no local degrees of freedom and they are also exactly solvable and quantizable [9]. As topological field theories, they can be used in the classification of three-dimensional manifolds [10]. The quantum CS theories are known to describe the quantum Hall effect [11]. CS theories can also be defined on three-dimensional manifolds *with* a boundary. In that case, their transformations under the "large" gauge transformations are non-trivial

[1]This is also called Wess-Zumino-Novikov-Witten (WZNW) theory.

and given by a closed 3-form, that has been used in the context of quantum anomalies. The fact that both WZW and CS theories are related to the quantum anomalies is not accidental. Their relation reflects a profound connection between them. For example, the gauge transformations of the CS action give a non-trivial contribution to the gauged WZW model describing the most general form of a two-dimensional chiral anomaly [12]–[15]. Any CS theory defined on a three-manifold with a boundary, induces a two-dimensional WZW model as a topological field theory [10, 16, 17].

In general, the dynamics at the boundary is determined by the asymptotic behavior of the fields. This is essential for a suitable definition of the global charges of the theory [18]–[20].

The most interesting aspect of these two classes of theories, which will be further investigated, is their deep connection with lower-dimensional gravity theories. For example, two-dimensional induced gravity can be obtained as a gauge extension of the WZW model [21, 22, 23], while the Liouville theory, describing the asymptotic dynamics of three-dimensional Einstein-Hilbert gravity with negative cosmological constant, is equivalent to the two-dimensional induced gravity in the conformal gauge [24, 25].

On the other hand, three-dimensional gravity, described by the Einstein-Hilbert action, which is linear in the curvature of space-time, can be formulated as a CS gauge theory invariant under de Sitter (dS), anti-de Sitter (AdS) or Poincaré groups [26, 27, 28]. Asymptotically locally AdS gravity, for example, has an infinite-dimensional algebra of asymptotic symmetries described by the Virasoro algebra, whose realization in terms of conserved charged requires a non-trivial classical central charge [29].

The need to look for alternative gravity theories arises from the fact that General Relativity, which gives a successful classical description of gravitational phenomena in four dimensions, does not admit a standard quantum description yet, while the other three fundamental forces are consistently unified and described by quantum theories of the Yang-Mills (YM) type. In this approach, the main obstruction for existence of a quantum theory of gravity is its nonrenormalizability, *i.e.*, the impossibility of removing all divergences which appear in the high-energy sector of the theory, due to the dimension of gravitational constant. While the renormalizability of YM theories is a consequence of their invariance under local gauge transformations, since the gauge principle provides a dimensionless coupling constant, the Einstein-Hilbert theory is invariant under general coordinate transformations $x \to x' = x'(x)$ (diffeomorphisms). This symmetry, however, does not guarantee the consistency of the quantum theory, because this symmetry does not have a fiber-bundle structure as in YM theories (which is sufficient, but not necessary condition for a theory to be renormalizable).

Supersymmetry is naturally introduced, since supersymmetric theories can lead to a non-trivial unification of space-time and internal symmetries within a relativistic quantum field theory (see, *e.g.*, [30]–[32]). In these theories, both bosons and fermions belong to the same representation of the supergroup. The gravitational interaction emerges naturally from *local supersymmetry*, since the anticommutator of two supersymmetry generators (supercharges) gives a generator of local translations. In that way, supergravity theories are obtained as supersymmetric extensions of the purely gravitional part.

Furthermore, supersymmetric extensions of chiral and conformal symmetries define supersymmetric WZW models, characterized by super KM and super Virasoro algebras [33]. In three dimensions, these superalgebras are obtained as the algebra of the classical charges for AdS supergravity models, with adequate asymptotic conditions [34, 35].

The existence of supersymmetry makes possible to construct non-negative quantities quadratic in the supercharges, which gives rise to the inequalities known as Bogomol'nyi bounds [36]. These bounds guarantee the stability of the ground state (vacuum) in supergravity theories, so that it remains a state of minimal energy after perturbations. The Bogomol'nyi bound ensures the positivity of energy in the standard supergravities [37, 38, 39], even in presence of other conserved quantities [40].

Higher-dimensional theories can be physically meaningful if one supposes that only four dimensions of space-time are observable, while others are "too small" to be visible at currently reachable energies. In that sense, a four-dimensional theory would be an effective theory. This can be realized by the procedure known as dimensional reduction, where one assumes that the radius of extra dimensions is compactified beyond sight (see, *e.g.*, [41, 42]).

It is interesting, thus, to consider higher-dimensional CS theories, which are defined in all odd dimensions, and have Lagrangians represented by CS forms [43]–[47]. CS gravity and supergravity theories are based on the anti-de Sitter [AdS, or $SO(D-1,2)$], de Sitter [dS, or $SO(D,1)$], and Poincaré [$ISO(D-1,1)$] gauge groups, as well as their supersymmetric extensions. They are by construction invariant under diffeomorphisms and provide a non-standard, consistent description of gravity as a gauge theory [48]–[52]. They are genuine gauge theories which are extensions of the Einstein-Hilbert action. Their actions are polynomials in the curvature R and they can also depend explicitly on the torsion T. Furthermore, they possess propagating degrees of freedom [54, 55], and have a very rich phase space structure. In CS supergravities, unlike in the standard supergravity theories, the supersymmetry algebra closes *off shell*, without addition of the auxiliary fields [56].

On the other hand, WZW and super WZW theories could be generalized to higher dimensions as field theories whose symmetries are described by (supersymmetric) extensions of KM and Virasoro algebras. They are not studied as much as in the two-dimensional case. For example, the WZW action is known in four dimensions, with the local symmetry described by a four-dimensional extension of the KM algebra, or WZW_4 algebra [46, 47, 57]. The relation between this four-dimensional WZW theory and a CS theory in five dimensions is established only at the level of algebras [55], while the actions has not been obtained explicitly. In general, the action of the four-dimensional super WZW model remains unknown.

Higher-dimensional CS theories have complex configurational space. In five-dimensional CS supergravity, for example, it was observed that the linearized action around an AdS background seems to have one more degree of freedom than the full nonlinear system [58]. This paradoxical behavior arises from the violation of the *regularity conditions* among the symmetry generators of the theory, or their functional dependence, in the region of phase space defined by the selected background. Therefore, CS theories in $D \geq 5$ dimensions are *irregular systems*, where Dirac's standard procedure of finding local symmetries and physical degrees of freedom [59] fails.[2] The problem of the regularity does not appear in lower-dimensional WZW or CS theories, but can occur in any physical system, independently of the dimension of space-time.

Constraints satisfying regularity conditions are sometimes referred to as *effective* con-

[2]A well-known example of an irregular system is a relativistic massless particle ($p^\mu p_\mu = 0$), which is irregular at the origin of momentum space ($p_\mu = 0$).

straints [60]. The issue of regularity (*effectiveness*) and its relevance for the equivalence between the Lagrangian and Hamiltonian formalisms has also been discussed in several references [61, 62, 63].

There are many different areas where the WZW and CS theories find their applications but, from the point of view of this thesis, the main motivation to study these theories is: (*i*) CS theories are alternative gravity and supergravity theories in odd dimensions, (*ii*) every *AdS*-CS theory induces a conformal field theory at the boundary, a WZW model, and (*iii*) WZW models should correspond to (super)gravity theories in even dimensions.

Therefore, three problems related to CS theories and to the topics (*i*)-(*iii*) are presented in this thesis, where each one can be analyzed independently.

- The first problem is the construction of the super WZW model coupled to supergravity, from the chosen canonical representation of the super Virasoro algebra. The model is obtained explicitly in two dimensions [64], what may provide some insight for finding the unknown four-dimensional super WZW theory.

- The second question arises from the need of dealing with irregular CS systems, and of generalizing the Dirac procedure. The problem is analyzed for classical mechanical systems with finite number of degrees of freedom [65, 66], and discussed in CS theories as well [66].

- The third part presents a work currently in progress, based on the study of the dynamical structure of five-dimensional *AdS*-CS supergravity, both in the bulk and at the boundary. The physical degrees of freedom and local symmetries of these theories depend on the symplectic form (defining the kinetic term). Since the symplectic form is a function of phase space coordinates whose rank can vary throughout the phase space, CS theories can be either *regular*, or *irregular* [66]. Moreover, they can be *generic*, with a minimal number of local symmetries [55], or *degenerate* if the symplectic form has a lower rank and additional symmetries emerge [67, 68]. In the asymptotic sector, the charge algebra of the *AdS*-CS supergravity is the supersymmetric extension of the WZW_4 algebra with a central charge [69].

The thesis is organized as follows.

In Chapter 2, the two-dimensional WZW model is reviewed. Two independent KM algebras are obtained as canonical symmetries of the WZW action. The conformal symmetry in this model is not independent, since the Virasoro generators can be expressed as bilinears of the KM generators. The super WZW model is also discussed, using superspace formalism.

In Chapter 3, two-dimensional WZW supergravity is constructed using the Hamiltonian method. The canonical (first order) action is defined from the phase space representation of the super Virasoro algebra, and the Lagrange multipliers corresponding to the super Virasoro generators appear as the components of the metric field and gravitini.

In Chapter 4, Dirac's procedure is extended to irregular classical mechanical systems with finite number of degrees of freedom. These systems are classified, and regularized when possible, by introducing dynamically equivalent regular Lagrangians. It is shown that a system cannot evolve in time from a regular phase space configuration into an irregular one, since regular and irregular configurations always belong to sectors of phase space that do not intersect.

In Chapter 5, the dynamical structure of the higher-dimensional CS theories is studied, where the symmetries are analyzed using canonical methods. Criteria which determine whether the regularity and genericity conditions are satisfied around the chosen background, are presented.

In Chapter 6, five-dimensional AdS-CS supergravity theory, based on $SU(2,2\,|N\,)$ group, is studied, as the simplest example of CS supergravity theories with propagating degrees of freedom. In the particular case of $N=4$, a class of generic and regular backgrounds is found, such that all charges can be defined at the boundary. Among them, there are configurations which are BPS states. The classical charge algebra is obtained as the supersymmetric extension of the WZW_4 algebra with a central extension.

The main results of the Thesis were published in Refs. [64, 65, 66], and the manuscript [69] is in preparation.

Chapter 2

Wess-Zumino-Witten model

Wess-Zumino-Witten (WZW) models are conformal field theories in which an affine, or Kac-Moody (KM) algebra gives the spectrum of the theory. The two-dimensional WZW model studied here is a system whose kinetic term is given by the nonlinear sigma model and the potential is the Wess-Zumino term [1, 2]. This model was originally studied by Witten in the context of two-dimensional bosonization, where it provides non-Abelian bosonization rules describing non-interacting massless fermions [3]. The WZW action was also used as a term cancelling quantum anomalies [4]–[8]. It is a chirally and conformally invariant theory, and exactly solvable.

The purpose of this chapter is to present the WZW model and its global and local symmetries in a systematic way, as well as to introduce the notation. In the next chapter, a general idea of finding an action, starting only from the algebra of its local symmetries, will be presented in two dimensions and for $N = 1$ superconformal group. It will be shown that, in that way, the super Virasoso algebra leads to the super WZW model coupled to supergravity.

2.1 The action

The two-dimensional WZW model has a fundamental field g belonging to a non-Abelian semi-simple compact Lie group G and the dynamics given by the action

$$I_{\text{WZW}}\left[g\right] = I_0\left[g\right] + \Gamma\left[g\right] = -a\int\limits_{\mathcal{M}}\left\langle {}^*\mathbf{V}\mathbf{V}\right\rangle - \frac{k}{3}\int\limits_{\mathcal{B}}\left\langle \mathbf{V}^3\right\rangle \qquad \left(\mathbf{V}\equiv g^{-1}dg\right), \tag{2.1}$$

where $I_0\left[g\right]$ is the action of the non-linear σ-model with a positive dimensionless coupling constant a, while $\Gamma\left[g\right]$ is the topological Wess-Zumino term defined over a three-manifold $\mathcal{B}$ whose boundary is the two-dimensional space-time, $\partial\mathcal{B} = \mathcal{M}$, and k is a dimensionless constant. Here $\mathbf{V}$ is a Lie-algebra-valued one-form, ${}^*\mathbf{V}$ is its Hodge-dual and $\langle\cdots\rangle$ stands for a trace. The exterior product $\wedge$ between the forms is understood.

The Wess-Zumino term has a non-local expression, involving integration over the three-manifold $\mathcal{B}$, but it depends only on its boundary $\mathcal{M}$ modulo a constant. Independence from the choice of $\mathcal{B}$ follows from the identity

$$\frac{1}{12\pi}\int\limits_{\mathcal{B}}\left\langle \mathbf{V}^3\right\rangle - \frac{1}{12\pi}\int\limits_{\mathcal{B}'}\left\langle \mathbf{V}^3\right\rangle = -2\pi n\,, \qquad (n\in\mathbb{Z})\,, \tag{2.2}$$

with $\mathcal{M} = \partial\mathcal{B} = \partial\mathcal{B}'$, which is a consequence of the index theorem. The set $\mathcal{B}\cup\mathcal{B}'$ is a closed oriented manifold, topologically equivalent to a three-sphere S^3, provided the boundaries of $\mathcal{B}$ and $\mathcal{B}'$ have opposite orientations. Thus, the term $2\pi n$ in (2.2) arises because of the topologically distinct possibilities to have the mapping $g(x) : S^3 \to G$, classified by $\pi_3\left(G\right)\simeq\mathbb{Z}$ (for G compact semi-simple), with the corresponding winding number n. Therefore, all dynamics happens on the two-dimensional space-time $\mathcal{M}$, provided the constant k is proportional to an integer,

$$k = \frac{n}{8\pi}\,. \tag{2.3}$$

Then, the quantum amplitude $\int\left[dA\right]\,e^{iI_{\text{WZW}}[A]}$ is independent of the choice of $\mathcal{B}$, provided the constant k is quantized as in (2.3).

Field equations. The local coordinates on $\mathcal{M}$ with the signature $(-,+)$ are introduced as x^μ $(\mu = 0,1)$. Then the differential forms can be expressed in the basis of

1-forms on $\mathcal{M}$ as $\mathbf{V} = g^{-1}\partial_\mu g\, dx^\mu$ and ${}^*\mathbf{V} = \varepsilon_\mu{}^\nu\, g^{-1}\partial_\nu g\, dx^\mu$ (with $\varepsilon^{01} = 1$). Under a small variation of the gauge field, $g \to g + \delta g$, the WZW action changes as

$$\delta I_{\text{WZW}}[g] = -a \int d^2x \left\langle g^{-1}\delta g\, \partial_\mu \left(g^{-1}\partial^\mu g\right)\right\rangle + k \int d^2x\, \varepsilon^{\mu\nu} \left\langle g^{-1}\delta g\, \partial_\mu \left(g^{-1}\partial_\nu g\right)\right\rangle . \quad (2.4)$$

In the *light-cone* coordinates $x^\pm = \frac{1}{\sqrt{2}}\left(x^0 \pm x^1\right)$, it gives the following field equations:[1]

$$(a+k)\ \partial_- \left(g^{-1}\partial_+ g\right) + (a-k)\, \partial_+ \left(g^{-1}\partial_- g\right) = 0\,. \quad (2.5)$$

Therefore, taking into account that a must be positive in order to have the right sign in the kinetic term of (2.1), while the sign of $k = \frac{n}{8\pi}$ depends on $n \in \mathbb{Z}$, there are two possible choices of a which give a theory with chiral symmetry. For $a = -k$ and $n < 0$, the equation of motion is $\partial_+ \left(g^{-1}\partial_- g\right) = 0$, with the general solution

$$g(x^+, x^-) = g_+\left(x^+\right) g_-\left(x^-\right)\ , \quad (2.6)$$

where g_+ and g_- are elements of G. For $a = k$ and $n > 0$, (2.5) becomes $\partial_- \left(g^{-1}\partial_+ g\right) = 0$ and the general solution is factorized as $g_-\left(x^-\right) g_+\left(x^+\right)$. Equation (2.6) is invariant under the *left chiral* transformations $g_+\left(x^+\right) \to \Omega_+\left(x^+\right) g_+\left(x^+\right)$ and *right chiral* transformations $g_-\left(x^-\right) \to g_-\left(x^-\right)\Omega_-^{-1}\left(x^-\right)$, or

$$g(x^+, x^-) \to \Omega_+\left(x^+\right) g(x^+, x^-)\Omega_-^{-1}\left(x^-\right)\ , \qquad (\Omega_+, \Omega_-) \in G \otimes G\,. \quad (2.7)$$

This symmetry is related to two independent Kac-Moody algebras, as will be seen using canonical methods.

2.2 Kac-Moody algebra

The symmetries of the WZW action (2.1) with $a = -k > 0$ can be found using the Hamiltonian formalism [3, 70]. A summary of the formalism is given in Appendix A.

Let the local coordinates q^i parametrize the group manifold, $g = g(q)$, where the number of coordinates is equal to the dimension of G, and the generators of the group

[1] In the *light-cone* coordinates, the antisymmetric tensor becomes $\varepsilon^{+-} = -\varepsilon^{-+} = 1$, and the metric has non-zero components $\eta_{+-} = \eta_{-+} = -1$.

satisfy the Lie algebra $[\mathbf{G}_a, \mathbf{G}_b] = f_{ab}^{\ \ c}\,\mathbf{G}_c$. Then the Lie-algebra-valued 1-form $\mathbf{V}$ can be expanded as

$$\mathbf{V} = g^{-1}dg = dq^i E_i^a\,\mathbf{G}_a\,, \tag{2.8}$$

where E_i^a is a vielbein on the group manifold. The Killing metric in the adjoint representation of the Lie algebra is $g_{ab} = \langle \mathbf{G}_a \mathbf{G}_b \rangle = f_{ad}^{\ \ c} f_{bc}^{\ \ d}$, and in the coordinate basis the metric is

$$\gamma_{ij}\,(q) = E_i^a\,(q)\,E_j^b\,(q)\,g_{ab}\,. \tag{2.9}$$

On the basis of Poincaré's lemma, the equation $d\,\langle \mathbf{V} \rangle^3 = 0$ can be locally written as exterior derivative of a two-form,

$$\langle \mathbf{V} \rangle^3 = -6\,d\rho\,, \tag{2.10}$$

where $\rho \equiv \frac{1}{2}\,\rho_{ij}\,dq^i dq^j$. Rewriting this in components, the following identities are obtained:

$$\frac{1}{2}\,f_{abc}\,E_i^a E_j^b E_k^c = \partial_i \rho_{jk} + \partial_j \rho_{ki} + \partial_k \rho_{ij}\,. \tag{2.11}$$

Then the local expression for the action (2.1) in the *light-cone* coordinates becomes:

$$I_{\mathrm{WZW}}\,[g\,(q)] = -2k \int d^2x\ [\gamma_{ij}\,(q) + 2\rho_{ij}\,(q)]\,\partial_- q^i \partial_+ q^j\,. \tag{2.12}$$

Taking $\tau = x^0$ as canonical time, one finds that (2.12) has no first class constraints, therefore has no local symmetries. But it is expected to find that this action possesses two types of chiral symmetry which, unlike standard local symmetries, depend only on one coordinate. The canonical approach to the chiral symmetries requires to take one of the *light-cone* coordinates as the time variable. In order to find the complete chiral invariance of (2.12), both possibilities $\tau = x^-$ and $\tau = x^+$ should be analyzed.

a) Local symmetries

The space-time coordinates are chosen as $(\tau, \sigma) = (x^-, x^+)$. Then, the momenta $p_i = \frac{\delta I}{\delta \dot{q}^i}$ canonically conjugated to coordinates q^i are not independent and define the constraints

$$K_{-i} \equiv p_i + 2k\,(\gamma_{ij} + 2\rho_{ij})\,\partial_\sigma q^j \approx 0\,, \tag{2.13}$$

where the additional subindex "–" in K_{-i} refers to the choice of τ, while $\approx$ denotes the weak equality, since the derivatives of K_{-i} do not vanish on the constraint surface $K_{-i}=0$. An equivalent set of constraints (defining the same constraint surface) is

$$K_{-a}=-E_a^i K_{-i}\approx 0\,, \tag{2.14}$$

where E_a^i is the inverse vielbein $\left(E_a^i E_i^b=\delta_a^b \text{ and } E_i^a E_a^j=\delta_i^j\right)$. Then, the Hamiltonian

$$H=\int d\sigma\, u^a K_{-a} \tag{2.15}$$

depends on the arbitrary multipliers $u^a(\tau,\sigma)$ and generates the time evolution of any phase space variable $F(q,p)$ by Poisson brackets (PB) as

$$\frac{dF}{d\tau}=\{F,H\}\ . \tag{2.16}$$

The constraints $K_{-a}\approx 0$ are preserved in time if $\partial_\sigma u=0$, thus they do not yield new constraints, and $u=u(\tau)$. The PB of constraints give rise to the affine, or KM, algebra

$$\{K_{-a}(x)\,,K_{-b}(x')\}=f_{ab}^{\ \ c}K_{-c}(x)\,\delta(\sigma-\sigma')-4kg_{ab}\partial_\sigma\delta(\sigma-\sigma')\ , \tag{2.17}$$

with the *central extension* $-4k$. Here $x=(\tau,\sigma)$ and the PBs are taken at the same τ. The presence of the Schwinger term proportional to $\partial_\sigma\delta$ in (2.17) indicates that not all constraints are first class, however, it is not clear how to identify first and second class constraints. Assuming that the space is compact and all variables on $\mathcal{M}$ are periodic functions, with period L, $F(\tau,\sigma+L)=F(\tau,\sigma)$, they can be Fourier-expanded as

$$F(\tau,\sigma)=\frac{1}{L}\sum_{n\in\mathbb{Z}}F_n(\tau)\,e^{-\frac{2\pi i}{L}n\sigma}\qquad\longleftrightarrow\qquad F_n(\tau)=\int_0^L d\sigma\,F(\tau,\sigma)\,e^{\frac{2\pi i}{L}n\sigma}\,. \tag{2.18}$$

Then, the KM algebra becomes

$$\{K_{-an},K_{-bm}\}=f_{ab}^{\ \ c}\,K_{-c(n+m)}-8\pi i\,kn\,g_{ab}\delta_{n+m,0}\,. \tag{2.19}$$

The result does not depend on the period L, and now it is straightforward to separate first and second class constraints. From

$$\{K_{-a0},K_{-bn}\}=f_{ab}^{\ \ c}K_{-cn}\approx 0\ , \tag{2.20}$$

it can be seen that the zero modes K_{a0} are first class constrains and that they close the PB subalgebra which is isomorphic to the algebra of G, while the non-zero modes K_{an} $(n \neq 0)$ are second class constraints,

$$\left\{K_{-an}, K_{-b(-n)}\right\} \approx -8\pi i\, kn\, g_{ab} \neq 0\,, \qquad (n \neq 0)\ . \tag{2.21}$$

Therefore, the generator of gauge transformations, containing all first class constraints, has the form

$$G_-\left[\lambda\right] \equiv \lambda_-^a K_{-a0}\,, \tag{2.22}$$

with a local parameter $\lambda_-^a\left(\tau\right)$. The dynamical field q^i changes under infinitesimal gauge transformations as

$$\delta_- q^i = \left\{q^i, G\left[\lambda\right]\right\} = -\lambda_-^a E_a^i\,, \tag{2.23}$$

which, with the help of the expansion $g^{-1}\delta_- g = \delta_- q^i E_i^a \mathbf{G}_a$, leads to the transformation law $g^{-1}\delta_- g = -\lambda_-$, or

$$\delta_- g = -g\lambda_-\ . \tag{2.24}$$

The transformations (2.24) are the infinitesimal form of the right chiral gauge transformations

$$g \rightarrow g\ \Omega_-^{-1}\left(x^-\right)\,, \tag{2.25}$$

with a group element defined as $\Omega_- \equiv e^{\lambda_-} \approx 1 + \lambda_-$.

Alternatively, choosing the space-time coordinates as $(\tau, \sigma) = (x^+, -x^-)$, where the minus sign is adopted to preserve the orientation between coordinate axes, and with the help of the identity

$$-g\mathbf{V}g^{-1} = gdg^{-1} \equiv dq^i \tilde{E}_i^a\, \mathbf{G}_a\,, \tag{2.26}$$

the constraints take the form

$$K_{+a} \equiv -\tilde{E}_a^i\ \left[p_i + 2k\left(\gamma_{ij} - 2\rho_{ij}\right)\partial_\sigma q^j\right] \approx 0\,. \tag{2.27}$$

They form an independent KM algebra,

$$\left\{K_{+a}\left(x\right), K_{+b}\left(x'\right)\right\} = f_{ab}^{\ \ c} K_{+c}\left(x\right)\delta\left(\sigma - \sigma'\right) + 4k\ g_{ab}\partial_\sigma\delta\left(\sigma - \sigma'\right)\,, \tag{2.28}$$

with the central extension $+4k$, which has the opposite sign to that in (2.17). The mode expansion gives the algebra (2.19) with $k \rightarrow -k$ and the corresponding first class gauge generator $G_+[\lambda] \equiv \lambda_+^a K_{+a0}$ leads to right chiral transformations $\delta_+ g = \lambda_+ g$, which are the infinitesimal form of

$$g \to \Omega_+ \left(x^+\right) g\,, \tag{2.29}$$

where $\Omega_+ \equiv e^{\lambda_+} \approx 1 + \lambda_+$. The constraints K_{+a} commute with K_{-b}. Both results (2.25) and (2.29) are independent gauge symmetries of the action. This means that the action (2.12) is invariant under the chiral transformations (2.7) generated by K_{+a0} acting from the right, and K_{-a0} acting from the left. The modes $K_{\pm an}$, $n \neq 0$, are not the generators of local symmetries (they are second class constraints), and they lead to the central extensions $\pm 4k$ in the corresponding KM algebras.

b) Global symmetries

Choosing the space-time coordinates as $\tau = x^-$ and $\sigma = x^+$, the infinitesimal chiral transformations (2.7) of $g \in G$ take a form

$$\delta_\pm g = \lambda_+(\sigma)\, g - g \lambda_-(\tau)\ , \tag{2.30}$$

with the Lie-algebra-valued parameters $\lambda_\pm = \lambda_\pm^a \mathbf{G}_a$ or, in terms of the local fields q^i, the transformations become

$$\delta_\pm q^i = -\lambda_+^a(\sigma)\, \tilde{E}_a^i - \lambda_-^a(\tau)\, E_a^i\,. \tag{2.31}$$

Therefore, *right* chiral transformations, given by the time-dependent parameter $\lambda_-(\tau)$, lead to *local symmetries* of the WZW action. It is already shown that these symmetries are generated by the first class constraints $K_{-a0} \approx 0$.

On the other hand, *left* chiral transformations correspond to *global symmetries* of the action, since they are given by infinite number of time-independent parameters $\lambda_+(\sigma)$. Conserved charges corresponding to these global symmetries are obtained from Noether currents,

$$J_+^a \equiv 4k \left(g\partial_+ g^{-1}\right)^a = 4k\, \tilde{E}_i^a \partial_\sigma q^i\,, \tag{2.32}$$

and they do not vanish on the constraint surface $K_{-a} = 0$.

In order to find the current algebra, it is convenient to introduce a Dirac bracket (DB) as

$$\{M, N\}^* \equiv \{M, N\} - \sum_{m,n\neq 0} \{M, K_{-an}\}\, \Delta_{nm}^{ab}\, \{K_{-bm}, N\}\ , \tag{2.33}$$

where $\Delta_{nm}^{ab} \equiv \frac{1}{8\pi i\, kn}\, g^{ab}\delta_{n+m,0}$ $(n, m \neq 0)$ is the inverse of the PB matrix of second class constraints K_{-an} $(n \neq 0)$, given by Eq. (2.21). Then it can be shown that the currents J_+^a satisfy the KM algebra with the central charge $4k$,

$$\{J_{+a}(x)\,, J_{+b}(x')\}^* = f_{ab}^{\ \ c} J_{+c}(x)\,\delta(\sigma - \sigma') + 4k\; g_{ab}\partial_\sigma \delta(\sigma - \sigma')\ . \tag{2.34}$$

Similarly, choosing $\tau = x^+$ as canonical time, there are Noether currents $J_- = 4k\, g^{-1}\partial_- g$, corresponding to the right chiral symmetries as global symmetries of the WZW model. The currents J_-^a satisfy a KM current algebra with the central charge $-4k$.

2.3 Virasoro algebra

WZW theory is also invariant under conformal transformations, or diffeomorphisms $x^\mu \to x'^\mu(x)$ which change the line element by a scale factor at each point of space-time,

$$ds'^2 = g'_{\mu\nu}(x')\; dx'^\mu dx'^\nu = \Lambda(x)\, g_{\mu\nu}(x)\; dx^\mu dx^\nu = \Lambda(x)\, ds^2\,. \tag{2.35}$$

Conformal transformations have the form of chiral and antichiral mappings $x^+ \to f_+(x^+)$ and $x^- \to f_-(x^-)$, and their generators are the *light-cone* components of the energy-momentum tensor $T_{++}(x^+)$ and $T_{--}(x^-)$, which is traceless $(T^\mu_\mu = 2T_{+-} = 0)$. They satisfy two independent Virasoro algebras,

$$[T(x), T(y)] = -\left[T(x) + T(y)\right]\delta'(\sigma_x - \sigma_y) - \frac{c}{12}\delta'''(\sigma_x - \sigma_y)\,, \tag{2.36}$$

without central charge $(c = 0)$ in a classical theory, or *with central charges* $c = c_0$ and $c = -c_0$ (for T_{++} and T_{--} respectively) in the quantum case. (The normalization of c is adopted from string theory.) The appearance of the Schwinger term in the quantum Virasoro algebra is called a *quantum anomaly,* but it can appear in a classical theory as well [29]. The physical meaning of $c \neq 0$ in a classical theory is the breaking of conformal symmetry by the introduction of a macroscopic scale into the system (by boundary conditions, for example).

The Fourier modes of the Virasoro generators, L_n $(n \in \mathbb{Z})$, in a compact space with period L, are given by

$$L_n = \frac{L}{2\pi i}\int_0^L d\sigma\, T(\sigma)\; e^{\frac{2\pi i}{L}n\sigma}, \tag{2.37}$$

where $L_n^\dagger = L_{-n}$ for unitary representations. They obey the well-known commutation rules

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}\, 2\pi i\; n^3 \delta_{n+m,0}\,. \tag{2.38}$$

This algebra contains a finite subalgebra, generated by $\{L_{-1}, L_0, L_1\}$, associated to global conformal invariance.

Conformal symmetry does not arise in the Hamiltonian analysis because it is not an independent symmetry. Virasoro generators can be expressed in terms of KM currents as [90, 91]

$$\begin{aligned} T_{--} &= \frac{1}{4k}\, g^{ab} J_{-a} J_{-b}\,, \\ T_{++} &= -\frac{1}{4k}\, g^{ab} J_{+a} J_{+b}\,, \end{aligned} \tag{2.39}$$

and they represent the *light-cone* components of the energy-momentum tensor. Conformal invariance leads to $T_{-+} = 0$. In terms of the Fourier modes, the relations (2.39) become

$$L_{\pm n} = \mp \frac{1}{8\pi i k} \sum_{m\in\mathbb{Z}} g^{ab} J_{\pm a m} J_{\pm b(n-m)}\,. \tag{2.40}$$

The observation that the energy-momentum tensor is a bilinear in the currents, is used to construct them using the procedure of Sugawara [71, 72]. More precisely, for a given KM algebra, there is always a Virasoro algebra, such that they form a semi-direct product

$$[L_n, J_{am}] = 2m\, J_{a(n+m)}\,, \qquad\qquad [L_n, K] = 0\,, \tag{2.41}$$

where K is the central extension of the KM algebra. More about Virasoro and KM algebras in the conformal filed theory can be found in Refs. [73]–[76].

2.4 Gauged WZW model

The two-dimensional WZW model is closely related to a three-dimensional Chern-Simons (CS) theory whose dynamical field is a Lie-algebra-valued 1-form, $\mathbf{A} = A^a \mathbf{G}_a$. Definition of the CS Lagrangian comes from the Chern character,

$$P(A) = \left\langle \mathbf{F}^2 \right\rangle \equiv g_{ab} F^a F^b, \tag{2.42}$$

where $\mathbf{F} = d\mathbf{A} + \mathbf{A}^2$ is the field-strength 2-form associated with the gauge field $\mathbf{A}$. Since the Chern character is a closed form ($dP = 0$), on the basis of algebraic Poincaré's lemma[2], it can be locally written as the exterior derivative of a 3-form, called the CS form, which defines the CS Lagrangian as $dL_{\text{CS}}(A) = kP(A)$. Then the CS action is given by

$$I_{\text{CS}}[A] = \int\limits_{\mathcal{B}} L_{\text{CS}}(A) = k \int\limits_{\mathcal{B}} \left\langle \mathbf{AF} - \frac{1}{3}\mathbf{A}^3 \right\rangle , \tag{2.43}$$

where $\mathcal{B}$ is a three-dimensional manifold (not necessarily without a boundary).

Under finite gauge transformations

$$\mathbf{A}^g = g(\mathbf{A} + d)\, g^{-1} = g(\mathbf{A} - \mathbf{V})\, g^{-1} , \qquad (g \in G) , \tag{2.44}$$

the field-strength transforms homogeneously ($\mathbf{F}^g = g\mathbf{F}g^{-1}$), the Chern character is invariant and the CS Lagrangian changes as $L^g_{\text{CS}} = L_{\text{CS}} + \omega$, where ω is a closed form ($d\omega = 0$) which need not be exact for nontrivial topology of $\mathcal{B}$. Explicitly, under the finite gauge transformations, the CS action changes as

$$I_{\text{CS}}[A^g] - I_{\text{CS}}[A] = \alpha[A, g] , \tag{2.45}$$

where

$$\alpha[A, g] = -k \int\limits_{\mathcal{M}=\partial\mathcal{B}} \langle \mathbf{AV} \rangle + \frac{k}{3} \int\limits_{\mathcal{B}} \langle \mathbf{V}^3 \rangle . \tag{2.46}$$

The last term is recognized as the Wess-Zumino term. The functional $\alpha[A, g]$ satisfies the so-called *cocycle equation*, or Polyakov-Wiegmann identity [77],

$$\alpha[A^g, h] - \alpha[A, gh] + \alpha[A, g] = 0 , \qquad (g, h \in G) . \tag{2.47}$$

Any quantity which satisfies the above equation is called Wess-Zumino term, or *anomaly*, and it describes the non-invariance of the quantum theory under a classical gauge sym-

[2]Poincaré lemma states that any closed form $P(A)$ can be locally written as exact form $d\alpha(A)$. The algebraic Poincaré lemma guarantees that a differential form $\alpha(A)$ is a local function of A (depending on finite number of derivatives ∂A, $\partial^2 A$, $\ldots$).

metry.[3] Any other object of the form $\beta[A^g]-\beta[A]$ also satisfies the cocycle equation. Since $\langle \mathbf{A}^2\rangle = 0$, a natural nontrivial possibility for β is

$$\beta[A] = a\int\limits_{\mathcal{M}} \langle {}^*\mathbf{A}\mathbf{A}\rangle \qquad (a\in\mathbb{R})\ , \tag{2.48}$$

where ${}^*\mathbf{A}$ is a Hodge-dual field. With this choice, the action which satisfies the cocycle equation (2.47), and therefore describes the anomaly of a classical theory, is

$$I_{\text{GWZW}}[A,g] = I_{\text{WZW}}[g] + 2a\int\limits_{\mathcal{M}} \langle {}^*\mathbf{A}\mathbf{V}\rangle + k\int\limits_{\mathcal{M}} \langle \mathbf{A}\mathbf{V}\rangle\ , \tag{2.49}$$

and is called the *gauged WZW action*. More about this model and its quantization can be found in Refs. [13, 14, 15, 78].

2.5 Supersymmetric WZW model

The supersymmetric generalization of the WZW model is defined in $(1,1)$ superspace parametrized by four real coordinates $z^A=(x^\mu,\theta_\alpha)$, where x^μ $(\mu=0,1)$ are local coordinates on a two-dimensional space-time with the signature $(-,+)$, and θ_α $(\alpha=+,-)$ is a Majorana spinor.[4] The super Wess-Zumino-Witten (SWZW) model is given by the action [79, 80],

$$I_{\text{SWZW}}[g] = -k\int d^4z\ \left\langle \bar{D}S^\dagger DS\right\rangle - \frac{2k}{3}\int d^4z\ \left\langle S^\dagger\dot{S}\bar{D}S^\dagger\gamma_3 DS\right\rangle\ , \tag{2.50}$$

where k is a positive dimensionless constant, S is a matrix superfield, $\langle\cdots\rangle$ stands for a supertrace and $D_\alpha = \bar{\partial}_\alpha + i\left(\gamma^m\theta\right)_\alpha\partial_m$ is the supercovariant derivative defined in the tangent Minkowski space with coordinates x^m $(m=0,1)$, where $\partial_m\equiv\partial/\partial x^m$ and $\bar{\partial}_\alpha\equiv\partial/\partial\bar{\theta}^\alpha$. The γ-matrices γ^m satisfy the Clifford algebra and $\gamma_3\equiv i\gamma^0\gamma^1$. Integration is

[3] The CS action (2.43) plays the role of an effective action for a theory with matter fields ϕ, whose quantum theory is described by the functional integral

$$Z=\int [dA]\ e^{iI_{\text{CS}}[A]} = \int [dA]\,[d\phi]\ e^{iI_0[\phi,A]}\ ,$$

where the classical action is invariant under gauge transformations, $I_0[\phi^g,A^g] = I_0[\phi,A]$, while the measure has the anomaly $[d\phi]^g=[d\phi]\ e^{i\alpha[A,g]}$.

[4] The notation $(1,1)$ refers to the superspace with two Grassmann odd variables (θ_+,θ_-).

carried out over the superspace variables, where $d^4z \equiv d^2x\, d^2\theta$ and basic integrals for Grassman odd numbers are $\int d\theta = 0$ and $\int d\theta\, \theta = 1$. All conventions and representations are given in Appendix B.

The supermatrix S is expanded in the superspace as

$$\begin{aligned} S &= g + i\theta\psi + \frac{i}{2}\,\bar{\theta}\theta\, F\,, \\ S^\dagger &= g^\dagger + i\theta\psi^\dagger + \frac{i}{2}\,\bar{\theta}\theta\, F^\dagger\,. \end{aligned} \tag{2.51}$$

Supposing, for simplicity, that S is unitary ($SS^\dagger = 1$), one obtains

$$\begin{aligned} g^\dagger &= g^{-1}\,, \\ \psi^\dagger &= -g^\dagger\psi g^\dagger\,, \\ F^\dagger &= -g^\dagger F g^\dagger - ig^\dagger\bar{\psi}g^\dagger\psi g^\dagger\,, \end{aligned} \tag{2.52}$$

where the identity $\bar{\theta}\psi\ \bar{\theta}\psi^\dagger = -\frac{1}{2}\ \bar{\theta}\theta\ \bar{\psi}\psi^\dagger$, valid for Majorana fermions, is used. The equations of motion obtained from the action (2.50) are

$$\bar{D}\left(S^\dagger\gamma^+ DS\right) = 0\,, \tag{2.53}$$

where $\gamma^\pm = \frac{1}{2}\ (1 \pm \gamma_3)$ are projective γ-matrices.

In order to be convinced that this model is indeed a supersymmetric extension of the WZW model (2.1), the action (2.50) can be written in components and the Berezin integrals over Grassman variables are performed. Then,

$$I_{\mathrm{SWZW}}\,[S] = I_{\mathrm{WZW}}\,[g] + + I_{\mathrm{f}}\,[g,\psi] + I_F\,[g,\psi,F]\ , \tag{2.54}$$

where the bosonic sector $I_{\mathrm{WZW}}\,[g]$ is the WZW action (2.1) with $a = k$, and the additional terms demanded by the supersymmetry are

$$\begin{aligned} I_{\mathrm{f}}\,[g,\psi] &= -ik\int d^2x\ \left\langle\bar{\psi}^\dagger\left(\partial\!\!\!/ + \tfrac{1}{2}\,\gamma_3\partial\!\!\!/ g g^\dagger\right)\psi\right\rangle\ , \\ I_F\,[g,\psi,F] &= k\int d^2x\ \left\langle F^\dagger F - \tfrac{i}{4}\,\bar{\psi}^\dagger\gamma_3\psi\left(F^\dagger g - g^\dagger F\right)\right\rangle\ , \end{aligned} \tag{2.55}$$

where $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$. The matrix field F is auxiliary (it does not have a kinetic term in I_F) and it can be integrated out by means of its equations of motion,

$$F = \frac{i}{2}\,g\left(\bar{\psi}^\dagger\psi + \frac{1}{2}\,\bar{\psi}^\dagger\gamma_3\psi\right)\ . \tag{2.56}$$

Putting the solution for F, given by (2.56), into the action $I_F\left[g,\psi,F\right]$, and after using the Fierz identity for Majorana fermions,

$$\left\langle\left(\bar{\psi}^{\dagger}\psi\right)^{2}+\left(\bar{\psi}^{\dagger}\gamma_{3}\psi\right)^{2}\right\rangle=0\,, \tag{2.57}$$

one obtains that this action vanishes,

$$I_F\left[g,\psi,F\left(g,\psi\right)\right]=0\,. \tag{2.58}$$

Although the fermions ψ in I_{f} are, in general, interacting, after making the following reparametrization

$$\chi=g^{\dagger}\gamma^{+}\psi+\gamma^{-}\psi g^{\dagger}\,, \tag{2.59}$$

the fermionic action reduces to the action of a free fermion

$$I_0\left[\chi\right]=I_{\mathrm{f}}\left[g,\psi\right]=-\frac{ik}{2}\int d^2x\,\left\langle\bar{\chi}\partial\!\!\!/\chi\right\rangle\,, \tag{2.60}$$

and the fermions are completely decoupled from the WZW term. With the reparametrization (2.59), the superfield S is factorized as:

$$S=\left(1+i\theta_+\chi_-\right)g\left(1-i\theta_-\chi_+\right)\,. \tag{2.61}$$

Equations of motion. The classical equations of motion following from the action

$$I_{\mathrm{SWZW}}\left[S\right]=I_{\mathrm{WZW}}\left[g\right]+I_0\left[\chi\right] \tag{2.62}$$

lead to the general solution in *light-cone* coordinates:

$$g\left(x^+,x^-\right)=g_-\left(x^-\right)g_+\left(x^+\right)\,, \tag{2.63}$$

and for fermions

$$\chi_+=\chi_+\left(x^+\right)\,,\qquad\qquad\chi_-=\chi_-\left(x^-\right)\,, \tag{2.64}$$

giving the factorization of the superfield as

$$S\left(x^+,x^-,\theta_+,\theta_-\right)=S_-\left(x^-,\theta_+\right)S_+\left(x^+,\theta_-\right)\,, \tag{2.65}$$

with the factors:

$$S_-=\left(1+i\theta_+\chi\right)\,g_-\,,\qquad\qquad S_+=g_+\,\left(1-i\theta_-\chi_+\right)\,. \tag{2.66}$$

Local symmetries. The general form of solutions (2.63 - 2.66) implies that the SWZW model possesses:

(i) KM or chiral symmetries $G\otimes G$, where the components of the superfield transform under $\Omega_-(x^-)$ and $\Omega_+(x^-)$, with $\Omega_-\Omega_-^\dagger=\Omega_+\Omega_+^\dagger=1$, as

$$\begin{aligned} g &\rightarrow \Omega_- g\Omega_+^{-1}\,, \\ \chi_- &\rightarrow \Omega_-\chi_-\Omega_-^{-1}\,, \\ \chi_+ &\rightarrow \Omega_+\chi_+\Omega_+^{-1}\,. \end{aligned} \tag{2.67}$$

(ii) The supersymmetry partner of the local KM transformations is an additional invariance, with local parameters Majorana spinors $\eta_+(x^+)$ and $\eta_-(x^-)$, where the fields transform as

$$\delta g=0\,, \qquad\qquad \delta\chi_\pm=\eta_\pm\,. \tag{2.68}$$

(iii) The action is, by construction, also invariant under the *conformal supersymmetry transformations* which change the line element of the superpace $ds^2=d\ell^+d\ell^-$ (where $d\ell^\pm\equiv dx^\pm-d\theta^\pm\theta^\pm$) by a scale factor: $ds'^2=\Omega\,ds^2$. The fields change under superconformal transformations as

$$\begin{aligned} \delta_\epsilon g &= i\bar{\epsilon}\psi\,, \\ \delta_\epsilon\psi &= \left(\partial\!\!\!/ g+ig\bar{\psi}\gamma^+\psi\right)\epsilon\,, \end{aligned} \tag{2.69}$$

where the local parameter ϵ is a Majorana spinor satisfying the constraint $\partial\!\!\!/\gamma^\mu\epsilon=0$, that has the solution $\epsilon^\pm\equiv\gamma^\pm\epsilon=\epsilon^\pm(x^\pm)$. The supersymmetry transformations (2.69) imply the following transformations of g and χ in components:

$$\begin{aligned} \delta_\epsilon g &= i\left(\epsilon^+\chi^-g-\epsilon^-g\chi^+\right)\,, \\ \delta_\epsilon\chi^+ &= \left(g^\dagger\partial_+g+i\chi_+\chi_+\right)\epsilon^-\,, \\ \delta_\epsilon\chi^- &= \left(g\partial_-g^\dagger+i\chi_-\chi_-\right)\epsilon^+\,. \end{aligned} \tag{2.70}$$

Similarly to the non-supersymmetric case, it is possible to find the chiral *supercurrents* which close two independent super KM algebras. The generators of the superconformal transformations, *i.e.*, the group invariants of the supercurrents which can be obtained by a generalized Sugawara construction, close two independent super Virasoro algebras without central charges. The details of this analysis can be found in Ref. [79]. Those algebras will be constructed in the next chapter following a different approach, in the context of the SWZW model coupled to supergravity.

The general form of a super Virasoro algebra has two central charges c and $\hat{c}$ which commute with all generators. The infinite set of super Virasoro generators L_n $(n \in \mathbb{Z})$ and G_r (where $r \in \mathbb{Z}$ for the *Ramond sector* [81], while $r \in \mathbb{Z}+\frac{1}{2}$ for the *Neveu-Schwarz sector* [82, 83]), obey the following (anti)commutation relations:

$$\begin{aligned}
[L_n, L_m] &= (n-m)\, L_{n+m} + \frac{c}{12}\, n\left(n^2-1\right)\delta_{n+m,0}\,, \\
[G_r, L_n] &= \left(r-\frac{n}{2}\right) G_{n+r}\,, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{\hat{c}}{3}\left(r^2-\frac{1}{4}\right)\delta_{r+s,0}\,.
\end{aligned} \tag{2.71}$$

The Ramond and Neveu-Schwarz sectors correspond to two different periodic conditions of fermions, $\theta\left(e^{2\pi i}z\right) = \theta\left(z\right)$ or $\theta\left(e^{2\pi i}z\right) = -\theta\left(z\right)$. In the Neveu-Schwarz sector, the five generators $\left\{L_{-1}, L_0, L_1, G_{-1/2}, G_{1/2}\right\}$ form a closed subalgebra $osp\left(1\left|2\right.\right)$, while in the Ramond sector a closed subalgebra does not exist.

Chapter 3

Supersymmetric WZW model coupled to supergravity

The extension of the WZW model has been studied with rigid supersymmetry [79, 80], and with local supersymmetry [84, 85, 86]. The models were constructed and analyzed using Lagrangian formalism in both superfield and component notations. In the previous chapter, it is shown that in a case of global supersymmetry it is possible to choose the fermionic field such that fermions are completely decoupled.

In this chapter, supersymmetric WZW model coupled to two-dimensional supergravity will be constructed [64] as a theory which appears as a Lagrangian realization of the super Virasoro algebra. In the Hamiltonian formalism, the components of the metric tensor and Rarita-Schwinger field appear naturally as Lagrange multipliers corresponding to the constraints satisfying the super Virasoro PB algebra. Similar approach has been used to find a diffeomorphisms invariant action for the spinning string [87]–[89].

3.1 Super Virasoro generators

The Hamiltonian formalism (see Appendix A) will be used to construct an action invariant under the gauge transformations for a given algebra of the group G. Consider a PB *representation* of the algebra in the form

$$\{\phi_r, \phi_s\} = f_{rs}{}^{p}\phi_p \, , \tag{3.1}$$

whose elements ϕ_r are functions of the coordinates q^i and their conjugate momenta p_i. By definition, ϕ_r are *first class* constraints, and the canonical Hamiltonian is assumed to be zero (there are no local degrees of freedom). Then, the canonical action

$$I\left[q,p,u\right]=\int dt\,\left(p_i\,\dot{q}^i-u^r\phi_r\right) \tag{3.2}$$

is invariant under the gauge transformations generated by the first class constraints ϕ_r. Any phase-space function $F(q,p)$ changes as

$$\delta_\varepsilon F=\left\{F,\varepsilon^r\phi_r\right\}, \tag{3.3}$$

and the Lagrange multipliers u^r transform as

$$\delta_\varepsilon u^r=\dot{\varepsilon}^r+f_{ps}{}^r\,u^s\varepsilon^p\,. \tag{3.4}$$

The multipliers will be identified as gauge fields, later.

A similar approach has been used for the construction of the action for W-strings propagating on a group manifold and on curved backgrounds [92, 93], and also for two-dimensional induced gravity [94]. The covariant extension of the WZW model with respect to an arbitrary internal group has been obtained in [95] and with respect to the $SL(2,\mathbb{R})$ internal group and diffeomorphisms in [90, 91], by the same method. Here, the last approach will be supersymmetrized.

Super Kac-Moody algebra. The representation of the supersymmetric KM algebra will be constructed starting from the known bosonic KM sector. The field $g\in G$ is a mapping from a two-dimensional Riemannian space-time $\mathcal{M}$ to a semi-simple Lie group G, parametrized by local coordinates q^i, $g=g(q)$, and generated by the anti-Hermitean generators $\mathbf{G}_a$ closing a Lie algebra with structure constants $f_{ab}^{\ \ c}$. Two Maurer-Cartan (Lie algebra valued) one-forms can be defined, $\mathbf{A}_+=g^{-1}dg$ and $\mathbf{A}_-=-g\mathbf{A}_+g^{-1}=gdg^{-1}$, whose expansions

$$\mathbf{A}_+=dq^iE_{+i}^a\,\mathbf{G}_a\,,\qquad\qquad \mathbf{A}_-\equiv dq^iE_{-i}^a\,\mathbf{G}_a\,, \tag{3.5}$$

define vielbeins $E_{\pm i}^a\left(q\right)$ on the group manifold, with inverses $E_{\pm a}^i\left(q\right)$ ($E_{\pm i}^aE_{\pm a}^j=\delta_i^j$ and $E_{\pm i}^aE_{\pm b}^i=\delta_b^a$). The Killing metric is $g_{ab}=\frac{1}{2}\left\langle\mathbf{G}_a\mathbf{G}_b\right\rangle=\frac{1}{2}\,f_{ac}^{\ \ d}f_{bd}^{\ \ c}$. The metric $\gamma_{ij}\left(q\right)$ in the coordinate basis does not depend on the choice of the Maurer-Cartan form,

$$\gamma_{ij} = E^a_{+i}\, E^b_{+j}\, g_{ab} = E^a_{-i}\, E^b_{-j}\, g_{ab}\,, \tag{3.6}$$

and has the inverse $\gamma^{ij}(q)$. The forms $\left\langle (\mathbf{A}_+)^3 \right\rangle$ and $\left\langle (\mathbf{A}_-)^3 \right\rangle$ are closed, and can be locally written as

$$\frac{1}{2}\, \left\langle (\mathbf{A}_\pm)^3 \right\rangle = -6\, d\rho\,, \tag{3.7}$$

where $\rho \equiv \frac{1}{2}\, \rho_{ij}\, dq^i dq^j$ is a two-form, independent on the choice of $\mathbf{A}_+$ or $\mathbf{A}_-$. Rewriting in components, the expressions (3.7) leads to the identities

$$\pm\frac{1}{2}\, f_{abc}\, E^a_{\mp i} E^b_{\mp j} E^c_{\mp k} = \partial_i \rho_{jk} + \partial_j \rho_{ki} + \partial_k \rho_{ij}\,. \tag{3.8}$$

Choosing the canonical representation of (bosonic) KM currents in 2D Minkowski space-time $x^\mu = (\tau, \sigma)$ $(\mu = 0, 1)$ is the form (2.13) and (2.27) [70, 95, 96], one has

$$j_{\pm i} \equiv p_i + k\, \omega_{\pm ij}\, \partial_\sigma q^j\,, \tag{3.9}$$

where the momentum-independent part is

$$\omega_{\pm ij} \equiv \rho_{ij} \pm \frac{1}{2}\, \gamma_{ij}\,. \tag{3.10}$$

The basic PB are $\{q^i(x), p_j(x')\} = \delta^i_j\, \delta(\sigma - \sigma')$, so that PB of currents (3.9) close two independent KM algebras of the group G, with the central charges $\pm k$,

$$\{j_{\pm a}(x), j_{\pm b}(x')\} = f_{ab}^{\ \ c}\, j_{\pm c}(x)\ \delta(\sigma - \sigma') \pm k\, g_{ab} \partial_\sigma \delta(\sigma - \sigma')\ , \tag{3.11}$$

where $j_{\pm a} = -E^i_{\pm a}\, j_{\pm i}$ and $\{j_{+a}(x), j_{-b}(x')\} = 0$. From now on, the δ-function will be denoted as $\delta \equiv \delta(\sigma - \sigma')$, always when its argument cannot be confused.

In order to supersymmetrize this algebra, the fermionic fields $\hat{\chi}_{\pm a}$ are introduced. Because the fermionic part of the Lagrangian should be linear in the time derivative, there always exist second class constraints $S_{\pm a} \equiv \pi_{\pm a} - ik\, \hat{\chi}_{\pm a} \approx 0$, linear in the coordinate $\hat{\chi}_{\pm a}$ and in the corresponding canonical momenta $\pi_{\pm a}$. The Dirac brackets for the fermionic fields are $\{\hat{\chi}_{\pm a}, \hat{\chi}_{\pm b}\}^* = -\frac{i}{2k}\, g_{ab}\, \delta$, while for the bosonic currents $j_{\pm a}$, they remain the same as the PB. So, one can start from the relation (3.11) and

$$\{\hat{\chi}_{\pm a}, \hat{\chi}_{\pm b}\} = -\frac{i}{2k}\, g_{ab}\, \delta\,, \tag{3.12}$$

where, from now, the star can be omitted for the sake of simplicity. Note that in both bosonic and fermionic cases all quantities of opposite chiralities commute.

It is easy to check that bilinears in the fermionic fields

$$\tilde{\jmath}_{\pm a} \equiv -ik\, f_{abc}\, \hat{\chi}^b_{\pm} \hat{\chi}^c_{\pm} \tag{3.13}$$

satisfy the KM algebra without central charges and have nontrivial brackets with $\hat{\chi}_{\pm b}$,

$$\begin{aligned} \{\tilde{\jmath}_{\pm a}, \tilde{\jmath}_{\pm b}\} &= f_{ab}^{\ \ c}\, \tilde{\jmath}_{\pm c} \delta\,, \\ \{\tilde{\jmath}_{\pm a}, \hat{\chi}_{\pm b}\} &= f_{ab}^{\ \ c}\, \hat{\chi}_{\pm c} \delta\,. \end{aligned} \tag{3.14}$$

Therefore, using $\tilde{\jmath}_{\pm a}$, the new currents can be introduced,

$$J_{\pm a} \equiv j_{\pm a} + \tilde{\jmath}_{\pm a}\,, \tag{3.15}$$

such that KM algebras remain unchanged, and which, with its supersymmetric partners $\hat{\chi}_{\pm a}$, satisfy two independent super KM algebras:

$$\begin{aligned} \{J_{\pm a}, J_{\pm b}\} &= f_{ab}^{\ \ c}\, J_{\pm c}\, \delta \pm k\, g_{ab}\, \partial_\sigma \delta\,, \\ \{J_{\pm a}, \hat{\chi}_{\pm b}\} &= f_{ab}^{\ \ c}\, \hat{\chi}_{\pm c}\, \delta\,, \\ \{\hat{\chi}_{\pm a}, \hat{\chi}_{\pm b}\} &= -\frac{i}{2k}\, g_{ab}\, \delta\,. \end{aligned} \tag{3.16}$$

A super KM current is defined as

$$I_{\pm a}(z) \equiv \sqrt{2}k \hat{\chi}_{\pm a}(x) + \theta_{\mp} J_{\pm a}(x)\,, \tag{3.17}$$

where θ_α ($\alpha = +, -$) is a Majorana spinor, and four real local coordinates $z^M = (x^\mu, \theta_\alpha)$ parametrize $(1,1)$ superspace. Then the algebra (3.16) can be rewritten in the form

$$\{I_{\pm a}(z_1), I_{\pm b}(z_2)\} = \delta_{\pm 12} f_{ab}^{\ \ c}\, I_{\pm c}(z_1) - ik\, g_{ab}\, D^{\pm} \delta_{\pm 12}\,, \tag{3.18}$$

where $D^{\pm} \equiv \frac{\partial}{\partial \theta_{\mp}} \mp i\theta_{\mp} \frac{\partial}{\partial \sigma}$ is the super covariant derivative, while $\delta_{\pm 12} = (\theta_{\pm 1} - \theta_{\pm 2})\delta(\sigma_1 - \sigma_2)$ is a generalization of the Dirac δ-function to the super δ-function. Derivatives are always taken over the first argument of δ-functions. Superspace notation is given in Appendix B.

Super Virasoro algebra. The next step is to construct a super energy-momentum tensor as a function of the super KM currents, which is a group invariant. Up to the third power of $I_{\pm a}$, there are only two group invariants

$$\mathcal{I}_{\pm 1} \equiv g_{ab}\, D^{\pm} I^a_{\pm} I^b_{\pm}\,, \qquad \mathcal{I}_{\pm 2} \equiv i f_{abc}\, I^a_{\pm} I^b_{\pm} I^c_{\pm}\,, \tag{3.19}$$

because $g_{ab}\, I^a_{\pm} I^b_{\pm}$ is identically equal to zero (since super currents are odd variables). Therefore, the super energy-momentum tensor is looked for in the form

$$T_{\pm} \equiv \alpha_{\pm}\mathcal{I}_{\pm 1} + \beta_{\pm}\mathcal{I}_{\pm 2}\,, \tag{3.20}$$

where the requirement for the superalgebra closure determines the ratio of coefficients $\alpha_{\pm}$ and $\beta_{\pm}$. In components notation, one has

$$T_{\pm} = \mp G_{\pm} + \theta_{\mp} L_{\pm}\,, \tag{3.21}$$

where $L_{\pm}$ is the bosonic part which closes in a (bosonic) Virasoro algebra, while $G_{\pm}$ is its supersymmetric partner. The explicit expressions can be found from the expansions of the invariants:

$$\begin{aligned} \mathcal{I}_{\pm 1} &= \sqrt{2}k J^a_{\pm}\hat{\chi}_a + \theta_{\mp}\left(J^a_{\pm}J_{\pm a} \mp 2ik^2 \partial_\sigma \hat{\chi}^a_{\pm}\hat{\chi}_{\pm a}\right)\,, \\ \mathcal{I}_{\pm 2} &= 2ik^2\left(\sqrt{2}k\, f_{abc}\,\hat{\chi}^a_{\pm}\hat{\chi}^b_{\pm}\hat{\chi}^c_{\pm} + 3\theta_{\mp} f_{abc}\,\hat{\chi}^a_{\pm}\hat{\chi}^b_{\pm} J^c_{\pm}\right)\,. \end{aligned} \tag{3.22}$$

The simplest way to find the coefficients in (3.20) is to use the fact that for every super KM algebra there is a super Virasoro algebra such that they form a semi-direct product, Eq. (2.41) or, in another words, that PB of a Virasoro generator and a current, gives a current. Therefore, it is easy to see that

$$\{G_{\pm}, \hat{\chi}_{\pm a}\} = \pm\frac{i\alpha_{\pm}}{\sqrt{2}} J_{\pm a}\delta \pm \sqrt{2}k\,\left(\alpha_{\pm} - 3i\beta_{\pm}k\right)\,, \tag{3.23}$$

and the *r.h.s.* of (3.23) closes on currents only if the last term vanishes. Therefore

$$\frac{\beta_{\pm}}{\alpha_{\pm}} = -3ik\,, \tag{3.24}$$

and the super energy-momentum tensor is normalized as

$$T_{\pm} \equiv \mp\frac{1}{2k}\left(g_{ab}\, D^{\pm} I^a_{\pm} I^b_{\pm} + \frac{i}{3k}\, f_{abc}\, I^a_{\pm} I^b_{\pm} I^c_{\pm}\right)\,, \tag{3.25}$$

where the bosonic part is

$$L_\pm = \mp\frac{1}{2k}\left(J^a_\pm J_{\pm a} \pm 2ik^2\,\hat\chi^a_\pm\,\partial_\sigma\hat\chi_{\pm a} + 2ik\,f_{abc}\,\hat\chi^a_\pm\hat\chi^b_\pm J^c_\pm\right)\,, \tag{3.26}$$

while its supersymmetric partner is

$$G_\pm = \frac{1}{\sqrt{2}}J^a_\pm\hat\chi_{\pm a} + \frac{i\sqrt{2}k}{3}f_{abc}\,\hat\chi^a_\pm\hat\chi^b_\pm\hat\chi^c_\pm\,. \tag{3.27}$$

It is useful to express (3.26) and (3.27) in terms of commuting quantities $j_{\pm a}$ and $\hat\chi_{\pm a}$:

$$\begin{aligned} L_\pm &= \mp\frac{1}{2k}\,j^a_\pm j_{\pm a} - ik\,\hat\chi^a_\pm\partial_\sigma\hat\chi_{\pm a}\,,\\ G_\pm &= \frac{1}{\sqrt{2}}\,j^a_\pm\hat\chi_{\pm a} - \frac{ik}{3\sqrt{2}}\,f_{abc}\,\hat\chi^a_\pm\hat\chi^b_\pm\hat\chi^c_\pm\,. \end{aligned} \tag{3.28}$$

Using the super KM algebra (3.16), the following brackets between the components of energy-momentum tensor and currents are obtained,

$$\begin{aligned} \{L_\pm, J_{\pm a}\} &= -J_{\pm a}\,\partial_\sigma\delta\,, & \{G_\pm, J_{\pm a}\} &= \pm\tfrac{k}{\sqrt{2}}\,\hat\chi_{\pm a}\,\partial_\sigma\delta\,,\\ \{L_\pm, \hat\chi_{\pm a}\} &= \tfrac{1}{2}\left(\partial_\sigma\hat\chi_{\pm a}\,\delta - \hat\chi_{\pm a}\partial_\sigma\delta\right), & \{G_\pm, \hat\chi_{\pm a}\} &= -\tfrac{i}{2\sqrt{2}k}\,J_{\pm a}\,\delta\,, \end{aligned} \tag{3.29}$$

as well as the brackets between the components of energy-momentum tensor themselves

$$\begin{aligned} \{L_\pm, L_\pm\} &= -(\partial_\sigma L_\pm\delta + 2L_\pm\partial_\sigma\delta) = -\left[L_\pm(x) + L_\pm(x')\right]\partial_\sigma\delta\,,\\ \{G_\pm, G_\pm\} &= \pm\frac{i}{2}\,L_\pm\delta\,,\\ \{L_\pm, G_\pm\} &= -\frac{1}{2}\,\left(\partial_\sigma G_\pm\delta + 3G_\pm\partial_\sigma\delta\right)\,,\\ \{G_\pm, L_\pm\} &= -\frac{1}{2}\,\left(2\partial_\sigma G_\pm\delta + 3G_\pm\partial_\sigma\delta\right)\,. \end{aligned} \tag{3.30}$$

In terms of the super fields, instead of (3.29) one has

$$\{T_\pm(z_1), I_{\pm a}(z_2)\} = \pm\frac{i}{2}\,\left(D^\pm I_{\pm a}\,D^\pm\delta_{\pm 12} + I_{\pm a}\,D^{\pm 2}\delta_{\pm 12}\right)\,, \tag{3.31}$$

and instead of (3.30)

$$\{T_\pm(z_1), T_\pm(z_2)\} = \pm\frac{i}{2}\,\left(2D^{\pm 2}T_\pm\,\delta_{\pm 12} + D^\pm T_\pm\,D^\pm\delta_{\pm 12} + 3T_\pm\,D^{\pm 2}\delta_{\pm 12}\right)\,. \tag{3.32}$$

This is a supersymmetric extension of the Virasoro algebra *without central charge*.

Since $L_\pm$ and $G_\pm$ are the first class constrains, one can apply the general canonical method to construct a theory invariant under *diffeomorphisms* generated by $L_\pm$ and under *local supersymmetry* generated by $G_\pm$. It is known that, in the bosonic case, the similar approach leads to a covariant extension of the WZW theory with respect to diffeomorphisms [90, 91], thus here one expects to obtain covariant extension of WZW theory with respect to local supersymmetry.

3.2 Effective Lagrangian and gauge transformations

In order to construct a covariant theory, one uses the generators

$$\phi_r = (L_-, L_+, G_-, G_+)\,, \tag{3.33}$$

as first class constraints, with the explicit expressions given in equations (3.28), and the PB algebra (3.30) instead of the equation (3.1).

The action. According to (A.1), the canonical Lagrangian is introduced as

$$\hat{L} = \dot{q}^i p_i + ik\,\dot{\hat{\chi}}_+^a \hat{\chi}_{+a} + ik\,\dot{\hat{\chi}}_-^a \hat{\chi}_{-a} - h^- L_- - h^+ L_+ - i\psi^- G_- - i\psi^+ G_+\,, \tag{3.34}$$

with multipliers $u^r = (h^-, h^+, \psi^-, \psi^+)$. Note that, on Dirac brackets, the second class constrains are zero ($S_{\pm a} = 0$), giving the Grassmann odd momenta as $\pi_{\pm a} = ik\,\hat{\chi}_{\pm a}$. The remaining momentum variables can be eliminated by means of their equations of motion,

$$p^i = \frac{k}{h^- - h^+}\left[\dot{q}^i + \left(h^+ {\omega_+}^i{}_j - h^- {\omega_-}^i{}_j\right)\partial_\sigma q^j + \frac{i}{\sqrt{2}}\left(\psi^- \hat{\chi}_-^i + \psi^+ \hat{\chi}_+^i\right)\right], \tag{3.35}$$

where $\hat{\chi}^i_\pm = E^i_{\pm a}\hat{\chi}^a_\pm$. On the equations of motion (3.35), the currents (3.9) become

$$j^i_\pm = \frac{k}{2}\left[\hat{\partial}_\pm q^i + \frac{i}{\sqrt{-2\hat{g}}}\left(\psi^- \hat{\chi}_-^i + \psi^+ \hat{\chi}_+^i\right)\right], \tag{3.36}$$

so that the Lagrangian (3.34) can be written as a sum of three terms

$$\hat{L} = \hat{L}_{\text{WZW}} + \hat{L}_{\text{f}} + \hat{L}_{\text{int}}\,, \tag{3.37}$$

which have the form:

$$
\begin{aligned}
\hat{L}_{\rm WZW} &= -\tfrac{k}{2}\sqrt{-\hat{g}}\,\omega_{-ij}\,\hat{\partial}_- q^i \hat{\partial}_+ q^j\,, \\
\hat{L}_{\rm f} &= -ik\sqrt{-\hat{g}}\,\left(\hat{\chi}^a_+ \hat{D}_- \hat{\chi}_{+a} + \hat{\chi}^a_- \hat{D}_+ \hat{\chi}_{-a}\right)\,, \\
\hat{L}_{\rm int} &= \tfrac{ik}{2\sqrt{2}}\,\left(\psi^+ \hat{\chi}_{+i} \hat{\partial}_+ q^i + \psi^- \hat{\chi}_{-i} \hat{\partial}_- q^i\right)\,.
\end{aligned}
\tag{3.38}
$$

Here, $\hat{L}_{\rm WZW}$ and $\hat{L}_{\rm f}$ are the WZW and fermion Lagrangians respectively, covariantized in super gravitational fields $\hat{g}_{\mu\nu}$ and $\psi^{\pm}$, while $\hat{L}_{\rm int}$ describes the interaction between bosonic fields q^i and fermionic fields $\hat{\chi}_{\pm i}$. The tensor $\hat{g}_{\mu\nu}$ is introduced instead of variables (h^+, h^-) (see Appendix C)

$$
\hat{g}_{\mu\nu} \equiv -\frac{1}{2}\begin{pmatrix} -2h^+h^- & h^+ + h^- \\ h^+ + h^- & -2 \end{pmatrix}\,. \tag{3.39}
$$

Covariant derivatives, acting on fermionic fields $\hat{\chi}^a_{\pm}$, are defined by

$$
\hat{D}_{\mp}\hat{\chi}^a_{\pm} \equiv \hat{\partial}_{\mp}\hat{\chi}^a_{\pm} + \frac{i}{3\sqrt{-2\hat{g}}}\,f_{bc}{}^a \psi^{\pm}\hat{\chi}^b_{\pm}\hat{\chi}^c_{\pm} \pm \frac{i}{8\hat{g}}\,\psi^-\psi^+\,\hat{\chi}^a_{\mp}\,, \tag{3.40}
$$

where $\hat{\partial}_{\pm} = \hat{e}^{\mu}{}_{\pm}\partial_{\mu}$, and $\hat{e}^{\mu}{}_{\pm}$ are also given in Appendix.

Local symmetries. The general canonical method provides a mechanism to write out gauge symmetries of the Lagrangian (3.37). Instead of relations (3.3), with the help of (3.28), the following gauge transformations of the fields are found,

$$
\begin{aligned}
\delta q^i &= \frac{1}{k}\left(\varepsilon^- j^i_- - \varepsilon^+ j^i_+\right) - \frac{i}{\sqrt{2}}\left(\eta^+ \hat{\chi}^i_+ + \eta^- \hat{\chi}^i_-\right)\,, \\
\delta\hat{\chi}^a_{\pm} &= -\varepsilon^{\pm}\partial_1 \hat{\chi}^a_{\pm} - \frac{1}{2}\,\left(\partial_1\varepsilon^{\pm}\right)\hat{\chi}^a_{\pm} - \frac{1}{2k\sqrt{2}}\,\eta^{\pm}\,J^a_{\pm}\,,
\end{aligned}
\tag{3.41}
$$

and instead of (3.4) using (3.30), one obtains the gauge transformations of the multipliers,

$$
\begin{aligned}
\delta h^{\pm} &= \partial_0\varepsilon^{\pm} + h^{\pm}\partial_1\varepsilon^{\pm} - \varepsilon^{\pm}\partial_1 h^{\pm} \pm \frac{i}{2}\,\psi^{\pm}\eta^{\pm}\,, \\
\delta\psi^{\pm} &= \frac{1}{2}\,\psi^{\pm}\partial_1\varepsilon^{\pm} - \left(\partial_1\psi^{\pm}\right)\varepsilon^{\pm} + \partial_0\eta^{\pm} + h^{\pm}\partial_1\eta^{\pm} - \frac{1}{2}\,\left(\partial_1 h^{\pm}\right)\eta^{\pm}\,.
\end{aligned}
\tag{3.42}
$$

The bosonic fields $\varepsilon^{\pm}$ and the fermionic fields $\eta^{\pm}$ are the parameters of diffeomorphisms and local supersymmetry transformations respectively.

3.3 Lagrangian formulation

It turns out that the Lagrangian (3.37) is invariant under the following rescaling of fields by two arbitrary parameters $F(x)$ and $f(x)$:

$$\begin{aligned}
\hat{e}^{\pm}{}_{\mu} &\rightarrow e^{\pm}{}_{\mu} \equiv e^{F \pm f}\, \hat{e}^{\pm}{}_{\mu}\,, \\
\psi^{\pm} &\rightarrow \psi_{\mp(\mp)} \equiv \tfrac{1}{2\sqrt{-\hat{g}}}\, e^{-\frac{1}{2}(F \mp 3f)}\, \psi^{\pm}\,, \\
\hat{\chi}^{a}_{\pm} &\rightarrow \chi^{a}_{\pm} \equiv e^{-\frac{1}{2}(F \pm f)}\, \hat{\chi}^{a}_{\pm}\,.
\end{aligned} \tag{3.43}$$

As a consequence, one has:

$$\begin{aligned}
\sqrt{-\hat{g}} &\rightarrow \sqrt{-g} = e^{2F}\sqrt{-\hat{g}}\,, \\
\hat{\partial}_{\pm} &\rightarrow \partial_{\pm} \equiv e^{-(F \pm f)}\, \hat{\partial}_{\pm}\,.
\end{aligned} \tag{3.44}$$

In terms of the rescaled fields, the rescaled Lagrangian has the same form as the original one (3.37),

$$\begin{aligned}
L &= L_{\mathrm{WZW}} + L_{\mathrm{f}} + L_{\mathrm{int}}\,, \\
L_{\mathrm{WZW}} &= -\tfrac{k}{2}\sqrt{-g}\,\omega_{-ij}\,\partial_{-}q^{i}\partial_{+}q^{j}\,, \\
L_{\mathrm{f}} &= -ik\sqrt{-g}\,\left(\chi^{a}_{+}D_{-}\chi_{+a} + \chi^{a}_{-}D_{+}\chi_{-a}\right)\,, \\
L_{\mathrm{int}} &= \tfrac{ik}{\sqrt{2}}\sqrt{-g}\,\left[\psi_{-(-)}\chi_{+i}\partial_{+}q^{i} + \psi_{+(+)}\chi_{-i}\partial_{-}q^{i}\right]\,,
\end{aligned} \tag{3.45}$$

where the covariant derivative acts on $\chi^{a}_{\pm}$ as

$$D_{\mp}\chi^{a}_{\pm} \equiv \partial_{\mp}\chi^{a}_{\pm} + \frac{i\sqrt{2}}{3}\, f_{ab}{}^{c}\psi_{\mp(\mp)}\chi^{b}_{\pm}\chi^{c}_{\pm} \mp \frac{i}{2}\,\psi_{+(+)}\psi_{-(-)}\,\chi^{a}_{\mp}\,. \tag{3.46}$$

Note that the term with derivatives over F and f vanishes because of nilpotency of the field $\hat{\chi}^{i}_{\pm}$.

The introduction of the new fields F and f gives additional gauge freedom to the Lagrangian (3.45). As a consequence of the transformation $\delta_{\Lambda}F = \Lambda$, the Lagrangian becomes invariant under the *local Weyl transformations*

$$\begin{aligned}
\delta_{\Lambda}e^{\pm}{}_{\mu} &= \Lambda e^{\pm}{}_{\mu}\,, \\
\delta_{\Lambda}\psi_{\pm(\pm)} &= -\tfrac{1}{2}\,\Lambda\psi_{\pm(\pm)}\,, \\
\delta_{\Lambda}\chi^{a}_{\pm} &= -\tfrac{1}{2}\,\Lambda\chi^{a}_{\pm}\,,
\end{aligned} \tag{3.47}$$

while all F independent fields remain Weyl invariant. Furthermore, the Lagrangian (3.45) does not change under the *local Lorentz transformations*

$$\begin{aligned}
\delta_\ell e^{\pm}{}_\mu &= \mp \ell e^{\pm}{}_\mu \,, \\
\delta_\ell \psi_{\pm(\pm)} &= \pm\tfrac{3}{2}\,\ell\,\psi_{\pm(\pm)} \,, \\
\delta_\ell \chi^a_\pm &= \pm\tfrac{1}{2}\,\ell\,\chi^a_\pm \,,
\end{aligned} \tag{3.48}$$

which are generated by the transformation $\delta_\ell f = -\ell$. The vielbein $e^{\pm}{}_\mu$ transforms as a Lorentz vector, the fields $\psi_{\pm(\pm)}$ and $\chi^a_\pm$ transform like components of a spinor field with the spin $\frac{3}{2}$ and $\frac{1}{2}$ respectively, while all f independent fields are Lorentz scalars.

The Lagrangian (3.45) depends only on two components $\psi_{\pm(\pm)}$ of the Rarita-Schwinger spinor field $\psi_{\mu\alpha} = e^a{}_\mu \psi_{a(\alpha)}$. It means that there is also the additional *local super Weyl symmetry*

$$\delta_\xi \psi_{\pm(\mp)} = \pm \xi_\pm \,, \tag{3.49}$$

where ξ_α is an odd parameter. All other fields are super Weyl invariant. Transformations (3.47) – (3.49) can be written in a covariant form,

$$\begin{array}{llll}
\text{Lorentz:} & \delta_\ell e^a{}_\mu = -\ell\,\varepsilon^a{}_b\,e^b{}_\mu\,, & \delta_\ell\psi_\mu = \frac{1}{2}\,\ell\,\gamma_5\,\psi_\mu\,, & \delta_\ell\chi^a = \frac{1}{2}\,\ell\gamma_5\,\chi^a\,, \\
\text{Weyl:} & \delta_\Lambda e^a{}_\mu = \Lambda e^a{}_\mu\,, & \delta_\Lambda\psi_\mu = \frac{1}{2}\Lambda\psi_\mu\,, & \delta_\Lambda\chi^a = \frac{1}{2}\,\Lambda\chi^a\,, \\
\text{super Weyl:} & \delta_\xi e^a{}_\mu = 0\,, & \delta_\xi\psi_\mu = \gamma_\mu\xi\,, & \delta_\xi\chi^a = 0\,,
\end{array} \tag{3.50}$$

where $\gamma_\mu \equiv e^a{}_\mu\gamma_a$. The representation of the γ-matrices is given in Appendix C. Note that the fields $\hat{e}^{\pm}{}_\mu$, $\psi^\pm$ and $\hat{\chi}^a_\pm$, introduced in the Hamiltonian approach are Lorentz, Weyl and super Weyl invariants.

The fields F and f do not enter the original Lagrangian (3.37) but are introduced by the rescaling of fields (3.43). Thus their change under diffeomorphisms and supersymmetry transformations cannot be found just applying the general Hamiltonian rules (3.3), (3.4). They are introduced in such a way that the new fields $e^a{}_\mu$ and $\psi_{\mu\alpha}$ have proper Lorentz, Weyl and super Weyl transformations. Now, one demands that $e^a{}_\mu$ transforms like a vector and $\psi_{\mu\alpha}$ transforms like a Rarita-Schwinger field under general coordinate transformations with a local parameter $\varepsilon^\mu(x)$ and under $\mathcal{N}=1$ supersymmetric transformations with a local spinor parameter $\zeta_\alpha(x)$. It means that (see for example [97])

$$\begin{aligned} \delta e^a{}_\mu &= -\varepsilon^\nu \partial_\nu e^a{}_\mu - e^a{}_\nu \partial_\mu \varepsilon^\nu - \frac{i}{2}\,\bar{\zeta}\gamma^a \psi_\mu\,, \\ \delta\psi_\mu &= -\varepsilon^\nu \partial_\nu \psi_\mu - \psi_\nu\,\partial_\mu \varepsilon^\nu + \frac{1}{2}\,\nabla_\mu \zeta\,, \end{aligned} \tag{3.51}$$

where $\nabla_\mu \zeta = e^a{}_\mu \nabla_a \zeta$ and $\nabla_a \zeta = \left(\partial_a + \frac{1}{2}\gamma_5 \omega_a\right)\zeta$. Writing out in components, it gives

$$\begin{aligned} \delta(F \pm f) &= h^\pm \partial_1 \varepsilon^0 - \partial_1 \varepsilon^1 - \varepsilon^\mu \partial_\mu (F \pm f) \pm i e^{-(F\pm f)} \zeta_\mp \psi_{1\mp}\,, \\ \delta h^\pm &= \partial_0 \varepsilon^1 - h^\pm(\partial_0\varepsilon^0 - \partial_1\varepsilon^1) - (h^\pm)^2 \partial_1 \varepsilon^0 - \varepsilon^\mu \partial_\mu h^\pm \mp \tfrac{i}{2}\, e^{-\frac{1}{2}(F\pm f)} \zeta_\mp \psi^\pm\,, \\ \delta\psi^\pm &= \left[\tfrac{1}{2}\,\partial_1(\varepsilon^1 - h^\pm\varepsilon^0) + \tfrac{1}{2}\,(\partial_1 h^\pm)\varepsilon^0 - \partial_0\varepsilon^0 - h^\pm \partial_1 \varepsilon^0\right] \psi^\pm - \varepsilon^\mu \partial_\mu \psi^\pm\,, \\ &+ \left(\partial_0 + h^\pm \partial_1 - \tfrac{1}{2}\,\partial_1 h^\pm\right)\left(\zeta_\mp e^{-\frac{1}{2}\,(F\pm f)}\right) + \tfrac{i}{4}\,\zeta_\mp \psi_{\pm(\mp)} \psi^\pm\,, \end{aligned} \tag{3.52}$$

where $\psi^\mp \equiv 2e^{-\frac{1}{2}\,(F\mp f)}\,(\psi_{0\pm} + h^\mp \psi_{1\pm})$ in according with (3.43).

In order to establish the relation between the Hamiltonian and Lagrangian transformations one should compare the Hamiltonian transformations (3.42) with the Lagrangian one, the last two equations of (3.52). One finds that these gauge transformations can be identified by choosing the following relation between the gauge parameters

$$\varepsilon^\pm \equiv \varepsilon^1 - h^\pm \varepsilon^0\,, \qquad \eta^\pm \equiv 2\zeta_\mp e^{-\frac{1}{2}\,(F\pm f)} - \varepsilon^0 \psi^\pm\,,$$

and imposing the gauge fixing $\psi_{\pm(\mp)} = 0$, because in the Hamiltonian approach all quantities are super Weyl invariant. Substituting the equations of motion (3.36) in (3.41), the momentum independent formulation of transformation law of matter variables q^i, $\hat{\chi}^a_\pm$ is obtained. In terms of the Lagrangian variables ε^μ and $\zeta_\pm$, they stand

$$\begin{aligned} \delta q^i &= -\varepsilon^\mu\,\partial_\mu q^i - i\sqrt{2}\,\left(\zeta_- \chi^i_+ + \zeta_+ \chi^i_-\right)\,, \\ \delta\hat{\chi}^a_\pm &= -\varepsilon^\mu\,\partial_\mu \hat{\chi}^a_\pm + \varepsilon^0 \sqrt{-\hat{g}}\hat{\nabla}_\mp \hat{\chi}^a_\pm + \frac{1}{2}\,\left(h^\pm \partial_1 \varepsilon^0 - \partial_1 \varepsilon^1\right)\hat{\chi}^a_\pm + \\ &+ \frac{1}{2k\sqrt{2}}\,\left(\varepsilon^0 \psi^\pm - 2\zeta_\mp e^{-\frac{1}{2}(F\pm f)}\right) J^a_\pm\,. \end{aligned} \tag{3.53}$$

In the flat space limit $e^a{}_\mu \to \delta^a_\mu$ and $\psi_\mu \to 0$, the Lagrangian (3.45) becomes

$$L_0 = -\frac{k}{2}\,\omega_{-ij}\,\partial_- q^i \partial_+ q^j - ik\,\left(\chi^a_+ \partial_- \chi_{+a} + \chi^a_- \partial_+ \chi_{-a}\right)\,, \tag{3.54}$$

and bosonic and fermionic parts are decoupled. The first term is the bosonic WZW action, while the second one is the Lagrangian of free spinor fields $\chi^a_\pm$. The Lagrangian (3.54) describes the $\mathcal{N} = 1$ supersymmetric WZW theory discussed in the previous chapter [79, 80].

3.4 Conclusions

Using the general canonical method, the Lagrangian for the super WZW model coupled to two-dimensional supergravity is constructed. The basic ingredients of this approach are the symmetry generators, which are functions of the coordinates and momenta and satisfy the super KM and super Virasoro algebras. The application of the Hamiltonian method naturally incorporates gauge fields (the metric tensor and Rarita-Schwinger fields) as Lagrange multipliers of the symmetry generators. This method also gives a prescription for finding gauge transformations for both the matter and gauge fields.

The Hamiltonian formalism deals with the Lagrangian multipliers $h^\pm$ and $\psi^\pm$, which are just the part of the gauge fields necessary to represent the symmetry of the algebra. In the Lagrangian formulation, the covariant description of the fields is needed. In order to complete the vielbeins $e^a{}_\mu$, it was necessary to introduce the new bosonic components F and f, while for completing the Rarita-Schwinger fields $\psi_{\mu\alpha}$, the new fermionic fields $\psi_{\pm(\mp)}$ are needed. The new components are not physical because they do not appear in the Lagrangian, but they give additional gauge freedom corresponding to the additional gauge symmetries. These are local Weyl and local Lorentz symmetries for the bosonic fields F and f respectively, and local super Weyl symmetry for the fermionic field $\psi_{\mp(\pm)}$. The fields F and f are not parts of the Hamiltonian formalism, so their transformation laws under reparametrizations and supersymmetry is found requiring that vielbeins $e^a{}_\mu$ transform as vectors, and $\psi_{\mu\alpha}$ transform as a Rarita-Schwinger field. Consequently, the complete relation between Hamiltonian and Lagrangian formulations is established. The connection between the corresponding fields, gauge parameters and gauge transformations is found. In the flat superspace limit, the result of Ref. [79] is reproduced.

Chapter 4

Irregular constrained systems

In the previous chapters, the Hamiltonian analysis was used to find all local symmetries of the WZW model and its supersymmetric extension. The canonical method was applied to construct the super WZW action coupled to supergravity, starting from the representation of the super Virasoro algebra.

In general, Dirac's Hamiltonian analysis provides a systematic technique for finding the gauge symmetries and the physical degrees of freedom of constrained systems like gauge theories and gravity [59]. These theories contain more dynamical variables than is minimally requested by physics and, consequently, they are not independent. Therefore, in those theories constraints arise as functions of local coordinates in phase space, and they are usually functionally independent, at least in the most common cases of physical interest. However, there are some exceptional cases in which functional independence is violated, when it is not always clear how to identify local symmetries and physical degrees of freedom. In this chapter, these *irregular* systems are analyzed. For the review of the Hamiltonian formalism see Appendix A.

4.1 Regularity conditions

Consider a constrained system on phase space Γ with local coordinates $z^n = (q,p)$ $(n = 1,\ldots,2N)$ and a complete set of constraints

$$\phi_r(z) \approx 0\,, \qquad (r = 1,\ldots,R)\,, \tag{4.1}$$

which define a constraint surface[1]

$$\Sigma = \{\bar{z} \in \Gamma \mid \phi_r(\bar{z}) = 0 \;\; (r = 1, \ldots, R) \;\; (R \leq 2N)\} \; . \tag{4.2}$$

Dirac's procedure guarantees that the system remains on the constraint surface during its evolution.

Choosing different coordinates on Γ may lead to different forms for the constraints whose functional independence is not obvious. The *regularity conditions* (RCs) were introduced by Dirac to test this [98]:

The constraints $\phi_r \approx 0$ are regular if and only if their small variations $\delta\phi_r$ evaluated on Σ define R linearly independent functions of δz^n.

To first order in δz^n, the variations of the constraints have the form

$$\delta\phi_r = J_{rn}\, \delta z^n \, , \qquad (r = 1, ..., R) \, , \tag{4.3}$$

where $J_{rn} \equiv \partial\phi_r/\partial z^n|_\Sigma$ is the Jacobian evaluated on the constraint surface. An equivalent definition of the RCs is [99]:

The set of constraints $\phi_r \approx 0$ is regular if and only if the Jacobian $J_{rn} = \partial\phi_r/\partial z^n|_\Sigma$ has maximal rank, $\Re(\mathbf{J}) = R$.

A system of just one constraint can also fail the test of regularity, as for the constraint $\phi = q^2 \approx 0$ in a 2-dimensional phase space (q, p). In this case, the Jacobian $\mathbf{J} = (2q, 0)_{q^2=0} = 0$ has zero rank. The same problem occurs with the constraint $q^k \approx 0$, for $k > 1$, which has a zero of k-th order on the constraint surface. Thus one *constraint* may be dependent on itself, while one *function* is, by definition, always functionally independent.

Equivalent constraints. Different sets of constraints are said to be *equivalent* if they define the same constraint surface. This definition refers to the locus of constraints, not to the equivalence of the resulting dynamics. Since the surface Σ is defined by the

[1] This terminology is standard in Hamiltonian analysis although, in general, the set Σ is not a manifold since it can contain discontinuities, be non-differentiable, etc.

zeros of the constraints, while the regularity conditions depend on their derivatives, it is possible to replace a set of irregular ϕ's by an *equivalent* set of *regular* constraints $\tilde{\phi}$.

In the following sections, the *nature* of the constraints that give rise to irregularity is analyzed, and *where* the irregularities can occur.

4.2 Basic types of irregular constraints

Irregular constraints can be classified according to their behavior in the vicinity of the surface Σ. For example, linearly dependent constraints have Jacobian with constant rank R' throughout Σ, and

$$\phi_r \equiv J_{rn}(\bar{z})\,(z^n - \bar{z}^n) \approx 0\,, \qquad \Re(\mathbf{J}) = R' < R\,. \tag{4.4}$$

These constraints are regular systems in disguise simply because $R - R'$ constraints are redundant and should be discarded. The subset with R' linearly independent constraints gives the correct description. For example, linearly dependent constraints are clearly in this category. Apart from this trivial case, two main types of truly irregular constraints, which do not possess a linear approximation in the vicinity of some points of Σ, can be distinguished:

Type I. Multilinear constraints. Consider the constraint

$$\phi \equiv \prod_{i=1}^{M} f_i(z) \approx 0\,, \tag{4.5}$$

where the functions f_i have simple zeros. Each factor defines a surface of codimension 1,

$$\Sigma_i \equiv \{\bar{z} \in \Gamma \mid f_i(\bar{z}) = 0\}\,, \tag{4.6}$$

and Σ is the collection of all surfaces, $\Sigma = \bigcup \Sigma_i$. The rank of the Jacobian of ϕ is reduced at intersections

$$\Sigma_{ij} \equiv \Sigma_i \bigcap \Sigma_j\,. \tag{4.7}$$

Thus, the RCs hold everywhere on Σ, except at the intersections Σ_{ij}, where ϕ has zeros of higher order. Note that the intersections (4.7) also include the points where more than two Σ's overlap.

Type II. Nonlinear constraints. Consider the constraint of the form

$$\phi \equiv [f(z)]^k \approx 0\,, \qquad (k > 1)\ , \tag{4.8}$$

where the function $f(z)$ has a simple zero. This constraint has a zero of order k in the vicinity of Σ, its Jacobian vanishes on the constraint surface and, therefore, the RCs fail[2]. It could seem harmless to replace ϕ by the equivalent regular constraint $f(z) \approx 0$, but it is allowed to do only if it does not change the dynamics of original system, what is discussed below.

Types I and II are the two fundamental generic classes of irregular constraints. In general, there can be combinations of them occurring simultaneously near a constraint surface, as in constraints of the form $\phi = [f_1(z)]^2\, f_2(z) \approx 0$, etc.

4.3 Classification of constraint surfaces

The previous classification refers to the way in which ϕ approaches zero. Now it is going to be discussed *where* regularity can be violated.

The example of $q^2 \approx 0$ showed that only one constraint can be irregular, even if, as a function, it is functionally independent. This is possible since functional independence of the *functions* $\phi^r(z)$ permits the Jacobian $\partial\phi^r/\partial z^i$ to have rank lower than maximal on a submanifold of measure zero,

$$\Xi = \left\{ w \in \Gamma \left| \Re \left(\frac{\partial \phi^r}{\partial z^i} \right)_{z=w} < R \right. \right\}\ , \tag{4.9}$$

while the *constraints* $\phi^r \approx 0$ have the Jacobian evaluated at the particular surface Σ. Their intersection

$$\Sigma_0 = \Sigma \cap \Xi \tag{4.10}$$

defines a submanifold on Σ where the RCs are violated, while on the rest of Σ the RCs are satisfied.

Thus, barring accidental degeneracies such as linearly dependent constraints, one of these three situation may present themselves:

[2]Here it is assumed $k > 1$, in order to avoid infinite values for $\frac{\partial \phi}{\partial z^i}$ on Σ.

A. *The RCs are satisfied everywhere on the constraint surface*: The surfaces Σ and Ξ do not intersect and $\mathbf{J}$ has maximal rank throughout Σ. These are regular systems.

B. *The RCs fail everywhere on the constraint surface:* $\Sigma_0 = \Sigma$ is a submanifold of Ξ and $\mathbf{J}$ has constant rank $R' < R$ on Σ_0. This irregular system contains nonlinear (type II) constraints.

C. *The RCs fail on* Σ_0: The intersection Σ_0 is a measure zero submanifold on Σ, so that $\Re\left(\mathbf{J}\right) = R' < R$ on Σ_0, while $\Re\left(\mathbf{J}\right) = R$ elsewhere on Σ. This irregular system contains multilinear (type I) constraints.

In the last case, the constraint surface can be decomposed into two non overlapping sets Σ_0 and Σ_R. Then, the rank of the Jacobian jumps from $\Re(\mathbf{J}) = R$ on Σ_R, to $\Re(\mathbf{J}) = R'$ on Σ_0 and the manifold Σ is not differentiable at Σ_0. Although the functions ϕ_r are continuous and differentiable, this is not sufficient for regularity.

For example, a massless relativistic particle in Minkowski space has phase space (q^μ, p_ν) with both regular and irregular sectors. The constraint $\phi \equiv p^\mu p_\mu \approx 0$ has Jacobian $\mathbf{J} = (0, 2p^\mu)_{\phi=0}$, and its rank is one everywhere, except at the apex of the cone, $p^\mu = 0$, where the light-cone is not differentiable and the Jacobian has rank zero. From the viewpoint of irreducible representations of the Poincaré group, the orbits with $p^\mu = 0$ correspond to the trivial representation of the group, and this point is excluded from the phase space of the massless particle (see, *e.g.*, [100]).

The lack of regularity, however, is not necessarily due to the absence of a well defined smooth tangent space for Σ. Consider for example the multilinear constraint

$$\phi(x, y, z) = (x - 1)(x^2 + y^2 - 1) \approx 0\,. \tag{4.11}$$

Here the constraint surface Σ is composed of two sub-manifolds: the plane $\Pi = \{(x, y, z) \mid x - 1 \approx 0\}$, and the cylinder $C = \{(x, y, z) \mid x^2 + y^2 - 1 \approx 0\}$, which are tangent to each other along the line $L = \{(x, y, z) \mid x = 1, y = 0, z \in \mathbb{R}\}$. The Jacobian on Σ is

$$\mathbf{J} = \left(3x^2 + y^2 - 2x - 1, 2y\,(x - 1)\,, 0\right)_{\phi=0}\,, \tag{4.12}$$

and its rank is one everywhere, except on L, where it is zero. The constraint ϕ is irregular on this line. However, the tangent vectors to Σ are well defined there. The irregularity arises because ϕ is a multilinear constraint of the type described by (4.5) and has two simple zeros overlapping on L. The equivalent set of regular constraints on L is $\{\phi_\Pi = x - 1 \approx 0,\, \phi_C = x^2 + y^2 - 1 \approx 0\}$.

4.4 Treatment of systems with multilinear constraints

In what follows regular systems and linearly dependent constraints will not be discussed. They are either treated in standard texts, or they can be trivially reduced to the regular case.

Consider a system of type I, as in Eq. (4.5). In the vicinity of an irregular point where only two surfaces (4.6) intersect, say Σ_1 and Σ_2, the constraint $\phi \approx 0$ is equivalently described by the set of regular constraints

$$f_1 \approx 0\,, \qquad f_2 \approx 0\,. \tag{4.13}$$

This replacement generically changes the Lagrangian of the system, and the orbits, as well. Suppose the original canonical Lagrangian is

$$L(q,\dot{q},u) = p_i\dot{q}^i - H(q,p) - u\phi(q,p)\,, \tag{4.14}$$

where H is the Hamiltonian containing all regular constraints. Replacing ϕ by (4.13), gives rise to an effective extended Lagrangian

$$L_{12}(q,\dot{q},v) = p_i\dot{q}^i - H(q,p) - v^1 f_1(q,p) - v^2 f_2(q,p)\,. \tag{4.15}$$

defined on Σ_{12}. Thus, instead of the *irregular* Lagrangian (4.14) defined on the whole Σ, there is a collection of *regularized* effective Lagrangians defined in the neighborhood of the different intersections of Σ_is. For each of these regularized Lagrangians, the Dirac procedure can be carried out to the end.

This can be illustrated with the example of a Lagrangian in a $(2+N)$-dimensional configuration space $(x, y, q^1, \ldots, q^N)$,

$$L = \frac{1}{2}\sum_{k=1}^{N}\left(\dot{q}^k\right)^2 + \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \lambda xy\,. \tag{4.16}$$

This Lagrangian describes a free particle moving on the set $\left\{ \left(x, y, q^k\right) \in \mathbb{R}^{N+2} \middle| \, xy = 0 \right\} \subset \mathbb{R}^{N+2}$, which is the union of two $(N+1)$-dimensional planes where x and y vanish respectively. The constraint surface defined by $xy = 0$ can be divided into the following sets:

$$\begin{aligned} \Sigma_1 &= \left\{ \left(x, 0, q^k; p_x, p_y, p_k\right) \middle| \, x \neq 0 \right\} , \\ \Sigma_2 &= \left\{ \left(0, y, q^k; p_x, p_y, p_k\right) \middle| \, y \neq 0 \right\} , \\ \Sigma_{12} &= \left\{ \left(0, 0, q^k; p_x, p_y, p_k\right) \right\} . \end{aligned} \tag{4.17}$$

The constraint is regular on $\Sigma_1 \bigcup \Sigma_2$, while on Σ_{12} it is irregular and can be exchanged by $\{\phi_1 = x \approx 0\,,\ \phi_2 = y \approx 0\}$. The corresponding regularized Lagrangians are

$$\begin{aligned} L_1 &= \frac{1}{2} \sum_{k=1}^{N} \left(\dot{q}^k\right)^2 + \frac{1}{2}\, \dot{x}^2 , \\ L_2 &= \frac{1}{2} \sum_{k=1}^{N} \left(\dot{q}^k\right)^2 + \frac{1}{2}\, \dot{y}^2 , \\ L_{12} &= \frac{1}{2} \sum_{k=1}^{N} \left(\dot{q}^k\right)^2 , \end{aligned} \tag{4.18}$$

and the Lagrange multipliers have dropped out, so the regularized Lagrangians describe physical degrees of freedom only – as expected.

The corresponding regularized Hamiltonians are

$$\begin{aligned} H_1 &= \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \frac{1}{2}\, p_x^2 , \\ H_2 &= \frac{1}{2} \sum_{k=1}^{N} p_k^2 + \frac{1}{2}\, p_y^2 , \\ H_{12} &= \frac{1}{2} \sum_{k=1}^{N} p_k^2 , \end{aligned} \tag{4.19}$$

which are defined in the corresponding reduced manifolds of phase space (obtained after completing the Dirac-Bergman procedure):

$$\begin{aligned} \tilde{\Sigma}_1 &= \left\{ \left(x, 0, q^k; p_x, 0, p_k\right) \middle| \, x \neq 0 \right\} , \\ \tilde{\Sigma}_2 &= \left\{ \left(0, y, q^k; 0, p_y, p_k\right) \middle| \, y \neq 0 \right\} , \\ \tilde{\Sigma}_{12} &= \left\{ \left(0, 0, q^k; 0, 0, p_k\right) \right\} . \end{aligned} \tag{4.20}$$

It is straightforward to generalize the proposed treatment when more than two surfaces Σ_i overlap.

Evolution of a multilinearly constrained system. In the presence of a multilinear constraint, there are regions of the phase space where the Jacobian has different rank. A question is, whether the system can evolve from a generic configuration in a region of maximal rank, to a configuration of lower rank, in finite time, and in the case that that were possible, what happens with the system afterwards.

To answer this question, consider the example (4.16) for $N = 1$,

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \lambda xy \,. \tag{4.21}$$

In the regions Σ_1 and Σ_2 [see Eqs. (4.17)], the rank is maximal and the free particle can move freely along the x- or y-axis, respectively.

Suppose that the initial state is

$$x(0) = a > 0\,, \quad y(0) = 0\,, \quad z(0) = 0\,, \quad \dot{x}(0) = -v < 0\,, \quad \dot{y}(0) = 0\,, \quad \dot{z}(0) = 0\,, \tag{4.22}$$

so that the particle is moving on Σ_1, with finite speed along the x-axis towards $x = 0$ on Σ_{12}. The evolution is given by $\bar{x}(t) = a - vt$, $\bar{y}(t) = 0$, $\bar{z}(t) = 0$ and the particle clearly reaches $x = 0$ in a finite time ($T = a/v$). What happens then? According to the evolution equation, for $x < 0$ the trajectory takes the form $\bar{x}(t) = a' - v't$, $\bar{y}(t) = 0$, $\bar{z}(t) = 0$, however the action would be infinite unless $a = a'$ and $v = v'$. Therefore, the particle continues unperturbed past beyond the point where the RCs fail. So, the irregular surface is not only reachable in a finite time, but it is crossed without any observable effect on the trajectory.

From the point of view of the trajectory in phase space, it is clear that the initial state $(a, 0, 0; -v, 0, 0)$ lies on the surface $\tilde{\Sigma}_1$, and at $t = T$ the system reaches the point $(0, 0, 0; -v, 0, 0)$, which *does not lie on* the surface $\tilde{\Sigma}_{12} = \{(0, 0, z; 0, 0, p_z)\}$.

While it is true that at $t = T$ the Jacobian changes rank, it would be incorrect to conclude that the evolution suffers a jump since the dynamical equations are perfectly valid there. In order to have a significant change in the dynamics, the Jacobian should change its rank in an open set.

Degenerate systems. The problem of evolution in irregular systems should not be confused with the issues arising in *degenerate systems*, in which the symplectic matrix of *regular* constraints $\{\phi_r, \phi_s\} = \Omega_{rs}(z)$ is a phase space function which changes its rank at $t = \tau$. In those systems, it *is* possible for a system in a generic initial configuration, to reach (in a finite time) a configuration where the symplectic form $\Omega_{rs}(z(\tau))$ has lower rank, leading to a *new* gauge invariance which cancels a number of degrees of freedom [67, 68].

That problem is unrelated to the one discussed in *irregular* systems and can be treated independently. In the irregular case it is the functional independence of the constraints that fails; in the degenerate dynamical systems it is the symplectic structure that breaks down.

4.5 Systems with nonlinear constraints

Consider the case of irregular systems of type II. It will be shown that a nonlinear irregular constraint can be replaced by an equivalent linear one without changing the dynamical contents of the theory, provided the linear constraint is second class. Otherwise, the resulting Hamiltonian dynamics will be, in general, inequivalent to that of the original Lagrangian system.

In order to illustrate this point, consider a system given by the Lagrangian

$$L(q,\dot{q},u) = \frac{1}{2}\,\gamma_{ij}\,\dot{q}^i\dot{q}^j - u\,[f(q)]^k\ , \tag{4.23}$$

where $k > 1$ and

$$f(q) \equiv c_i q^i \neq 0, \quad i = 1, ..., N\,. \tag{4.24}$$

Here it is assumed that the metric γ_{ij} to be constant and invertible, and the coefficients c_i are also constant. The Euler-Lagrange equations describe a free particle in an N-dimensional space, with time evolution $\bar{q}^i(t) = v_0^i\,t + q_0^i$, where $u(t)$ is a Lagrange multiplier. This solution is determined by $2N$ initial conditions, $q^i(0) = q_0^i$ and $\dot{q}^i(0) = v_0^i$ subject to the constraints $c_i q_0^i = 0$ and $c_i v_0^i = 0$. Thus, the system possesses $N - 1$ physical degrees of freedom.

In the Hamiltonian approach this system has a primary constraint $\pi \equiv \frac{\partial L}{\partial \dot{u}} \approx 0$ whose preservation in time leads to the secondary constraint

$$\phi \equiv [f(q)]^k \approx 0\,. \tag{4.25}$$

According to (4.8), this is a nonlinear constraint and there are no further constraints. As a consequence, the system has only two first class constraints $\left\{\pi \approx 0,\ f^k \approx 0\right\}$, and $N-1$ degrees of freedom, as found in the Lagrangian approach.

On the other hand, if one chooses instead of (4.25), the equivalent linear constraint

$$f(q) = c_i q^i \approx 0\,, \tag{4.26}$$

then its time evolution yields a *new* constraint,

$$\chi(p) \equiv \gamma^{ij} c_i\, p_j \approx 0\,. \tag{4.27}$$

Now, since

$$\{f, \chi\} = \gamma^{ij} c_i\, c_j \equiv \|c\|^2\ , \tag{4.28}$$

two cases can be distinguished:

- If $\|c\| = 0$, there are three first class constraints, $\pi \approx 0$, $f \approx 0$ and $\chi \approx 0$, which means that the system has $N-2$ physical degrees of freedom. In this case, substitution of (4.25) by the equivalent linear constraint (4.26), yields a *dynamically inequivalent* system[3].

- If $\|c\| \neq 0$, then $f \approx 0$ and $\chi \approx 0$ are second class, while $\pi \approx 0$ is first class, which leaves $N-1$ physical degrees of freedom and the substitution does not change the dynamics of the system.

Thus, if $f^k \approx 0$ is irregular, replacing it by the regular constraint $f \approx 0$ changes the dynamics if f is a first class function, but it gives the correct result if it is a second class function.

[3]The expression "substitution of constraints" always refers to two steps: (i) the exchange of a set of constraints by an equivalent set, and (ii) the introduction of a corresponding effective Lagrangian of type (4.15).

Note that in the Lagrangian description there is no room to distinguish first and second class constraints, so it would seem like the value of $||c||$ didn't matter. However, the inequivalence of the substitution can be understood in the Lagrangian analysis as well. Suppose that it were permissible to exchange the constraint $f^k \approx 0$ by $f \approx 0$ in the Lagrangian. Then, instead of (4.23), one would have

$$\tilde{L}(q,u) = \frac{1}{2}\,\gamma_{ij}\,\dot{q}^i\dot{q}^j - uf(q)\,. \tag{4.29}$$

It can be easily checked that (4.29) yields $N-2$ degrees of freedom when $\|c\| = 0$, and $N-1$ degrees of freedom when $\|c\| \neq 0$, which agrees with the results obtained in the Hamiltonian analysis. Note that the substitution of f^k by f modifies the dynamics only if $\gamma^{ij} c_i\, c_j = 0$, but this can happen nontrivially only if the metric γ_{ij} is not positive definite.

In general, a nonlinear irregular constraint $\phi \approx 0$ has a multiple zero on the constraint surface Σ, which means that its gradient vanishes on Σ as well. An immediate consequence of $\frac{\partial\phi}{\partial z^i} \approx 0$, is that ϕ commutes with all *finite* functions on Γ,

$$\{\phi, F(z)\} \approx 0\,. \tag{4.30}$$

As a consequence, $\phi \approx 0$ is first class and is always preserved in time, $\dot{\phi} \approx 0$. On the other hand, a nonlinear constraint cannot be viewed as a symmetry generator simply because it does not generate any transformation,

$$\delta_\varepsilon z^i = \left\{z^i, \varepsilon\phi\right\} \approx 0\,. \tag{4.31}$$

Consistently with this, ϕ cannot be gauge-fixed, as there is no finite function $\mathcal{P}$ on Γ such that $\{\phi, \mathcal{P}\} \neq 0$.

In this sense, a nonlinear first class constraint that cannot be gauge-fixed, cancels only half a degree of freedom. The other half degree of freedom cannot be cancelled because the gauge-fixing function does not exist and, in particular, it cannot appear in the Hamiltonian. Although it allow counting the degrees of freedom in a theory, these systems are pathological and their physical relevance is questionable since their Lagrangians cannot be regularized.

When a nonlinear constraint $\phi \approx 0$ can be exchanged by a regular one, the Lagrangian is regularized as in the case of multilinear constraints. For example, the system (4.23) with $\|c\| \neq 0$ has Hamiltonian

$$H = \frac{1}{2}\gamma^{ij}p_i p_j + \lambda\pi + uf(q)\,, \tag{4.32}$$

where $f = c_i q^i$ will turn out to be a second class constraint. The corresponding regularized Lagrangian coincides with $\tilde{L}$, Eq. (4.29), as expected.

In Refs. [61, 63] irregular systems of the type II were discussed. It was pointed out that there was a possible loss of dynamical information in some cases. From our point of view, it is clear that this would occur when f is a first class function.

4.6 Some implications of the irregularity

a) Linearization

It has been observed in five dimensional Chern-Simons theory, that the effective action for the linearized perturbations of the system around certain backgrounds seems to have more degrees of freedom than the fully nonlinear theory [58]. This is puzzling since the heuristic picture is that the degrees of freedom of a system correspond to the small perturbations around a local minimum of the action, and therefore the number of degrees of freedom should not change when the linearized approximation is used.

In view of the discussion in the previous section, it is clear that a possible solution of the puzzle lies in the fact that substituting a nonlinear constraint by a linear one may change the dynamical features of the theory. But the problem with linear approximations is more serious: the linearized approximation retains only up to quadratic and bilinear terms in the Lagrangian, which give linear equations for the perturbations. Thus, irregular constraints in the vicinity of the constraint surface are erased in the linearized action. The smaller number of constraints in the effective theory can lead to the wrong conclusion that the effective system possess more degrees of freedom than the unperturbed theory. The lesson to be learned is that the linear approximation is not valid in the part of the phase space where the RCs fail.

This is illustrated by the same example discussed earlier (4.23). One can choose as a background the solution $(\bar{q}^1, \ldots, \bar{q}^N, \bar{u})$, where $\bar{q}^i(t) = q_0^i + v_0^i t$ satisfies the constraint

$$c_i \bar{q}^i = 0\,, \tag{4.33}$$

and $\bar{u}(t)$ is an arbitrarily given function. This describes a free particle moving in the $(N-1)$-dimensional plane defined by (4.33). The linearized effective Lagrangian, to second order in the small perturbations $s^i = q^i - \bar{q}^i(t)$ and $w = u - \bar{u}(t)$, has the form

$$L_{\texttt{eff}}\,(s,w) = \frac{1}{2}\,\gamma_{ij}\left(v_0^i + \dot{s}^i\right)\left(v_0^j + \dot{s}^j\right) - \bar{u}\left(c_i s^i\right)^2\,, \tag{4.34}$$

and the equations of motion are

$$\ddot{s}^i + \Gamma^i_j(t)\,s^j = 0\,, \qquad (i = 1,\ldots,N)\,, \tag{4.35}$$

where $\Gamma^i_j \equiv 2\bar{u}\,\gamma^{ik}c_k c_j$ is the eigen-frequency matrix. Since $\bar{u}$ is not a dynamical variable, it is not varied and the nonlinear constraint $\left(c_i s^i\right)^2 = 0$ is absent from the linearized equations. The system described by (4.35) possesses N physical degrees of freedom, that is, one degree of freedom more than the original nonlinear theory (4.23).

The only indication that one of these degrees of freedom has a nonphysical origin is the following: If $\|c\| \neq 0$, splitting the components of s^i along c_i and orthogonal to c_i as

$$s^i(t) \equiv s(t)\gamma^{ij}c_j + s^i_\perp(t)\,, \tag{4.36}$$

gives rise to the projected equations

$$\ddot{s}^i_\perp \;=\; 0\,, \tag{4.37}$$

$$\ddot{s} + 2\bar{u}(t)\|c\|^2 s \;=\; 0\,. \tag{4.38}$$

The $N-1$ components of $s^i_\perp(t)$ obey a deterministic second order equation, whereas $s(t)$ satisfies an equation which depends on an indeterminate arbitrary function $\bar{u}(t)$. The dependence of $s = \bar{s}(t, \bar{u}(t))$ on the background Lagrange multiplier $\bar{u}$ is an indication that s is a nonphysical degree of freedom, since u was an arbitrary function to begin with. This is not manifest in Eq. (4.38), where $\bar{u}$ is a fixed function and, from a naive point of view, $s(t)$ is determined by the same equation, regardless of the physically obscure origin of the function $\bar{u}$. It is this naive analysis that leads to the wrong conclusion indicated above.

Note that a linearized theory may be consistent by itself, but it is not necessarily a faithful approximation of a nonlinear theory.

One way to avoid the inconsistencies between the original theory and the linearized one would be to first regularize the constraints (if possible) and then linearize the corresponding regular Lagrangian.[4]

b) Dirac conjecture

Dirac conjectured that *all* first class constraints generate gauge symmetries. It was shown that Dirac's conjecture is not true for first class constraints of the form f^k $(k > 1)$, and following from $\dot{f} \approx 0$ [101, 102]. Therefore, for systems with nonlinear constraints, the conjecture does not work and they generically provide counterexamples of it [99, 109, 110].

From the point of view of irregular systems, it is clear that Dirac's conjecture fails for nonlinear constraints because they do not generate any local transformation, c.f. Eq. (4.31). In Refs. [61, 63] it was observed that Dirac's conjecture may not hold in the presence of irregular constraints of type II.

In the case of multilinear constraints, however, Dirac's conjecture holds. The fact that at irregular points the constraints do not generate any transformation only means that these are fixed points of the gauge transformation.

c) Quantization

Although it is possible to deal systematically with classical theories containing irregular constraints, there may be severe problems in their quantum description. Consider a path integral of the form

$$Z \sim \int [dq][dp][du] \exp i \left[p\dot{q} - H(q,p) - u\phi(q,p) \right] , \tag{4.39}$$

where $\phi = \left[f(q,p) \right]^k$ is a nonlinear constraint. Integration on u yields to $\delta\left(f^k\right)$, which is not well-defined for a zero of order $k > 1$, making the quantum theory ill defined. Only if the nonlinear constraint could be exchanged by the regular one, $f(q,p) \approx 0$, the quantum theory could be saved. An example of this occurs in the standard approach for QED, where it is usual practice to introduce the nonlinear (Coulomb) gauge fixing

[4]There may be also other problems in linear approximation (*e.g.*, when a topology is non-trivial), but these cases are not discussed here.

term $u(\partial_i A^i)^2$ in order to fix the Gauss law $\phi = \partial_i \pi^i \approx 0$ (where $\pi^i \equiv \frac{\delta I_{ED}}{\delta \dot{A}_i}$). Since the function $f(A) = \partial_i A^i$ is second class, the substitution of $f^2(A) \approx 0$ by a regular constraint $f(A) \approx 0$ does not change its dynamical structure.

The other possibility to quantize a system with a nonlinear constraint is to modify the original Lagrangian so that it becomes regular, leading to the the change in dynamics as well. This possibility is considered in the Siegel model of the chiral boson [103], whose quantum theory is analyzed in [104].

There are other examples of irregular systems whose quantum theories are discussed in the literature. For example, it is shown that for the models of relativistic particles with higher spin ($s > \frac{1}{2}$), called *systems admitting no gauge conditions*, since they contain irregular first class constraints, different quantization methods can lead to different physical results [105, 106]. Other examples of quantum irregular systems are planar gauge field theories [107] and topologically massive gauge fields [108].

4.7 Summary

The dynamics of a system possessing constraints which may violate the regularity conditions (functional independence) on some subsets of the constraint surface Σ, was discussed. These so-called irregular systems are seen to arise generically because of nonlinearities in the constraints and can be classified into two families: multilinear (type I) and nonlinear (type II).

- Type I constraints are of the form $\phi = \prod f_i(z)$, where f_i possess simple zeros. These constraints violate the regularity conditions (RCs) on sets of measure zero on the constraint surface Σ.

- Type I constraints can be exchanged by equivalent constraints which are regular giving an equivalent dynamical system.

- Type II constraints are of the form $\phi = f^k$ $(k > 1)$ where f has a simple zero. They violate the RCs on sets of nonzero measure on Σ.

- A type II constraint can be replaced by an equivalent linear one only if the latter is second class; if the equivalent linear constraint is first class, substituting it for the original constraint would change the system.

- In general, the orbits can cross the configurations where the RCs are violated without any catastrophic effect for the system. If the symplectic form degenerates at the irregular points, additional analysis is required.

- The naive linearized approximation of an irregular constrained system generically changes it by erasing the irregular constraints. In order to study the perturbations around a classical orbit in an irregular system, it would be necessary to first regularize it (if possible) and only then do the linearized approximation.

Chapter 5

Higher-dimensional Chern-Simons theories as irregular systems

In this chapter the Hamiltonian dynamics of Chern-Simons (CS) theories in $D \geq 5$ is analyzed. It is known that regular and generic CS theories are invariant under the gauge symmetries and diffeomorphisms [54, 55], but these theories also possess sectors in phase space which are irregular [66]. Here a criterion for the choice of a regular background in CS theories is found, and the implications of the irregularity in CS theories are discussed.

The dynamics arising at the boundary is not considered here.

5.1 Chern-Simons action

A CS theory is one where the fundamental field is a Lie-algebra-valued connection one-form $\mathbf{A} = A^a \mathbf{G}_a$, where the corresponding anti-Hermitean generators close a N-dimensional algebra $[\mathbf{G}_a, \mathbf{G}_b] = f_{ab}^{\ \ c}\, \mathbf{G}_a$. The connection field defines a covariant derivative D that acts on a p-form ω as $D\omega = d\omega + [\mathbf{A}, \omega]$.[1]

The CS Lagrangian is a $(2n+1)$ - form such that its exterior derivative (in $2n+2$ dimensions) is

$$dL_{\mathrm{CS}} = k\ g_{a_1 \cdots a_{n+1}}\, F^{a_1} \cdots F^{a_{n+1}}\,. \tag{5.1}$$

[1]A p-form is the object $\omega = \frac{1}{p!}\, \omega_{n_1 \cdots n_p}\, dx^{n_1} \cdots dx^{n_p}$, and the commutator of p-form ω and q-form Ω is given by $[\omega, \Omega] = \omega\Omega - (-)^{pq}\, \Omega\omega$.

Here k is a dimensionless constant, $g_{a_1\cdots a_{n+1}}$ is a completely symmetric, invariant tensor $(Dg_{a_1\cdots a_{n+1}}=0)$ and $\mathbf{F} = F^a\mathbf{G}_a = d\mathbf{A}+\mathbf{A}^2$ is the field-strength 2-form which satisfies the *Bianchi identity* $D\mathbf{F}=0$.

The definition (5.1) is based on the fact that the form on the *r.h.s.* is *closed* (its exterior derivative vanishes due to the Bianchi identity) and therefore can *locally* be written as dL_{CS}. Applying Stokes' theorem to (5.1), the integral of the total derivative becomes the integration over the D-dimensional manifold $\mathcal{M}_D=\partial\mathcal{M}_{D+1}$ without boundary and one obtains the integral identity $\int_{\mathcal{M}_{D+1}} dL_{\text{CS}}(A)=\int_{\mathcal{M}_D} L_{\text{CS}}(A)$. In general, a CS theory is defined on an arbitrary (not necessary closed) manifold $\mathcal{M}$,

$$I_{\text{CS}}[A]=\int\limits_{\mathcal{M}} L_{\text{CS}}(A)\,. \tag{5.2}$$

Variation of the expression (5.1) with respect to the gauge field A^a gives (up to an exact form)

$$\delta L_{\text{CS}}=k\,(n+1)\;g_{aa_1\cdots a_n}\,F^{a_1}\cdots F^{a_n}\delta A^a\,, \tag{5.3}$$

from where, supposing the suitable boundary conditions which give a well defined extremum for I_{CS}, the Euler-Lagrange equations are obtained,

$$g_{aa_1\cdots a_n}\,F^{a_1}\cdots F^{a_n}=0\,. \tag{5.4}$$

Written with all indices, the equations of motion are

$$\varepsilon^{\mu\mu_1\nu_1\cdots\mu_n\nu_n}g_{aa_1\cdots a_n}\,F^{a_1}_{\mu_1\nu_1}\cdots F^{a_n}_{\mu_n\nu_n}=0\,, \tag{5.5}$$

where x^μ $(\mu=0,1,\ldots,2n)$ are local coordinates at $\mathcal{M}$.

The CS action possesses the following local symmetries:

- By construction, it is invariant under general coordinate transformations or *diffeomorphisms*

$$\begin{aligned} x^\mu &\;\rightarrow\; x'^\mu=x^\mu+\xi^\mu(x)\,, \\ \mathbf{A}(x) &\;\rightarrow\; \mathbf{A}'(x')=\mathbf{A}(x)\,, \qquad\qquad (\delta_\xi\mathbf{A}=-\pounds_\xi\mathbf{A})\ , \end{aligned} \tag{5.6}$$

  where $\pounds_\xi A^a_\mu\equiv\partial_\mu\xi^\nu A^a_\nu+\xi^\nu\partial_\nu A^a_\mu$ stands for a Lie derivative;

- It has an infinitesimal *gauge symmetry*,

$$\delta_\lambda \mathbf{A} = -D\lambda\,, \tag{5.7}$$

since the *r.h.s.* of (5.1) is an explicitly gauge invariant expression.

Under *large* gauge transformations, a CS action changes for a closed form. In this chapter the local dynamics is analyzed, and all effects arising at the boundary are neglected.

5.2 Hamiltonian analysis of Chern-Simons theories

Here the features of the sectors with *regular* dynamics of $(2n+1)$-dimensional CS theories are reviewed, where the dimension is larger than three [55].

In the Hamiltonian approach, one supposes that the space-time manifold $\mathcal{M}$ has topology $\mathbb{R}\times\sigma$, where σ is a $2n$-dimensional spatial manifold parametrized by local coordinates x^i. The CS action I_{CS}, defined by (5.1) and (5.2), is linear in velocities (first order formalism) and can be written as

$$I_{\text{CS}}\left[A\right] = \int d^{2n+1}x\,\left(\mathcal{L}_a^i\,\dot{A}_i^a - A_0^a\chi_a\right)\,, \tag{5.8}$$

where[2]

$$\chi_a \equiv -\frac{k}{2^n}\,\left(n+1\right)\,\varepsilon^{i_1j_1\cdots i_nj_n}\,g_{aa_1\cdots a_n}F_{i_1j_1}^{a_1}\cdots F_{i_nj_n}^{a_n}\,, \tag{5.9}$$

and $\varepsilon^{i_1\cdots i_{2n}} \equiv \varepsilon^{0i_1\cdots i_{2n}}$. The explicit form of $\mathcal{L}_a^i$ is not necessary since it defines the kinetic term through the *symplectic form* Ω_{ab}^{ij} :

$$\begin{aligned}\Omega_{ab}^{ij}\left(x,x'\right) &= \frac{\delta\mathcal{L}_b^j\left(x'\right)}{\delta A_i^a\left(x\right)} - \frac{\delta\mathcal{L}_a^i\left(x\right)}{\delta A_j^b\left(x'\right)} \\ &= -\frac{kn}{2^{n-1}}\,\left(n+1\right)\,\varepsilon^{iji_2j_2\cdots i_nj_n}g_{aba_2\cdots a_n}F_{i_2j_2}^{a_1}\cdots F_{i_nj_n}^{a_n}\,\delta\left(x-x'\right)\,. \end{aligned}\tag{5.10}$$

(For derivation see Appendix D.) This matrix plays a central role in the dynamics of CS theories, as it will be seen later. The gauge field A_i^a and its canonically conjugated

[2]The expression for χ_a can be obtained directly from (5.3) for a special choice of variations $\delta A^a = \delta A_0^a\,dt$, and the definition $\chi_a = -\frac{\delta I_{\text{CS}}}{\delta A_0^a}$ following from (5.8).

momenta $\pi_a^i \equiv \delta I_{\rm CS}/\delta \dot{A}_i^a$ defines the phase space Γ, with the basic PB

$$\left\{ A_i^a\left(x\right), \pi_b^j\left(x'\right)\right\} = \delta_i^j \delta_b^a \delta\left(x-x'\right) \tag{5.11}$$

taken at the same time $x^0 = x'^0 = t$, where $\delta\left(x-x'\right)$ stands for a Dirac δ-function at the spatial section. The field A_0^a is a Lagrange multiplier.

Constraints. Constraints in the theory are

$$\begin{aligned} \phi_a^i &\equiv \pi_a^i - \mathcal{L}_a^i(A) \approx 0\,, \\ \mathcal{G}_a &\equiv -\chi_a + D_i \phi_a^i \approx 0\,, \end{aligned} \tag{5.12}$$

where the covariant derivative acts on ϕ_a^i as $D_i\phi_a^i \equiv \partial_i \phi_a^i + f_{ab}{}^c A_i^b \phi_c^i$. The total Hamiltonian is

$$H = \int d^{2n}x\, \left(-A_0^a \mathcal{G}_a + u_i^a \phi_a^i\right)\,, \tag{5.13}$$

and it depends on N arbitrary functions A_0^a and $2nN$ arbitrary functions u_i^a. Constraints define the constraint surface $\Sigma \subset \Gamma$ and they close the following PB algebra:

$$\begin{aligned} &\{\pi_a^0,\ \text{all}\ \} = 0\,, && \{\mathcal{G}_a, \mathcal{G}_b\} = f_{ab}{}^c \phi_c^i \delta\,, \\ &\left\{\phi_a^i, \phi_b^j\right\} = \Omega_{ab}^{ij} \delta\,, && \left\{\mathcal{G}_a, \phi_b^i\right\} = f_{ab}{}^c \phi_c^i \delta\,, \end{aligned} \tag{5.14}$$

with $\Omega_{ab}^{ij}(x,x') = \Omega_{ab}^{ij}\left(x\right)\delta\left(x-x'\right)$. The constraints ϕ_a^i and $\mathcal{G}_a$ evolve as

$$\dot{\phi}_a^i \approx \left\{\phi_a^i, H\right\} = \Omega_{ab}^{ij} u_j^b - f_{ab}{}^c \phi_c^i \approx \Omega_{ab}^{ij} u_j^b = 0\,, \tag{5.15}$$

$$\dot{\mathcal{G}}_a \approx \left\{\pi_a^0, H\right\} = f_{ab}{}^c \left(A_0^b \mathcal{G}_c - u_i^b \phi_c^i\right) \approx 0\,, \tag{5.16}$$

and, therefore, there are no new constraints. The equation (5.15) gives restrictions on a number of multipliers u_a^i, depending on the rank of Ω_{ab}^{ij}.

Generic theories. The dynamics of CS theories basically depends on the symplectic matrix Ω_{ab}^{ij}. This matrix is degenerate for any CS theory, due to the identity (Appendix D)

$$\Omega_{ab}^{ik} F_{kj}^b = -\delta_j^i \chi_a \approx 0\,. \tag{5.17}$$

Therefore, there are at least $2n$ zero modes of the symplectic matrix, $(\mathbf{V}_i)^a_j = F^a_{ij}$.

Among all CS theories based on a gauge group of dimension N, there is a family of *generic* theories for which:

(a) the zero modes of Ω^{ij}_{ab} are the $2n$ *linearly independent* vectors $(\mathbf{V}_i)^a_j = F^a_{ij}$;

(b) there are no other zero modes and the rank of symplectic matrix is the largest,

$$\Re\left(\mathbf{\Omega}\right) = 2n\left(N-1\right) . \tag{5.18}$$

For $D \geq 5$, a CS theory cannot be generic on the whole phase space. For example, any CS theory possess a *pure gauge* solution ($F = 0$) which does not satisfy generic conditions (a) and (b). Therefore, a generic CS theory is determined by (i) an invariant tensor $g_{a_1 \cdots a_{n+1}}$ and (ii) a domain, or a sector of the phase space on which the generic conditions (a) and (b) are satisfied. This sector can be chosen as an open set around a solution $\bar{A}$ of the constraints $\chi(\bar{A}) = 0$ such that (a) and (b) are fulfilled for $\bar{F}$ and $\bar{\Omega}$.

First and second class constraints. In order to separate first from second class constraints, a (non-singular) transformation of ϕ^i_a has to be made, which diagonalize the symplectic form in (5.14):

$$\mathcal{H}_i \equiv F^a_{ij}\phi^j_a \approx 0\,, \qquad\qquad \theta_\alpha \equiv S^a_{\alpha i}\phi^i_a \approx 0\,, \tag{5.19}$$

where $\alpha = 1, \ldots, 2n\left(N-1\right)$. The constraints $\mathcal{H}_i$ correspond to the zero modes of Ω^{ij}_{ab} and satisfy the algebra

$$\left\{\mathcal{H}_i\left(x\right), \mathcal{H}_j\left(x'\right)\right\} = \left[\mathcal{H}_i\left(x'\right)\partial_j + \mathcal{H}_j\left(x\right)\partial_i - F^a_{ij}\mathcal{G}_a\left(x\right)\right]\delta\left(x - x'\right), \tag{5.20}$$

thus they are first class. Constraints θ_α are second class. The explicit form of the tensor $S^a_{\alpha j}$ is not always possible to find, and the condition

$$\Re\left(\Delta_{\alpha\beta}\right) = 2n(N-1)\,, \qquad\qquad \Delta_{\alpha\beta} \equiv S^a_{\alpha_i}\Omega^{ij}_{ab}S^b_{\beta j}\,, \tag{5.21}$$

provides that the only non-vanishing brackets on Σ are

$$\left\{\theta_\alpha, \theta_\beta\right\} \approx \Delta_{\alpha\beta}\,\delta\,. \tag{5.22}$$

Therefore, in a generic CS theory with $2nN$ dynamical fields A_a^i, N first class constraints $\mathcal{G}_a$, $2n$ first class constraints $\mathcal{H}_i$, and $2n\,(N-1)$ second class constraints θ_α, the number of physical degrees of freedom is:

$$f_{2n+1}(N) = nN - n - N\,, \qquad (n, N > 1)\,. \tag{5.23}$$

Local symmetries. A CS theory is invariant under the following local transformations:

- *Gauge transformations,* generated by

  $$G\,[\lambda] \equiv \int d^{2n}x\,\lambda^a \mathcal{G}_a\,, \tag{5.24}$$

  which act on gauge fields as

  $$\delta_\lambda A_i^a = \{A_i^a, G\,[\lambda]\} = -D_i \lambda^a\,; \tag{5.25}$$

- *Improved spatial diffeomorphisms,* generated by

  $$\mathcal{H}\,[\varepsilon] \equiv \int d^{2n}x\,\varepsilon^i \mathcal{H}_i\,, \tag{5.26}$$

  which change the gauge field as

  $$\delta_\varepsilon A_i^a = \{A_i^a, \mathcal{H}\,[\varepsilon]\} = \varepsilon^j F_{ji}^a\,. \tag{5.27}$$

Local symmetries which are not independent from the above ones, are *spatial diffeomorphisms,* which differ from improved spatial diffeomorphisms by a gauge transformation,

$$\delta_\xi A_i^a = -\pounds_\xi A_i^a = -\xi^j F_{ji}^a - D_i\left(\xi^i A_i^a\right)\,, \tag{5.28}$$

and *(improved) time reparametrizations*, which change the gauge fields as

$$\delta_\varepsilon A_i^a = \varepsilon^0 F_{0i}^a\,. \tag{5.29}$$

Due to the equations of motion $\Omega_{ab}^{ij} F_{0j}^b = 0$, the vector F_{0j}^a must be a linear combination of zero modes of Ω_{ab}^{ij}, therefore in a generic theory $F_{0i}^a = C^j F_{ji}^a$. In consequence, the time-like diffeomorphisms can be obtained from spatial diffeomorphisms *on-shell* through the redefinition of local parameter $\varepsilon^i = \varepsilon^0 C^i$.

Non-generic theories. The above analysis can be easily generalized to degenerate (non-generic) theories, in which the symplectic matrix Ω_{ab}^{ij} has K independent zero modes $(\mathbf{U}_\rho)_i^a = U_{\rho i}^a$,

$$\Omega_{ab}^{ij} U_{\rho j}^b \approx 0\,, \qquad (\rho = 1, \ldots, K) \ (2n < K \leq 2nN)\ , \tag{5.30}$$

where $K = 2n$ corresponds to the generic case. Then the first class constraints are

$$\mathcal{H}_\rho \equiv U_{\rho\, j}^a \phi_a^i \approx 0\,, \tag{5.31}$$

which, apart from $2n$ *improved spatial diffeomorphisms* also generate an *additional* $(K-2n)$*-parameter symmetry.* The number of physical degrees of freedom is given by

$$f_{2n+1}(N) = nN - N - \frac{K}{2}\,, \qquad (n, N > 1)\,. \tag{5.32}$$

This formula is a generalization of (5.23) obtained for generic theories, and it enables the counting of the degrees of freedom in the sectors of phase space which have more than minimal number of local symmetries. However, during its evolution, the degenerate system can change the number of degrees the freedom (loosing them) if it reaches the point where the symplectic form has lower rank [67, 68]. Therefore, the formula (5.32) is valid only on an open set not containing degenerate points.

5.3 Regularity conditions

In the previous chapter it was implicitly supposed that all constraints were regular. Now the regularity conditions in CS theories are going to be analyzed.

First, consider the original set of constraints obtained from Dirac-Bergman algorithm, $\phi_a^i \approx 0$ and $\chi_a \approx 0$. Constraints ϕ_a^i are regular since they are linear in momenta. Thus, the regularity of CS theories is determined by momentum-independent constraints χ_a. It is convenient to write χ_a in the basis of spatial 1-forms dx^i as

$$K_a \equiv d^{2n}x\, \chi_a = -k\,(n+1)\ g_{aa_1\cdots a_n} F^{a_1} \cdots F^{a_n} \approx 0\,. \tag{5.33}$$

Their small variations evaluated at $K_a = 0$ are

$$\delta K_a = \mathbf{J}_{ab} D\delta A^b\,, \tag{5.34}$$

and they define the $(2n-2)$-form $\mathbf{J}_{ab}$, which can be identified as the Jacobian,

$$\mathbf{J}_{ab} \equiv -kn\,(n+1)\; g_{aba_1\cdots a_{n-1}}\;\; F^{a_1}\cdots F^{a_{n-1}}|_{K=0}\ . \tag{5.35}$$

According to Dirac's definition, a sufficient and necessary condition for K_a to be regular is

$$\Re\left(\mathbf{J}_{ab}\right) = N\,. \tag{5.36}$$

Since $\mathbf{J}_{ab}$ is field dependent, its rank may change in phase space. In particular, for a pure gauge configuration $F^a = 0$, the Jacobian $\mathbf{J}_{ab}$ has rank zero. For other configurations, the rank of Jacobian can range from zero to N, and the irregularities are always of *multilinear* type because in the expression (5.33), thanks to the antisymmetric tensor $\varepsilon^{i_1 j_1\cdots i_n j_n}$, the phase space coordinate A_i^a, for one particular choice of i, occurs only linearly.

Regularity conditions of first and second class constraints. Suppose that the set of constraints (ϕ_a^i, χ_a) is regular. A new equivalent set, with separated first and second class constraints, is introduced via the transformation

$$\mathcal{T} : \left(\phi_b^j, \chi_c\right) \rightarrow \left(\mathcal{H}_i, \theta_\alpha, \mathcal{G}_a\right)\ , \tag{5.37}$$

where the matrix of transformation is given by

$$\mathcal{T} = \left[\begin{array}{cc} \left(\begin{array}{cc} F_{ij}^b & 0 \\ 0 & S_{\alpha j}^b \end{array}\right) & 0 \\ \delta_a^b D_j & -\delta_a^c \end{array}\right]. \tag{5.38}$$

This transformation preserves the regularity of the original constraints only if it is invertible at Σ, *i.e.*, its rank is equal to the number of constraints, $\Re\left(\mathcal{T}\right) = 2n(N+1)$. For the rank of $\mathcal{T}$, one obtains[3]

$$\Re\left(\mathcal{T}\right) = \Re\left(F_{ij}^b\right) + \Re\left(S_{\alpha j}^b\right) + N\,, \tag{5.39}$$

[3]If a matrix $M = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right)$ has an invertible submatrix A, then $\Re\left(M\right) = \Re\left(A\right) + \Re\left(D - CA^{-1}B\right)$. This is a consequence of the fact that the zero modes of M have the form $\left(\begin{array}{c} -A^{-1}B\,v \\ v \end{array}\right)$, where v is a zero mode of $D - CA^{-1}B$.

implying that N constraints $\mathcal{G}_a$ are always regular (if χ_a are regular), while the regularity of $\mathcal{H}_i$ and θ_α requires two extra conditions:

$$\Re\left(F_{ij}^b\right) = 2n\,, \qquad\qquad \Re\left(S_{\alpha j}^b\right) = 2n(N-1)\,, \tag{5.40}$$

which mean that $C^i F_{ij}^b = 0$ and $C^\alpha S_{\alpha j}^b = 0$ have to have the unique solution $C^i = C^\alpha = 0$.

Generic condition and the regularity. The study of dynamics in CS theories requires not only the analysis of regularity, but also of genericity. A generic configuration has the symplectic form Ω_{ab}^{ij} with maximal rank, while a regular configuration has the invertible Jacobian $\mathbf{J}_{ab}$. The relation between Ω_{ab}^{ij} and $\mathbf{J}_{ab}$, Eqs. (5.10) and (5.35), is given by

$$2\,d^{2n}x\,\Omega_{ab}^{ij} \equiv dx^i dx^j\,\mathbf{J}_{ab}\,. \tag{5.41}$$

In spite of the fact that both conditions are expressed in terms of the same matrix Ω_{ab}^{ij}, they are independent.

This is illustrated with four different examples.

1. *An irregular and non-generic configuration* is the *pure gauge,* $F = 0$, which occurs in any non-Abelian CS theory in $D \geq 5$.

2. *Irregular and generic configurations* can be found in five-dimensional AdS_5-CS supergravity, see Ref. [58].

3. *Regular and generic configurations* can also be found in five-dimensional AdS_5-CS supergravity, as it is discussed in Chapter 6.

4. *Regular and non-generic configurations* occur in a five-dimensional CS theory based on a direct product $G_1 \otimes G_2$, for a particular choice of the invariant tensor.

In order to demonstrate the last example, the group indices can be taken as $a = (r, \alpha)$ corresponding to G_1 and G_2 respectively, and the invariant tensor chosen to have non vanishing components $g_{rs1} = g_{rs}$ $(r, s = 1, 2, \ldots)$ and $g_{\alpha\beta\bar{1}} = g_{\alpha\beta}$ $(\alpha, \beta = \bar{1}, \bar{2}, \ldots)$, with tensors g_{rs} and $g_{\alpha\beta}$ invertible. Then the configuration

$$F^a = \left(f^1 dx^1 dx^2,\ h^{\bar{1}} dx^3 dx^4\right) \tag{5.42}$$

is regular and non-generic. Indeed, the Jacobian evaluated at (5.42) has maximal rank

$$\mathbf{J}_{ab} = \left(\begin{array}{cc} -6k\, g_{rs}\, f^1 dx^1 dx^2 & 0 \\ 0 & -6k\, g_{\alpha\beta}\, h^{\bar{1}} dx^3 dx^4 \end{array} \right) , \tag{5.43}$$

while Ω^{ij}_{ab} with non-vanishing components

$$\Omega^{34}_{rs} = -3k\, g_{rs}\, f^1, \qquad\qquad \Omega^{12}_{\alpha\beta} = -3k\, g_{\alpha\beta}\, h^{\bar{1}}, \tag{5.44}$$

has $2N$ zero modes

$$\mathbf{V}^a_i = \left(\begin{array}{c} u^r \delta^1_i + v^r \delta^2_i \\ u^\alpha \delta^3_i + v^\alpha \delta^4_i \end{array} \right) , \tag{5.45}$$

with $2N$ arbitrary functions $u^a(x)$ and $v^a(x)$, and is therefore degenerate. This particular CS theory has no physical degrees of freedom, according to the formula (5.32) applicable to regular non-generic theories, for $n = 2$ and $K = 2N$.

As a consequence of the existence of both regularity and genericity issues, the regularization problem is much more delicate in CS theories. For example, in a *pure gauge* sector, $F^a = 0$, the constraint surface is defined by the *pure gauge* configurations $A^a_i = -D\lambda^a_i$, and the multilinear constraints $\chi_a \approx 0$ can be exchanged by the equivalent regular set $\mathcal{A}^a_i \equiv A^a_i + D\lambda^a_i \approx 0$. In that case, all constraints $\{\mathcal{A}^a_i \approx 0,\ \phi^i_a \approx 0\}$ are second class and there are no physical degrees of freedom in the *pure gauge* sector , as expected.

A more general situation occurs around the background $F^a = (F^r, F^\alpha)$, where only one block of the field-strength vanishes, $F^\alpha = 0$, leading to the irregular constraints χ_α. It is supposed that the rest of constraints χ_r are regular. In that case, $\chi_\alpha \approx 0$ are exchanged by $\mathcal{A}^\alpha_i \equiv A^\alpha_i + D\lambda^\alpha_i \approx 0$, and the constraints $\{\mathcal{A}^\alpha_i \approx 0,\ \phi^i_\alpha \approx 0\}$ are second class, so that the variables $(A^\alpha_i,\ \pi^i_\alpha)$ can be eliminated from the corresponding reduced phase space. In consequence, the dynamics of this sector is effectively determined by the regular constraints $\{\chi_r \approx 0,\ \phi^i_r \approx 0\}$, *i.e.*, by the submatrix of the symplectic form $\Omega^{ij}_{rs} = \{\phi^i_r, \phi^j_s\}^*$. Therefore, the problems of irregularity and genericity are decoupled, and the whole dynamics follows only from Ω^{ij}_{rs}. The underlying reason for the decoupling of two problems, is that regular and irregular sectors do not intersect in the phase space, and that there is an effective symplectic form determining the dynamics of the each sector.

Although a wide class of irregular CS sectors are of the type described above (when the irregularity is a consequence of $F^\alpha = 0$), there may be another "accidental" irregular configurations, specific only for a certain CS theory, where an independent analysis is required.

5.4 Conclusions: the phase space of CS theories

The dynamical structure of higher-dimensional CS theories invariant under a Lie group with more than one generator, is complex and crucially depends on the symplectic form Ω. Since the rank of this matrix, which determines the number and character of constraints in a theory, changes throughout the phase space, CS theories have the following general features:

- There exist *regular* and *irregular* sectors of phase space, where the constraints χ_a are functionally independent or not, respectively. A system cannot spontaneously, in finite time, evolve from one sector to another.

- If the rank of the symplectic form is maximal, the theory is *generic*, otherwise, it is *degenerate.* This classification is independent from the regularity, although both conditions are expressed in terms of the same matrix Ω_{ab}^{ij}.

- During its evolution a regular system can reach a point in *configurational* space where it is irregular, but it passes this point without any effect. The reason is that these two sectors do not have intersections in the *phase space.*

- During its evolution a regular system can reach the point where it is degenerate, *i.e.,* the symplectic form has lower rank. Then the system cannot leave this sector of lower rank since it gains additional local symmetry and looses physical degrees of freedom there.

- In Chern-Simons theories, it is not possible to separate first from second class constraints explicitly, for all configurations.

The features mentioned above make the dynamics of CS theories more complicated and with rich structure.

Chapter 6

AdS-Chern-Simons supergravity

It is known that gravity in 2+1 dimensions, described by the Einstein-Hilbert action with or without cosmological constant, is exactly soluble and quantizable [9]. This is possible because it has no local degrees of freedom, and because its action is a CS form and can be seen as a gauge theory for the $(A)dS$ or Poincaré groups. This is an insight that motivates the study of similar theories, represented by CS forms, in all odd dimensions [45, 48, 50, 51] (see Appendix E). In these theories, gravity becomes a truly gauge theory, where the vielbein (e^a) and the spin-connection (ω^{ab}) are components of the same connection field $\mathbf{A}$, for the $(A)dS$ or Poincaré algebra. In higher dimensions ($D \geq 5$), the CS actions are not equivalent to the Einstein-Hilbert actions, as they contain terms nonlinear in curvature, and torsion is a dynamical field [53].

In this chapter, CS supergravities based on the supersymmetric extensions of the AdS group, are studied. These theories contain, apart from the gravitational fields (e, ω) and the gravitini (ψ), a number of the additional bosonic gauge fields. The supersymmetry algebra closes *off shell*, without the need to introduce auxiliary fields. The number of boson and fermion components are not equal [56].

The simplest higher-dimensional CS supergravity with propagating degrees of freedom occurs in five dimensions. An additional simplification is that the Lagrangian of the five-dimensional theory does not contain torsion explicitly in the purely gravitational sector, which is just a polynomial in the curvature and vielbein.

6.1 $D = 5$ supergravity

a) Algebra

Five-dimensional AdS-Chern-Simons supergravity is based on the supersymmetric extension of the AdS group, $SU(2,2\,|N\,)$. The superalgebra $su(2,2\,|N\,)$ is generated by the following anti-Hermitean generators [58]:

$$\begin{array}{llll} so(2,4): & \mathbf{J}_{AB}\,, & (A,B=0,\ldots,5)\;, & (15\text{ generators})\,,\\ su(N): & \mathbf{T}_\Lambda\,, & (\Lambda=1,\ldots,N^2-1)\;, & (N^2-1)\,,\\ \text{SUSY}: & \mathbf{Q}_r^\alpha,\ \bar{\mathbf{Q}}_\alpha^r\,, & (\alpha=1,\ldots,4;\ r=1,\ldots,N)\;, & (8N)\,,\\ u(1): & \mathbf{G}_1\,, & & (1)\,, \end{array} \tag{6.1}$$

where $\eta_{AB} = \text{diag}\,(-,+,+,+,+,-)$. Lorentz rotations and AdS translations are generated by $\mathbf{J}_{ab}$ and $\mathbf{J}_a \equiv \mathbf{J}_{a5}$ $\;(a,b = 0,\ldots,4)$ respectively, and supersymmetry (SUSY) generators transform as Dirac spinors in a vector representation of $SU(N)$. The dimension of this superalgebra is

$$\mathcal{N}\left(SU(2,2\,|N\,)\right) = N^2 + 8N + 15\,. \tag{6.2}$$

b) Field content

The fundamental field is the Lie-algebra-valued connection 1-form, $\mathbf{A} = A^M \mathbf{G}_M$, with components

$$\mathbf{A} = \frac{1}{\ell}\, e^a \mathbf{J}_a + \frac{1}{2}\,\omega^{ab}\mathbf{J}_{ab} + a^\Lambda \mathbf{T}_\Lambda + \left(\bar{\psi}^r \mathbf{Q}_r - \bar{\mathbf{Q}}^r \psi_r\right) + \phi\,\mathbf{G}_1\,. \tag{6.3}$$

Apart from the purely gravitational part $\left(\frac{1}{\ell}\,e^a, \omega^{ab}\right)$, where ℓ is the AdS radius, this theory contains a fermionic sector $\left(\psi_r^\alpha,\ \bar{\psi}_\alpha^r\right)$ (gravitini), and bosonic fields a^Λ and ϕ demanded by supersymmetry. The components of the field-strength $\mathbf{F} = d\mathbf{A} + \mathbf{A}^2 = F^M \mathbf{G}_M$ along the bosonic generators are

$$\begin{aligned} F^{a5} &= \tfrac{1}{\ell}\,T^a + \tfrac{1}{2}\,\bar{\psi}^r \Gamma^a \psi_r\,,\\ F^{ab} &= R^{ab} + \tfrac{1}{\ell^2}\,e^a e^b - \tfrac{1}{2}\bar{\psi}^r \Gamma^{ab}\psi_r\,,\\ F^\Lambda &= \mathcal{F}^\Lambda + \bar{\psi}^s \left(\tau^\Lambda\right)_s^{\ r} \psi_r\,,\\ F^\phi &= d\phi - i\bar{\psi}^r \psi_r\,, \end{aligned} \tag{6.4}$$

where $N \times N$ matrices τ^{Λ} are generators of $su(N)$, and $a \equiv a^{\Lambda}\tau_{\Lambda}$ and $\mathcal{F} = da + a^2$ are the corresponding field and its field-strength, $d\phi$ is the $u(1)$ field-strength and the torsion (T^a) and curvature (R^{ab}) two-forms are given in Appendix E. The component of the field-strength along the fermionic generator $\left(-\bar{\mathbf{Q}}^r F_r \text{ term}\right)$, is

$$F_r = D\psi_r \,, \tag{6.5}$$

where the super covariant derivative of a p-form $\mathbf{X}$ is $D\mathbf{X} = \mathbf{X} + [\mathbf{A}, \mathbf{X}]$ (see Appendix G). In particular, for the spinor ψ_r, it is

$$D\psi_r = (\nabla + \not{e})\,\psi_r - a_r^s \psi_s + i\left(\frac{1}{4} - \frac{1}{N}\right)\phi\,\psi_r \,, \tag{6.6}$$

where $\nabla\psi_r = (d + \not{\omega})\,\psi_r$ is the Lorentz covariant derivative, with $\not{\omega} \equiv \frac{1}{4}\,\omega^{ab}\Gamma_{ab}$ and $\not{e} \equiv \frac{1}{2\ell}\,e^a\Gamma_a$.

c) Action

The CS Lagrangian, defined by (5.1), in five dimensions becomes

$$dL_{\text{CS}} = ik\left\langle \mathbf{F}^3 \right\rangle = ik\, g_{MNK}\, F^M F^N F^K \,, \tag{6.7}$$

where $\langle \ldots \rangle$ stands for a symmetrized invariant supertrace (symmetric in bosonic and antisymmetric in fermionic indices) which is explicitly given in Appendix F.[1] The constant k is dimensionless and real, and antisymmetric wedge product acts between the forms. The CS action can be explicitly written as

$$I_{\text{CS}}[A] = \int\limits_{\mathcal{M}} L_{\text{CS}}(\mathbf{A}) = ik \int\limits_{\mathcal{M}} \left\langle \mathbf{A}\,(d\mathbf{A})^2 + \frac{3}{2}\,\mathbf{A}^3 d\mathbf{A} + \frac{3}{5}\,\mathbf{A}^5 \right\rangle + B[A] \,, \tag{6.8}$$

where $B[A]$ is a boundary term which must be added so that the action is stationary on the classical orbits. In components, the CS supergravity action takes the form originally obtained by Chamseddine [48],

$$L_{\text{CS}} = L_{\text{g}}(\omega, e) + L_{\mathtt{SU(N)}}(a) + L_{\mathtt{U(1)}}(\omega, e, \phi) + L_{\mathtt{f}}(\omega, e, a, \phi, \psi) \,, \tag{6.9}$$

[1] The symmetrized supertrace $\langle\ \rangle \equiv \langle\ ,\ ,\ \rangle$ has three Lie-algebra-valued entries. For example, $\left\langle \mathbf{A}^3 d\mathbf{A} \right\rangle \equiv \left\langle \mathbf{A}^2, \mathbf{A}, d\mathbf{A} \right\rangle = \left\langle \mathbf{A}, \mathbf{A}^2, d\mathbf{A} \right\rangle$.

where

$$
\begin{aligned}
L_{\mathrm{g}} &= \tfrac{k}{8}\,\varepsilon_{abcde}\left(\tfrac{1}{\ell}\,R^{ab}R^{cd}e^{e}+\tfrac{2}{3\ell^{3}}\,R^{ab}e^{c}e^{d}e^{e}+\tfrac{1}{5\ell^{5}}\,e^{a}e^{b}e^{c}e^{d}e^{e}\right)\,,\\
L_{\mathtt{SU(N)}} &= -ik\,\mathrm{Tr}_{N}\left(a\mathcal{F}^{2}-\tfrac{1}{2}\,a^{3}\mathcal{F}+\tfrac{1}{10}\,a^{5}\right)\,,\\
L_{\mathtt{U(1)}} &= k\left(\tfrac{1}{4^{2}}-\tfrac{1}{N^{2}}\right)\phi\,(d\phi)^{2}+\tfrac{3k}{4\ell^{2}}\left(T^{a}T_{a}-\tfrac{\ell^{2}}{2}\,R^{ab}R_{ab}-R^{ab}e_{a}e_{b}\right)\phi+\tfrac{3k}{N}\,\mathcal{F}^{\Lambda}\mathcal{F}_{\Lambda}\phi\,,\\
L_{\mathtt{f}} &= -\tfrac{3ik}{4}\,\bar{\psi}^{r}\left[\tfrac{1}{\ell}\,T^{a}\Gamma_{a}+\tfrac{1}{2}\left(R^{ab}+e^{a}e^{b}\right)\Gamma_{ab}+2i\left(\tfrac{1}{N}+\tfrac{1}{4}\right)d\phi-\bar{\psi}^{s}\psi_{s}\right]D\psi_{r}\\
&\quad-\tfrac{3ik}{2}\bar{\psi}^{r}\left(\mathcal{F}_{r}^{s}-\tfrac{1}{2}\,\bar{\psi}^{s}\psi_{r}\right)D\psi_{s}+\text{c.c.}\,,
\end{aligned}
\tag{6.10}
$$

and $\mathcal{F}_{r}^{s}=\mathcal{F}^{\Lambda}\left(\tau_{\Lambda}\right)_{r}^{s}$. On the basis of the form of $D\psi_{r}$ given by (6.6), one can see that the fermions carry a $U(1)$ charge $q=\frac{1}{4}-\frac{1}{N}$. The pure gravitational Lagrangian L_{g} contains the standard Einstein-Hilbert Lagrangian with negative cosmological constant, plus an additional term quadratic in curvature which is a dimensional continuation of the Gauss-Bonnet density from four to five dimensions.

From the Lagrangian $L_{\mathtt{U(1)}}$ and the relation (6.6) it can be seen that the case $N=4$ is exceptional since the dynamics of ϕ changes (it looses the cubic term in the Lagrangian) and fermions become neutral ($q=0$). It is shown in the Appendix F that for $N=4$, the $U(1)$ generator becomes a central extension.

d) Local symmetries

As all CS theories, the *AdS*-CS supergravity action is invariant under diffeomorphisms, $\delta x^{\mu}=\xi^{\mu}(x)$, $\delta_{\xi}\mathbf{A}=-\pounds_{\xi}\mathbf{A}$, and infinitesimal gauge transformations, $\delta_{\lambda}\mathbf{A}=-D\lambda$ (see the Section 5.1). In particular, under local supersymmetry transformations with parameter $\epsilon_{r}^{\alpha}(x)$, from $\lambda=\bar{\epsilon}^{r}\mathbf{Q}_{r}-\bar{\mathbf{Q}}^{r}\epsilon_{r}$ one obtains

$$
\begin{aligned}
\delta_{\epsilon}e^{a} &= -\tfrac{1}{2}\left(\bar{\psi}^{r}\Gamma^{a}\epsilon_{r}-\bar{\epsilon}^{r}\Gamma^{a}\psi_{r}\right)\,, & \delta_{\epsilon}\psi_{\alpha}^{r} &= -D\epsilon_{\alpha}^{r}\,,\\
\delta_{\epsilon}\omega^{ab} &= \tfrac{1}{4}\left(\bar{\psi}^{r}\Gamma^{ab}\epsilon_{r}-\bar{\epsilon}^{r}\Gamma^{ab}\psi_{r}\right)\,, & \delta_{\epsilon}\bar{\psi}_{r}^{\alpha} &= -D\bar{\epsilon}_{r}^{\alpha}\,,\\
\delta_{\epsilon}a^{\Lambda} &= \bar{\psi}^{r}\left(\tau^{\Lambda}\right)_{r}^{s}\epsilon_{s}-\bar{\epsilon}\left(\tau^{\Lambda}\right)_{r}^{s}\psi_{s}\,, & \delta_{\epsilon}\phi &= -i\left(\bar{\psi}^{r}\epsilon_{r}-\bar{\epsilon}\psi_{r}\right)\,.
\end{aligned}
\tag{6.11}
$$

Unlike standard supergravities, here the supersymmetry algebra closes *off shell* by construction, without requiring auxiliary fields [56].

The CS action is not invariant under *large* gauge transformations

$$
\mathbf{A}^{g}=g\left(\mathbf{A}+d\right)g^{-1}\,,\qquad g\in SU\left(2,2\,|N\right)\,,
\tag{6.12}
$$

where $g = e^{\lambda}$. Using the fact that the field-strength transforms homogenously, $\mathbf{F}^{g} = g\mathbf{F}g^{-1}$, leading to $dL_{\mathsf{CS}}^{g} = dL_{\mathsf{CS}}$, one obtains that the Lagrangian changes as $L_{\mathsf{CS}}^{g} = L_{\mathsf{CS}} + \omega$, where ω is a closed form ($d\omega = 0$) which is not exact for nontrivial topology.

***e*) Field equations**

Varying (6.8) with respect to the connection, yields

$$\delta I_{\mathrm{CS}} = 3ik \int\limits_{\mathcal{M}} \left\langle \mathbf{F}^{2}\, \delta\mathbf{A} \right\rangle \,, \tag{6.13}$$

provided δB and the boundary conditions are chosen so that the boundary term in δI_{CS} vanishes. Then the action has an extremum if the equations of motion are satisfied,

$$\left\langle \mathbf{G}_{M} \mathbf{F}^{2} \right\rangle = g_{MNK} F^{N} F^{K} = 0 \,. \tag{6.14}$$

One solution of these equations is a connection which is locally flat, a *pure gauge* field, $\mathbf{A} = gdg^{-1}$. The theory with configurations in the sector around a *pure gauge* field has no propagating degrees of freedom and the entire dynamics is contained in the nontrivial topology, but in that case the theory is irregular. In general, there are physical degrees of freedom in the bulk, as well as at the boundary.

The asymptotic dynamics is sensitive to the choice of boundary conditions and the topology of $\partial\mathcal{M}$. The boundary conditions must be chosen so that δB can be integrated to get B, or, if $B = 0$, so that δI_{CS} does not contain additional boundary terms in (6.13). Here it is supposed that the adequate boundary conditions exist, and they will be discussed later, in connection with the conserved charges.

6.2 Conserved charges

In Chapter 5, the Hamiltonian dynamics of CS theories in $D \geq 5$ was analyzed. It was shown that, in regular sectors of phase space, gauge invariance is expressed by the presence of first class constraints $\mathcal{G}_{M}$ which generate gauge transformations. There is also diffeomorphism invariance in these theories. Spatial diffeomorphisms are generated by first class constraints $\mathcal{H}_{i}$, while time-like diffeomorphisms are not independent, but realized as *on-shell* symmetries. Generic sectors of these theories do not have additional independent local symmetries. The symplectic form, which defines the kinetic term in CS theories, was also discussed. It was emphasized that it can change its rank throughout configuration space. Depending on the rank of the symplectic form, which in general depends on the background, CS theories can be either regular or irregular, generic or degenerate.

The study of the boundary dynamics has been intentionally left out of the analysis. In 2+1 dimensions, where there are no locally propagating degrees of freedom, the boundary dynamics is purely topological. In higher dimensions, besides purely topological degrees of freedom, there also exist local ones. In this chapter, the boundary dynamics is analyzed (see [55] for a review of the bosonic case). From now on, only *regular* and *generic* CS theories are considered, and they in general have local degrees of freedom.

In the Hamiltonian formalism it is assumed that space-time is $\mathcal{M} \simeq \mathbb{R} \times \sigma$, where σ is an Euclidean manifold. The PB of the canonical fields $\left(A_i^M, \pi_M^i\right)$ on the phase space Γ is given by

$$\left\{\pi_M^i, A_j^N\right\} = -\delta_j^i \delta_M^N \, \delta = -\left(-\right)^{\varepsilon_M} \left\{A_j^N, \pi_M^i\right\} \,, \tag{6.15}$$

where δ denotes the Dirac's δ-function at the spatial section, A_0^M is a Lagrange multiplier, and the number $\varepsilon_M = 0, 1$ (mod 2) is the Grassmann parity of A_μ^M and π_M^μ (see Appendix G for the conventions). The Hamiltonian for the action (6.8) is given by

$$H_T = \int d^4x \, \left(A_0^M \mathcal{G}_M + u_i^M \phi_M^i\right) \,, \tag{6.16}$$

where boundary terms have been neglected for the moment. The constraints are given by

$$\begin{aligned} \phi_M^i &= \pi_M^i - \mathcal{L}_M^i \approx 0 \,, \\ \mathcal{G}_M &= -\chi_M + D_i \phi_M^i \approx 0 \,, \end{aligned} \tag{6.17}$$

where the covariant derivative acts on ϕ_M^i as $D_i \phi_M^i = \partial_i \phi_M^i + f_{MN}{}^K A_i^K \phi_M^i$ and, for $D = 5$,

$$\begin{aligned} \mathcal{L}_M^i &= ik \, \varepsilon^{ijkl} \left(g_{MNK} \, F_{jk}^N A_l^K - \frac{1}{4} \, g_{MNL} \, f_{KS}{}^L A_j^N A_k^K A_l^S\right) \,, \\ \chi_M &= -\frac{3ik}{4} \, \varepsilon^{ijkl} g_{MNK} \, F_{ij}^N F_{kl}^K \approx 0 \,. \end{aligned} \tag{6.18}$$

The symplectic form defining the kinetic term in the action is a function on phase space

$$\Omega_{MN}^{ij} = -3ik \, \varepsilon^{ijkl} g_{MNK} \, F_{kl}^K \,, \tag{6.19}$$

whose rank can vary throughout Γ. In the regular and generic sectors, the action does not have other independent symmetries apart from spatial diffeomorphisms and gauge transformations, and the constraints satisfy the Poisson brackets algebra

$$\begin{aligned} \{\mathcal{G}_M, \mathcal{G}_N\} &= f_{MN}{}^K \, \mathcal{G}_K \, \delta \, , \\ \left\{\mathcal{G}_M, \phi^i_N\right\} &= f_{MN}{}^K \, \phi^i_K \, \delta \, , \\ \left\{\phi^i_M, \phi^j_N\right\} &= \Omega^{ij}_{MN} \, \delta \, , \end{aligned} \tag{6.20}$$

where $\mathcal{G}_M$ are first class constraints (generators of gauge transformations), while among ϕ's there are four first class constraints (generators of spatial diffeomorphisms) and the rest are second class constraints. The number of locally propagating degrees of freedom in this theory, according to (5.23), is

$$f_5\,(N) = N^2 + 8N + 13\,. \tag{6.21}$$

Regular generic background. In the following, a class of backgrounds (solutions of the constraints) is chosen so that they provide a regular and generic theory. They also allow separating first and second class constraints among ϕ's, which is in general a difficult task. However, in the case of $N = 4$, the invariant tensor takes the same form as invariant tensor of $g \otimes u\,(1)$. Explicitly, the supersymmetric algebra $su(2,2\,|4)$ is generated by the $u(1)$ generator $\mathbf{G}_1$ and the $psu(2,2\,|4)$ generators $\mathbf{G}_{M'} = \left(\mathbf{J}_{AB}, \mathbf{T}_\Lambda, \mathbf{Q}^\alpha_r, \bar{\mathbf{Q}}^r_\alpha\right)$, so that the invariant tensor g_{MNK} , decomposed as $\mathbf{G}_M \to (\mathbf{G}_{M'}, \mathbf{G}_1)$, takes the simpler form (see Appendix F):

$$g_{MNK} \quad \to \quad \left\{ g_{M'N'K'}\,, \quad g_{M'N'1} = -\frac{i}{4}\,\gamma_{M'N'}\,, \quad g_{M'11} = 0\,, \quad g_{111} = 0 \right\}. \tag{6.22}$$

Here $\gamma_{M'N'}$ is the invertible Killing metric of $PSU(2,2\,|4)$.

(i) Regular background. The background has to be such that the Jacobian matrix

$$J_{MN} = -6ik\, g_{MNK}\, F^K = \begin{pmatrix} J_{M'N'} & \frac{3}{2}\, k\, \gamma_{N'K'}\, F^{K'} \\ \frac{3}{2}\, k\, \gamma_{M'K'}\, F^{K'} & 0 \end{pmatrix} \tag{6.23}$$

is invertible. (From now on, the forms are defined on the spatial section.) When the submatrix $J_{M'N'}$ is invertible, the regularity conditions require that $F^{K'}$ be non-zero, for at least one K'. The simplest solution occurs when the fermionic field vanishes ($\psi_r^\alpha = 0$) and the space-time is locally AdS ($F^{AB} = 0$), while the $su(4)$ field-strength $\mathcal{F}^\Lambda$ has a component only along $dx^1 dx^2$,

$$\mathcal{F}^\Lambda = \mathcal{F}^\Lambda_{12}\, dx^1 dx^2 \neq 0\,. \tag{6.24}$$

This configuration is on the constraint surface if the $u(1)$ field ϕ has a field-strength satisfying $f_{34} = 0$, while the remaining components $f_{ij} = \partial_i \phi_j - \partial_j \phi_i$ are arbitrary.

(ii) Generic background. The background is generic if the symplectic form

$$\mathbf{\Omega} = \Omega^{ij}_{MN} = \begin{pmatrix} \Omega^{ij}_{M'N'} & \Omega^{ij}_{M'1} \\ \Omega^{ij}_{N'1} & 0 \end{pmatrix} \tag{6.25}$$

has maximal rank. Since $\mathbf{\Omega}$ has always four zero modes

$$\Omega^{ij}_{MN} F^N_{jk} = -\delta^i_k \chi_M \approx 0\,, \tag{6.26}$$

the maximal rank is $\Re\left(\mathbf{\Omega}\right) = 4\left(\mathcal{N} - 1\right)$. This is satisfied if the following two conditions are fulfilled:

$$\begin{array}{lccl} \text{(I)} & \Omega^{ij}_{M'N'} & = & \text{non-degenerate}, \\ \text{(II)} & \det f_{ij} & \neq & 0 \qquad (f_{34} = 0)\,. \end{array} \tag{6.27}$$

Consequently, the inverse $\Delta^{M'N'}_{ij}$ and f^{ij} exist,

$$\Delta^{M'K'}_{ik}\, \Omega^{kj}_{K'N'} = \delta^j_i \delta^{M'}_{N'}\,, \qquad\qquad f_{ij}\, f^{jk} = \delta^i_k\,, \tag{6.28}$$

and the rank of the symplectic form is:[2]

$$\Re\left(\Omega^{ij}_{MN}\right) = \Re\left(\Omega^{ij}_{M'N'}\right) = 4\left(\mathcal{N} - 1\right)\,. \tag{6.29}$$

[2]If the symplectic matrix Ω^{ij}_{MN} has invertible submatrix $\Omega^{ij}_{M'N'}$, then its rank is $\Re\left(\Omega^{ij}_{MN}\right) = \Re\left(\Omega^{ij}_{M'N'}\right) + \Re\left(M^{ij}\right)$, where $M^{ij} \equiv \Omega^{ik}_{M'1}\, \Delta^{M'N'}_{kl} \Omega^{lj}_{N'1}$ (see the footnote at p. 57). However, on the basis of the identity (6.26), the matrix M^{ij} weakly vanishes,

$$M^{ij} = f^{ij}\chi_1 - \Omega^{ik}_{M'1}\, \Delta^{M'N'}_{kl}\, f^{lj} \chi_{N'} \approx 0\,,$$

and therefore it has zero rank on the constraint surface.

(iii) First and second class constraints. Since the submatrix $\Omega^{ij}_{M'N'}$ is invertible, it is possible to separate the first and second class constraints. The second class constraints are $\phi^i_{M'}$, while

$$\tilde{\phi}^i_1 = \phi^i_1 - \Delta^{M'N'}_{jk}\Omega^{ki}_{N'1}\phi^j_{M'} \approx 0 \tag{6.30}$$

are first class constraints related to the generators of spatial diffeomorphisms,

$$\mathcal{H}_i \equiv f_{ij}\tilde{\phi}^j_1 = F^M_{ij}\phi^j_M \approx 0\,. \tag{6.31}$$

In general, one could introduce Dirac brackets and eliminate unphysical degrees of freedom coming from the second class constraints. But it is more convenient not to do so in order to maintain explicitly covariant expressions.

Improved generators. The generators of gauge symmetry are given by

$$G[\lambda] = \int d^4x\, \lambda^M \mathcal{G}_M = \int d^4x\, \lambda^M \left(-\chi_M + D_i\phi^i_M\right)\,, \tag{6.32}$$

and their action on phase space functions F is

$$\delta_\lambda F = \{F, G[\lambda]\} = (-)^{\varepsilon_F\varepsilon_M} \int d^4x\, \lambda^M \{F, \mathcal{G}_M\}\,. \tag{6.33}$$

The generators can be made to have local functional derivatives. This means that the variation of the generator $G[\lambda, z]$ takes the form $\int dx\, \delta z^A \frac{\delta G}{\delta z^A}$, without derivatives $\partial_j\left(\delta z^A\right)$ which would give rise to boundary terms. The generators (6.32), however, vary as

$$\delta G[\lambda] = \delta G_Q[\lambda] - \delta Q[\lambda]\,, \tag{6.34}$$

where $\delta G_Q[\lambda]$ is the bulk term and $\delta Q[\lambda]$ is a boundary term. The generators of local transformations should be the so-called *improved generators*, which differ from the original ones by boundary terms (the Regge-Teitelboim approach [18]),

$$G_Q[\lambda] \equiv G[\lambda] + Q[\lambda]\,, \tag{6.35}$$

such that their functional derivatives are local functions. In order find the explicit expressions for the improved generators, it is more convenient to rewrite the original generators (6.32) as

$$G[\lambda] = \int_\sigma \langle \lambda\left(-\mathbf{K} + D\mathbf{\Phi}\right)\rangle\,, \tag{6.36}$$

where the 4-form $\mathbf{K}$ ($K_{Mijkl} = \chi_M \, \varepsilon_{ijkl}$), and the 3-form $\mathbf{\Phi}$ ($\Phi_{Mjkl} = \phi^i_M \, \varepsilon_{ijkl}$) are defined on the 4-dimensional spatial manifold σ by

$$\begin{aligned} \mathbf{K} &\equiv -3ik\, \mathbf{F}^2 \approx 0\,, \\ \mathbf{\Phi} &\equiv \mathbf{\Pi} - ik \left(\{\mathbf{A}, \mathbf{F}\} - \frac{1}{2}\,\mathbf{A}^3 \right) \approx 0\,. \end{aligned} \tag{6.37}$$

Here $\Pi_{Mjkl} = \pi^i_M \, \varepsilon_{ijkl}$. Then, the generators (6.36) vary as

$$\delta G\left[\lambda\right] = \int\limits_\sigma \left[- \left\langle \lambda \delta \mathbf{K} \right\rangle + \left\langle \lambda D \delta \mathbf{\Phi} \right\rangle + \left\langle \left[\lambda, \mathbf{\Phi}\right] \delta \mathbf{A} \right\rangle \right] \,, \tag{6.38}$$

with the following variations of the constraints:

$$\begin{aligned} \delta \mathbf{K} &= -3ik \, \left\{ \mathbf{F}, D\delta \mathbf{A} \right\} \,, \\ \delta \mathbf{\Phi} &= \delta \mathbf{\Pi} - ik \, \left\{ \mathbf{F}, \delta \mathbf{A} \right\} - ik \, \left\{ \mathbf{A}, D\delta \mathbf{A} \right\} + \frac{ik}{2}\, \delta \mathbf{A}^3. \end{aligned} \tag{6.39}$$

Therefore, the bulk term in (6.38) has the form

$$\delta G_Q\left[\lambda\right] = \int\limits_\sigma \left\langle \delta \mathbf{A} \mathbf{X}\left(\lambda\right) + \delta \mathbf{\Pi} D\lambda \right\rangle \,, \tag{6.40}$$

where the 3-form $\mathbf{X}$ is given by the expression

$$\mathbf{X}\left(\lambda\right) = ik \left\{ D\lambda, \mathbf{F} \right\} - \frac{3ik}{2} \left\{ D\lambda, \mathbf{A}^2 \right\} + ik \left\{ \left[\mathbf{F}, \lambda\right], \mathbf{A} \right\} - \left[\lambda, \mathbf{\Phi}\right] \,. \tag{6.41}$$

From Eqs. (6.40) and (6.41), the functional derivatives of G_Q are

$$\begin{aligned} \frac{\delta G_Q\left[\lambda\right]}{\delta A^M_i} &= \frac{1}{3!}\, \varepsilon^{ijkl} X_{Mjkl}\left(\lambda\right) \,, \\ \frac{\delta G_Q\left[\lambda\right]}{\delta \pi^i_M} &= -\left(-\right)^{\varepsilon_M} D_i \lambda^M \,. \end{aligned} \tag{6.42}$$

Thus, the improved generators indeed generate local gauge transformations,

$$\delta_\lambda A^M_i = \left\{ A^M_i, G_Q\left[\lambda\right] \right\} = \left(-\right)^{\varepsilon_M} \frac{\delta G_Q\left[\lambda\right]}{\delta \pi^i_M} = -D_i \lambda^M \,, \tag{6.43}$$

and it can also be shown that momenta transform as

$$\delta_\lambda \pi^i_M = \left\{ \pi^i_M, G_Q\left[\lambda\right] \right\} = -\frac{\delta G_Q\left[\lambda\right]}{\delta A^M_i} = -\frac{1}{3!}\, \varepsilon^{ijkl} X_{Mjkl}\left(\lambda\right) \,. \tag{6.44}$$

The improved generators, however, are not constraints but take the value $G_Q \approx Q$ on the constraint surface. $Q\left[\lambda\right]$ generates gauge transformations at the boundary and is called the *charge*.

Conserved charges. The boundary term in (6.38), denoted by $-\delta Q\left[\lambda\right]$, is

$$\delta Q\left[\lambda\right]=-6ik\int\limits_{\partial\sigma}\left\langle \lambda\mathbf{F}\delta\mathbf{A}\right\rangle -2ik\int\limits_{\partial\sigma}\left\langle D\lambda\mathbf{A}\delta\mathbf{A}\right\rangle -\int\limits_{\partial\sigma}\left\langle \lambda\delta\mathbf{\Phi}\right\rangle \,. \tag{6.45}$$

This expression can be integrated out provided the connection is fixed at the boundary,

$$\mathbf{A}\;\longrightarrow\;\bar{\mathbf{A}}\,,\qquad\text{at }\;\partial\sigma, \tag{6.46}$$

where $\bar{\mathbf{A}}$ is a regular generic configuration. The choice of boundary conditions is not unique, and (6.46) is the simplest one which still gives a non trivial asymptotic dynamics. More general possibility is to solve Q from (6.45) without fixing of all components of the connection at the boundary.

Using (6.45) and the boundary conditions (6.46), the charge is obtained as

$$Q\left[\lambda\right]=-6ik\int\limits_{\partial\sigma}\left\langle \lambda\bar{\mathbf{F}}\mathbf{A}\right\rangle -2ik\int\limits_{\partial\sigma}\left\langle \bar{D}\lambda\bar{\mathbf{A}}\mathbf{A}\right\rangle -\int\limits_{\partial\sigma}\left\langle \lambda\mathbf{\Phi}\right\rangle \,. \tag{6.47}$$

The charge can be explicitly written as

$$Q\left[\lambda\right]\approx-2ik\int\limits_{\partial\sigma}g_{MNK}\,\left(3\lambda^{M}\bar{F}^{N}+\bar{D}\lambda^{M}\bar{A}^{N}\right)\,A^{K}, \tag{6.48}$$

where the term proportional to the constraints $\mathbf{\Phi}$ vanishes *on-shell,* and it does not give contributions to the charge. The second term at the *r.h.s.* of (6.47), comes from the variation of the constraints $\phi_{M}^{i}=(\phi_{1}^{i},\phi_{M'}^{i})$, where $\phi_{M'}^{i}$ are second class. On the other hand, one should not expect to have a contribution of the second class constraints, which naturally do not appear in the gauge generator (6.36) on the reduced phase space, where $\phi_{M'}^{i}=0$, what can be explicitly provided by introducing the appropriate Dirac brackets. Both approaches should be equivalent, because the second class constraints do not generate gauge transformations, and there are no conserved charges associated to them.

One can see that the term $\left\langle \bar{D}\alpha,\bar{\mathbf{A}},\mathbf{A}\right\rangle$ vanishes for the local $PSU\left(2,2\left|4\right.\right)$ parameter $\alpha\equiv\alpha^{M'}\mathbf{G}_{M'}$, once the asymptotic conditions are used. Evaluated at the background ($\mathbf{A}=\bar{\mathbf{A}}$), this term is indeed zero, due to the identity $g_{M'NK}\,\bar{A}^{N}\bar{A}^{K}\equiv0$. The fields asymptotically tend to the background, so that the connection behaves as $\mathbf{A}\sim\bar{\mathbf{A}}+\Delta\mathbf{A}$, while the local parameter is $\alpha \sim \bar{\alpha}+\Delta\alpha$, where $\bar{\alpha}$ is a covariantly constant vector ($\bar{D}\bar{\alpha}=0$) which describes symmetries of the vacuum at the boundary. Then the term

$$\left\langle \bar{D}\alpha, \bar{\mathbf{A}}, \mathbf{A}\right\rangle \sim \left\langle \bar{D}\Delta\alpha, \bar{\mathbf{A}}, \Delta\mathbf{A}\right\rangle \ , \tag{6.49}$$

gives a contribution of second order, since $\Delta\mathbf{A}$ and $\Delta\alpha$ are subleading compared to $\bar{\mathbf{A}}$ and $\bar{\alpha}$, in the limit in which the boundary is taken to infinity.

Charge algebra. The PB algebra of improved gauge generators can be found directly from the definition of the functional derivatives (6.42), and it has the form

$$\left\{G_Q\left[\lambda\right], G_Q\left[\eta\right]\right\} = \int\limits_{\sigma} \left\langle \mathbf{X}\left(\lambda\right) D\eta - X\left(\eta\right) D\lambda\right\rangle \ , \tag{6.50}$$

where the expression for $\mathbf{X}$ is given by (6.41). This algebra closes, and it has the general form

$$\left\{G_Q\left[\lambda\right], G_Q\left[\eta\right]\right\} = G_Q\left[\left[\lambda,\eta\right]\right] + C\left[\lambda,\eta\right] \ , \tag{6.51}$$

where $\left[\lambda,\eta\right]^M = -f_{KN}{}^M\lambda^N\eta^K$, and $C\left[\lambda,\eta\right]$ is a boundary term. Thus, the classical algebra acquires a central extension C, called the *central charge*, which emerges as a consequence of working with the improved generators. In order to calculate C, rather than starting from (6.50), it is more convenient to find it from the gauge transformation of the charge Q,

$$\delta_\eta Q\left[\lambda\right] = \left\{Q\left[\lambda\right], Q\left[\eta\right]\right\} = Q\left[\left[\lambda,\eta\right]\right] + C\left[\lambda,\eta\right] \ . \tag{6.52}$$

On the basis of the Brown-Henneaux theorem [112], the central extension obtained from the gauge transformation of the charge, Eq. (6.52), is the same as the central charge in the algebra of improved generators, Eq. (6.51), evaluated on the background. This follows from the fact that the charge algebra (6.52) is valid only on the reduced phase space, after gauge fixing of all first class constraints $\mathcal{G}_M$, so that the original generators $G\left[\lambda\right]$ are all strongly zero.

The central charge can be evaluated as follows. In the class of generic regular configurations $\bar{A}$ given by (6.24) and (6.27), the vacuum $\bar{A}_v$ is the one for which the charges vanish, $\bar{Q}\left[\lambda\right] \equiv Q\left[\lambda\right]_{\bar{A}_v} = 0$. Then, the central charges are obtained as $C\left[\lambda,\eta\right] = \delta_\eta Q\left[\lambda\right]_{\bar{A}_v}$. This will be calculated in the next section.

6.3 Killing spinors and BPS states

In supergravity theories, the anticommutation relation are of the form $\left\{Q, Q^{\dagger}\right\} \sim P + J + \cdots$, giving that the sum of the total charges is bounded from below, since it is proportional to $\sum\left(QQ^{\dagger} + QQ^{\dagger}\right) \geq 0$ (Bogomol'nyi bound [36]). In standard supergravity theories it leads to the positivity of energy [37, 38, 39]. When the bound is saturated, the corresponding states, the so-called BPS states, have some unbroken supersymmetries. Therefore, the existence of BPS states is important for the stability of the theory. In what follows, a BPS state will be constructed which is also regular and generic.

For the bosonic BPS states $\psi_r^{\alpha} = 0$, and the supersymmetry transformations $\delta_{\epsilon}\psi_r^{\alpha} = -D\epsilon_r^{\alpha}$ leave this condition invariant if the local fermionic parameter satisfies

$$D\epsilon_r^{\alpha} = 0\,. \tag{6.53}$$

This is the Killing equation for the spinor ϵ_r^{α}, and its solutions are Killing spinors.

Assuming a generic regular configuration, conditions (6.24) and (6.27) are satisfied if:

$$\begin{array}{lllllll} \psi_r & = & 0\,, & \mathcal{F}^{\Lambda} & = & \mathcal{F}_{12}^{\Lambda}\, dr d\varphi^2 \neq 0\,, & \\ F^{AB} & = & 0\,, & \det\, f_{ij} & \neq & 0\,, & f_{34} = 0\,. \end{array} \tag{6.54}$$

The local coordinates on $\mathcal{M}$ are chosen as $x^{\mu} = (t, r, x^n)$, where x^n $(n = 2, 3, 4)$ parametrize the boundary $\partial\sigma$, placed at the infinity of the radial coordinate r.

Locally *AdS* space-time. The AdS space-time ($F^{AB} = 0$) can be described by the metric

$$ds^2_{\mathrm{AdS}} = \ell^2\,\left(dr^2 + e^{2r}\eta_{\bar{n}\bar{m}}\, dx^{\bar{n}} dx^{\bar{m}}\right)\,, \tag{6.55}$$

where $x^{\bar{n}} = (t, x^n)$ and $\eta_{\bar{n}\bar{m}} = \mathrm{diag}\,(-,+,+,+,+)$. The vielbein (e^a) and the spin connection (ω^{ab}) are given by

$$\begin{array}{ll} e^1 = \ell dr\,, & \omega^{\bar{n}1} = \frac{1}{\ell}\, e^{\bar{n}} = e^r dx^{\bar{n}}\,, \\ e^{\bar{n}} = \ell e^r dx^{\bar{n}}\,, & \omega^{\bar{n}\bar{m}} = 0\,. \end{array} \tag{6.56}$$

It is easy to check that this space-time is indeed torsionless ($T^a = 0$) and with constant negative curvature ($R^{ab} = -\frac{1}{\ell^2}\, e^a e^b$). Then the AdS connection is

$$\mathbf{W} = \frac{1}{4}\, W^{AB}\Gamma_{AB} = \frac{1}{2\ell}\,\left[e^1\Gamma_1 + e^{\bar{n}}\Gamma_{\bar{n}}\,(1 + \Gamma_1)\right]\,. \tag{6.57}$$

The Killing spinors ε^{α} for the metric (6.55), which are solutions of

$$(d+\mathbf{W})\,\varepsilon = 0\,, \tag{6.58}$$

and they have the form [135]

$$\varepsilon = e^{-\frac{r}{2}\Gamma_1}\,\left[1 - x^{\bar{n}}\Gamma_{\bar{n}}\,(1+\Gamma_1)\right]\,\varepsilon_0\,, \tag{6.59}$$

where ε_0 is a constant spinor. The derivation of (6.59) is given in the Appendix H. Changing the topology of $\partial\sigma$ by the identification of the coordinates x^n gives a locally AdS space-time, and it eliminates the $x^{\bar{n}}$-dependence of ε, which therefore must be (anti)chiral under Γ_1. Some examples of Killing spinors in the locally AdS space-time are given in [136].

Consistency of the Killing equation. The Killing equation (6.53) for the $su(4)$ spinor ϵ_r^{α} can be written in components as

$$\left[\delta_r^s\,(d+\mathbf{W}) - a_r^s\right]\epsilon_s = 0\,, \tag{6.60}$$

where $W^{AB} = \left(\frac{1}{\ell}\,e^a, \omega^{ab}\right)$ is the AdS connection, a is the $su(4)$ connection, and (for $N=4$) the $u(1)$ field is decoupled and it does not appear in the covariant derivative. The consistency of this equation requires $DD\epsilon = [\mathbf{F},\epsilon] = 0$, or in components,

$$\left(\frac{1}{4}\,\delta_r^s\,F^{AB}\Gamma_{AB} - \mathcal{F}_r^s\right)\epsilon_s = 0\,. \tag{6.61}$$

For the configurations (6.54), this equation reduces to

$$\mathcal{F}_r^s\,\epsilon_s = 0\,. \tag{6.62}$$

This equation has nonvanishing solution provided the matrix $\mathcal{F}_r^s$ has zero eigenvalues and ϵ_r are its zero modes. In order to have a nontrivial $su(4)$ curvature, $\mathcal{F}^{\Lambda}$ must be nonvanishing for more than one value of the index Λ, so that the contributions of all components cancel. Using the local isomorphism $su(4)\simeq so(6)$, it is more convenient to represent the $su(4)$ curvature as $\mathcal{F}_r^s = \frac{1}{2}\mathcal{F}^{IJ}\,(\tau_{IJ})_r^{\ s}$, where

$$\tau_{IJ} = \frac{1}{2}\,\hat{\Gamma}_{IJ}\,, \qquad (I,J = 1,\ldots,6) \tag{6.63}$$

are the $so(6)$ generators, $\hat{\Gamma}_{IJ} = \frac{1}{2}\left[\hat{\Gamma}_I, \hat{\Gamma}_J\right]$ and $\hat{\Gamma}_I$ are the Euclidean $\hat{\Gamma}$-matrices. Furthermore, the commuting matrices τ_{12} and τ_{34} generate the $u(1) \otimes u(1)$ subalgebra of $so(6)$. Since $(\tau_{12})^2 = (\tau_{34})^2 = -\frac{1}{4}$, the eigenvalues of τ_{12} and τ_{34} are $\pm\frac{i}{2}$. Considering the "twisted" configuration, *i.e.*, assuming

$$\begin{aligned} (\tau_{12})_r^{\ s}\,\epsilon_s &= \frac{i}{2}\,\epsilon_r\,, \\ (\tau_{34})_r^{\ s}\,\epsilon_s &= -\frac{i}{2}\,\epsilon_r\,, \end{aligned} \tag{6.64}$$

and that the only $u(1)$ curvature components are $\mathcal{F}^{12} = da^{12}$ and $\mathcal{F}^{34} = da^{34}$, then Eq. (6.62) becomes $\frac{i}{2}\left(\mathcal{F}^{12} - \mathcal{F}^{34}\right)\epsilon_r = 0$, whose solution is

$$\mathcal{F}^{12} = \mathcal{F}^{34}. \tag{6.65}$$

In terms of the connection, this implies

$$a^{12} = a^{34} + d\eta\,, \tag{6.66}$$

where $\eta\,(r, x^n)$ is an arbitrary function. Since $\mathcal{F}^{12}_{12}\,(r, x^2)$ can only depend on two coordinates by virtue of the Bianchi identity, the simplest solution reads

$$\begin{aligned} a^{12}\,(x^2) &= \rho\,(x^2)\;dr\,, \\ \mathcal{F}^{12}\,(x^2) &= -\rho'\,(x^2)\;dr dx^2\,. \end{aligned} \tag{6.67}$$

The Killing spinors. Since $a_r^{s}\,\epsilon_s = \frac{i}{2}\,d\eta\,\epsilon_r$, the $su(2,2\,|4)$ Killing equation (6.60) reduces to

$$\left(d + \mathbf{W} - \frac{i}{2}\,d\eta\right)\,\epsilon_s = 0\,, \tag{6.68}$$

where $\mathbf{W}$ is given by (6.57). The spinor ϵ_s can be factorized as

$$\epsilon_s^{\alpha} = u_s\,\varepsilon^{\alpha}\,, \tag{6.69}$$

where the spinor ε^{α} is the AdS Killing spinor (6.59), while the $su(4)$ vector u_s is a solution of the equation

$$\left(d - \frac{i}{2}\,d\eta\right)\,u_s = 0\,, \tag{6.70}$$

and it has the form $u_s = e^{\frac{i}{2}\eta}u_{0s}$. Therefore, The spinor ϵ_s is

$$\epsilon_s = e^{\frac{i}{2}\eta}e^{-\frac{r}{2}\Gamma_1}\,\left[1 - x^{\bar{n}}\Gamma_{\bar{n}}\left(1 + \Gamma_1\right)\right]\,\epsilon_{0s}\,, \tag{6.71}$$

where the constant spinor ϵ^{α}_{0s} satisfies the conditions (6.64)

$$(\tau_{12})_r^s\,\epsilon_{0s} = \frac{i}{2}\,\epsilon_{0r}\,, \qquad\qquad (\tau_{34})_r^s\,\epsilon_{0s} = -\frac{i}{2}\,\epsilon_{0r}\,. \tag{6.72}$$

The norm $\|\epsilon\|^2 = \bar{\epsilon}\epsilon$ is constant and positive,

$$\|\epsilon\|^2 = \|\epsilon_0\|^2 > 0\,, \tag{6.73}$$

where $\bar{\epsilon}^r_\alpha = \epsilon^{r\dagger}_\beta\,(\Gamma_0)^\beta_\alpha$ (see Appendix H).

Therefore, the existence of configurations with some unbroken supersymmetries, which saturate the Bogomol'nyi bound, is important for the stability of the theory. Among them, there is the ground state $\bar{A}_v$.

The central charge. A pending issue is to find the explicit expression of the central charge, C, for the charge algebra (6.52). This can be done for the configurations that asymptotically tend to the background solution which is a BPS state. Since $\bar{F}^{M'} = 0$ at the boundary, the only nonvanishing field-strength is $\bar{f}$, and the charge (6.47) becomes

$$\bar{Q}\,[\lambda] = -\frac{3k}{4}\int\limits_{\partial\sigma} \bar{f}\,\left(\lambda_{IJ}\bar{a}^{IJ} + \lambda_{AB}\bar{W}^{AB}\right)\,. \tag{6.74}$$

The goal is to find the vacuum, for which all charges vanish, $\bar{Q}\,[\lambda] = 0$. The $su(4)$ fields are given by (6.66) and (6.67), so that the first term in the expression for the charge (6.74) is proportional to the *pure gauge* $d\eta$. Therefore, for the vacuum state, η can be chosen so that $d\eta = 0$. On the basis of (6.56), the second term in the charge (6.74) vanishes if the AdS parameters λ_{n5}, λ_{n1} obey the asymptotic conditions:

$$\varepsilon^{nmk}\bar{f}_{nm}\,\left(\lambda_{k5} + \lambda_{k1}\right)e^r \to 0\,, \qquad (r \to \infty)\,. \tag{6.75}$$

Supposing that all conditions are fulfilled and $\bar{Q} = 0$, the charge $Q\,[\lambda]$ in (6.47) is found to change under the gauge transformations $\delta_\eta \mathbf{A} = -D\eta$, as

$$C\,[\lambda,\eta] = 2ik\int\limits_{\partial\sigma} \left\langle \left(3\lambda\bar{\mathbf{F}} + \bar{D}\lambda\bar{\mathbf{A}}\right)\bar{D}\eta\right\rangle\,. \tag{6.76}$$

Finally, after substitution of the BPS background, the charge becomes

$$C\,[\lambda,\eta] = k\int\limits_{\partial\sigma} \bar{f}\,\eta^{M'}d\lambda_{M'}\,. \tag{6.77}$$

The algebra (6.51), with the central charge (6.77), is a supersymmetric extension of the WZW_4 algebra [46, 47, 54, 57]. This algebra has a nontrivial central extension for $PSU(2,2\,|4)$, which depends on the 2-form $\bar{f}$. In the Ref. [54], the WZW_4 algebra describing the asymptotic symmetries of a CS theory based on $g\otimes u(1)$ algebra was studied. The background chosen there was irregular, and it was not possible to define the $u(1)$ charge, so that the WZW_4 algebra was associated only to the subalgebra g. In the case of the superalgebra (6.51), all generators are well defined because the chosen background is regular and generic. In contrast, the super WZW_4 algebra obtained here is associated to the full gauge group $SU(2,2\,|4)$.

6.4 Conclusions

In this chapter, CS supergravities based on the supersymmetric extension of the AdS_5 algebra, $su(2,2\,|N)$ were analyzed. This supergravity theories contain the gravitational fields, $2N$ gravitini, $su(N)$ bosonic fields, and $u(1)$ field. The action, apart from the Einsten-Hilbert term, contains terms nonlinear in curvature and is torsionless. The sypersymmetry algebra closes *off-shell*, without bringing in the auxiliary fields.

In the case of $N=4$, the invariant tensor of $su(2,2\,|4)$ algebra has the same form as the invariant tensor of $g\otimes u(1)$. The theory has rich local dynamics, with 61 locally propagating degrees of freedom in the regular generic sector, as well as nontrivial asymptotic dynamics.

The asymptotic dynamics depends on the choice of boundary conditions and is determined by the subset of the gauge transformations which preserve these conditions at the boundary. A result of this analysis was to show that adequate boundary conditions exist, and to find the corresponding symmetries and the charges at the boundary.

The following results are obtained:

- A class of backgrounds is found, which provide a regular and generic theory. They are locally AdS space-times with bosonic $su(4)$ and $u(1)$ matter. It is impossible to have a regular and generic AdS space-time without both types of bosonic matter fields. On the other hand, if AdS space-time is not required, then the $su(4)$ field-strength can vanish (*pure-gauge*).

- Around the chosen backgrounds, the first and second class constraints are explicitly separated. This, in general, is an extremely difficult task for an arbitrary CS theory.
- The charges corresponding to the complete gauge symmetry are obtained for the simplest choice of the asymptotic conditions $A \to \bar{A}$. This problem is not trivial since higher-dimensional CS theories are irregular systems and there are sectors in the phase space where it is not possible to define some generators.
- BPS states exist among the considered backgrounds, which is important for the stability of the theory.
- The supersymmetric extension of the classical WZW_4 algebra, associated to $su(2,2\,|4\,)$, is obtained. The nontrivial central extension occurs only for $psu(2,2\,|4\,)$ subalgebra.

In addition to these results, work on the asymptotics of the CS supergravity is in progress, because there are still many questions to be answered. One of them is related to the finding of the asymptotic symmetries and the physical interpretation of the conserved charges. For example, is the mass associated to the generator of the local time translations $Q\,[\lambda^{a5}]$? In Ref. [113], the energy and angular momentum of the black hole embedded in this supergravity theory were found, but the considered solution belonged to the irregular sector. There are also other black hole solutions in the five-dimensional CS supergravity which may be considered [114]–[117]. Then, the natural question is to analyze which charges (or their combinations) are bounded from below by the Bogomol'nyi bound. For example, in $3D$ supergravity, the black hole solution with the zero mass and zero angular momentum is a BPS state, as well as the extreme case $M = \ell\,|J|$ [118].

Furthermore, for the BPS states obtained in the last section, it is straightforward to make the mode expansion of the super WZW_4 algebra, for instance when a boundary has a topology $S^1 \times S^1 \times S^1$, or $S^1 \times S^2$. It is interesting to see which are the implications of different topologies to the asymptotic symmetry.

One of the interesting questions which remains to be investigated, is whether there is a regular and generic background (for example, when the $su(4)$ field does not vanish at the boundary) which leads to the super WZW_4 with the $u(1)$ central extension, as well. It seems, in general, possible.

Chapter 7

List of main results and open problems

Main results

- The action for the two-dimensional super Wess-Zumino-Witten model coupled to supergravity is obtained by canonical methods, so that it is invariant under local supersymmetry transformations by construction.
- Standard Dirac's procedure for dealing with constrained systems is generalized to the cases when the constraints are functionally dependent (irregular). These irregular systems are classified, and regularized when possible, for classical theories with finite number of degrees of freedom.
- Higher-dimensional Chern-Simons theories are analyzed in the context of irregular systems, and the criterion which recognizes irregular sectors in their phase space is presented.
- The dynamical content of AdS_5-Chern-Simons supergravity theory in five dimensions, based on the group $SU(2,2\,|4)$, is studied and, among them, a class of BPS states is found.
- The classical super WZW$_4$ algebra associated to $su(2,2\,|4)$, with nontrivial central extension, is obtained as the charge algebra for AdS_5-CS supergravity.

Open problems

- The method of construction of a super WZW action coupled to supergravity can be generalized to the cases of $N > 1$ supersymmetry, in $D = 2$.
- It is not clear whether all types of irregular systems can be consistently quantized, and how irregular sectors will present themselves after the quantization.
- The dynamic of irregular and degenerate Chern-Simons theories, in which the irregular sectors intersect with the surfaces of phase space with lower rank of the symplectic form, is not well understood.
- The pending questions in the AdS_5-Chern-Simons supergravity are the identification of the asymptotic symmetries and conserved charges in terms of observables (the mass, angular momentum, electric charge, etc.), as well as analysis of its boundary dynamics for different choices of topology of the space-time.
- One of more general problems is to obtain an action for a super WZW model in higher dimensions. It can be done, for example, solving the CS theory in $D = 5$ based on the supersymmetric extension of $U(1) \otimes U(1)$, and finding its theory at the boundary.

Appendix A

Hamiltonian formalism

In this chapter, the Hamiltonian formalism for the systems with bosonic degrees of freedom is reviewed [98, 101], [119]–[123]. A generalization of the formalism to the systems with fermionic degrees of freedom can be found in Refs. [124]–[127], while for the reviews see Refs. [59, 99, 102], [128]–[131].

Consider a *classical* system with *finite* number of degrees of freedom, described by the action

$$I[q,\dot{q}] = \int_{t_0}^{t_1} dt\, L(q^i,\dot{q}^i)\,, \qquad (i = 1,\cdots,N)\,, \tag{A.1}$$

which depends at most on first derivatives of the local coordinates $q^i(t)$ (up to divergence terms) and it does not depend on time explicitly. The classical dynamics is derived from the Hamilton's variational principle, as a stationary point of the action under variations $\delta q(t)$ with fixed endpoints $\delta q(t_0) = \delta q(t_1) = 0$.

a) Dirac-Bergman algorithm

In order to pass to the Hamiltonian formalism, momenta are defined in the usual way as

$$p_i = \frac{\delta I}{\delta \dot{q}^i}\,, \qquad (i = 1,\cdots,N)\,, \tag{A.2}$$

and Hamiltonian dynamics happens on $2N$-dimensional phase space

$$\Gamma = \{z^n \mid n = 1,\ldots,2N\}\,, \tag{A.3}$$

with local coordinates $z^n = (q^i, p_j)$. When all velocities can be solved from the equations (A.2) in terms of canonical variables, the evolution of a system is uniquely determined from its initial configuration by means of Hamilton equations. When all velocities $\dot{q}^i$ cannot be uniquely solved from equations (A.2) in terms of canonical variables, all momenta are not independent. In consequence, there are *primary constraints,*

$$\varphi_\alpha(z) = 0\,, \qquad (\alpha = 1, \cdots, N_P)\,, \tag{A.4}$$

which define the *primary constraint surface*

$$\Sigma_P = \{\bar{z} \in \Gamma \mid \varphi_\alpha(\bar{z}) = 0 \ (\alpha = 1, \ldots, N_P) \ (N_P \leq 2N)\} \ . \tag{A.5}$$

Although the primary constraints vanish on Σ_P, their derivatives do not, thus it is useful to make difference between the concepts of weak and strong equalities. A function $F(z)$, defined and differentiable in a neighborhood of $z \in \Gamma$, is *weakly* equal to zero if it vanishes on Σ_P,

$$F(z) \approx 0 \quad \Longleftrightarrow \quad F(z)|_{\Sigma_P} = 0\,, \tag{A.6}$$

and it is *strongly* equal to zero if the function F and its first derivatives vanish on Σ_P,

$$F(z) = 0 \quad \Longleftrightarrow \quad F,\ \left.\frac{\partial F}{\partial z^n}\right|_{\Sigma_P} = 0\,. \tag{A.7}$$

With this conventions, the primary constraints are

$$\varphi_\alpha(z) \approx 0\,, \qquad (\alpha = 1, \cdots, N_P)\,. \tag{A.8}$$

Primary constraints are functionally independent if the *regularity conditions* (RCs) [98] are fulfilled: the constraints $\varphi_\alpha \approx 0$ are regular if and only if their small variations $\delta\varphi_\alpha$ evaluated on Σ_p define N_P linearly independent functions of δz^n.

When the RCs are satisfied, there is relation between the strong and weak equalities. If a phase space function F is weakly equal to zero, then it is strongly equal to a linear combination of constraints,

$$F \approx 0 \quad \Longleftrightarrow \quad F = u^\alpha \varphi_\alpha\,. \tag{A.9}$$

Therefore, the existence of primary constraints naturally leads to appearance of arbitrary functions of time $u^\alpha(t)$ in a theory.

Total Hamiltonian and time evolution. The canonical Hamiltonian, obtained by Legendre transformation of the Lagrangian,

$$H_0(q,p) = p_i \dot{q}^i - L(q,\dot{q})\,, \tag{A.10}$$

depends only on canonical variables. Due to presence of primary constraints, canonical variables are not independent and H_0 is not unique, therefore the Legendre transformation is not invertible. Using the relation between the weak and strong equalities (A.9), one introduces *total Hamiltonian* as

$$H = H_0 + u^\alpha \varphi_\alpha\,. \tag{A.11}$$

It gives an invertible Legendre transformation, but the dynamics following from (A.11) depends on N_P arbitrary functions $u^\alpha(t)$. Introducing a Poisson bracket (PB) on the phase space Γ as

$$\{F,G\} = \frac{\partial F}{\partial z^n}\,\omega^{nm}\,\frac{\partial G}{\partial z^m}\,, \tag{A.12}$$

where ω^{nm} is the antisymmetric symplectic form which determines the basic PB $\{z^n, z^m\} = \omega^{nm}$, Hamilton equations can be written as

$$\dot{z}^n = \{z^n, H_0\} + u^\alpha\,\{z^n, \varphi_\alpha\} \approx \{z^n, H\}\ . \tag{A.13}$$

Time evolution of phase space functions F is determined by

$$\dot{F} = \{F, H_0\} + u^\alpha\,\{F, \varphi_\alpha\} \approx \{F, H\}\ , \tag{A.14}$$

and in general it is not unique for given initial conditions $F\,(t_0)$.

Consistency conditions. Consistency of the theory during its time evolution requires that the primary constraints are preserved in time,

$$\dot{\varphi}^\alpha = \{\varphi^\alpha, H_0\} + u^\beta\,\{\varphi_\alpha, \varphi_\beta\} \approx 0\,. \tag{A.15}$$

These *consistency conditions* reduce to one of the following possibilities[1]:

[1]The systems in which the consistency conditions are not satisfied are excluded because, in such inconsistent models, the action has no stationary points.

- If $\{\varphi_\alpha, \varphi_\beta\} \approx 0$ and $\{\varphi_\alpha, H_0\} \approx 0$, then Eq. (A.15) is automatically satisfied (it reduces to $0 \approx 0$).
- If $\{\varphi_\alpha, \varphi_\beta\} \approx 0$ and $\{\varphi_\alpha, H_0\} \neq 0$, Eq. (A.15) is independent on multipliers and gives a *secondary constraint*.
- If $\{\varphi_\alpha, \varphi_\beta\} \neq 0$, Eq. (A.15) becomes an algebraic equation in multipliers, leading to restrictions on some of them.

Therefore, consistency conditions can give *secondary constraints*, the evolution of secondary constraints can give a new generation of constraints, and so on, until it stops after finite number of generations (because the dimension of Γ is finite). This procedure results with the complete set of constraints

$$\phi_r(z) \approx 0\,, \qquad (r = 1, \ldots, R)\ , \tag{A.16}$$

and a number of determined multipliers u.

A system is *regular* if all constraints satisfy the RCs; otherwise, the system is *irregular*. Here only regular systems are considered. Then the conditions (A.16) define $(2n - R)$-dimensional *constraint surface*

$$\Sigma = \{\bar{z} \in \Gamma \mid \phi_r(\bar{z}) = 0\ \ (r = 1, \ldots, R)\ \ (R \leq 2N)\}\ , \tag{A.17}$$

and weak and strong equalities are defined with respect to Σ. Dirac-Bergman procedure guarantees that the system remains on the constraint surface during its evolution.

First and second class functions. A function $F(z)$ is said to be *first class* if it has a PB with all constraints weakly vanished,

$$\{F, \phi_r\} \approx 0\,, \qquad (r = 1, \ldots, R)\,. \tag{A.18}$$

A function that is not first class, is called *second class*. As a consequence of Dirac-Bergman procedure, the total Hamiltonian H is a first class function.

In particular, the constraints (A.16) can be divided into first and second class constraints. While distinction of primary and secondary constraints is of minor importance in the final form of the Hamiltonian theory, the classification on first and second class constraints has important dynamical consequences.

First class constraints commute with all other constraints, $\{\phi_{\,\mathrm{I\ class}}, \phi_r\} \approx 0$, thus their consistency conditions give no restrictions on multipliers. Consequently, the final dynamics is not uniquely determined by initial conditions, and unphysical difference is related to the existence of *local symmetries* in a theory.

Second class constraints have $\{\phi_{\,\mathrm{II\ class}}, \phi_r\} \neq 0$, and their consistency conditions solve a number multipliers. The total Hamiltonian with solved multipliers is

$$H' = H_0' + v^a \phi_a \,, \tag{A.19}$$

where ϕ_a are *primary first class* constraints. Both Hamiltonians H' and H_0' are first class functions.

b) Dirac brackets

Consider a set of all second class constraints,

$$\theta_m(z) \approx 0 \,, \qquad (m = 1, \dots, N_2) \,, \tag{A.20}$$

where the corresponding PB matrix, or *Dirac matrix*, is non-degenerate,

$$\{\theta_m, \theta_k\} = \Delta_{mk} \,, \tag{A.21}$$

with an inverse Δ^{mk} $\left(\Delta_{mn}\Delta^{nk} = \delta_m^k\right)$. The rank of Dirac matrix is even, as it is antisymmetric, thus there are always even number of second class constraints.

The *Dirac bracket* (DB) of two phase space functions is defined by

$$\{F, G\}^* = \{F, G\} - \{F, \theta_m\}\, \Delta^{mk} \,\{\theta_k, G\} \ , \tag{A.22}$$

and it has all properties of a PB: antisymmetry, bilinearity and it obeys product rule and Jacobi identity. By construction, DB of second class constraints with any function F vanishes,

$$\{\theta_m, F\}^* = 0 \,. \tag{A.23}$$

Therefore, using DB instead of PB, the second class constraints can be exchanged by strong equalities (set to zero before evaluating DB) on the *reduced phase space* $\tilde{\Gamma} \subset \Gamma$, where $\theta_m = 0$.

The construction of DB has *iterative property*: a subset of second class constraints can be used to define a preliminary DB $\{\ \}^*$. The next set of secondary constraints define a new DB $\{\ \}^{**}$, where a preliminary DB is used instead of a PB in the definition (A.22); and so on, until all second class constraints are exhausted.

Gauge conditions. Unobservable degrees of freedom can be eliminated by imposing *gauge conditions*

$$\Psi_a(z) \approx 0\,. \tag{A.24}$$

The choice of functions Ψ_a has to be such that: (a) gauge conditions are *accessible*, or that the equation $\Psi_a(z) + \delta_\varepsilon \Psi_a(z) = 0$ has a solution in ε, and (b) this solution is *unique*, *i.e.*, there is no a residual gauge symmetry which preserves gauge conditions (A.24). When (a) and (b) are fulfilled, then the number of the gauge conditions is equal to the number of first class constraints

$$\chi_a(z) \approx 0\,, \qquad (a = 1, \ldots, N_1)\,, \tag{A.25}$$

and the matrix

$$\{\chi_a, \Psi_b\} = \sigma_{ab} \tag{A.26}$$

is invertible. The gauge conditions must be preserved in time and their consistency conditions determine all multipliers. Ψ_a are treated as any other constraints in a theory and, together with χ_a, they form a set of second class constraints. One can define DB which treat all constraints and gauge conditions as strong equalities at reduced phase space Γ^*, containing only physical degrees of freedom which number is given by the formula

$$N^* = 2N - (2N_1 + N_2)\ . \tag{A.27}$$

This formula applies only to the regular systems.

c) Local symmetries

Due to the presence of arbitrary multipliers v^a in the Hamiltonian (A.19), the evolution of a variable $F(t)$ cannot be uniquely determined from the given initial values $F(t_0)$. After a short interval $\delta t = t - t_0$, the difference of two F's, for two different choices of multipliers v_1^a and v_2^a, has the form

$$\Delta F\left(\delta t\right) = \varepsilon^a \left\{F, \phi_a\right\} \;, \tag{A.28}$$

where $\varepsilon^a = \left(v_2^a - v_1^a\right) \delta t$. This difference is unphysical and corresponds to a *gauge transformation* generated by *primary first class* constraints ϕ_a. Two successive gauge transformations of type (A.28), with parameters ε_1^a and ε_2^a, gives

$$\left(\Delta_1 \Delta_2 - \Delta_2 \Delta_1\right) F\left(\delta t\right) = \varepsilon_1^a \varepsilon_2^b \left\{F, \left\{\phi_a, \phi_b\right\}\right\} \;, \tag{A.29}$$

thus $\{\phi_a, \phi_b\}$, containing *secondary* first class constraints, also generates gauge transformations. Dirac conjectured that *all* first class constraints are generators of gauge symmetries, what will be discussed below.

Physical *observables* are quantities independent on arbitrary multipliers. These are gauge invariant objects.

Generator of local symmetries. The Hamiltonian formalism provides an algorithm to construct the generators of all gauge symmetries of the equations of motion (A.13). If the gauge transformations have canonical form

$$\delta_\varepsilon z^n = \left\{z^n, G\left[\varepsilon\right]\right\} \;, \tag{A.30}$$

where $\varepsilon(t)$ is an infinitesimal local parameter, and the generator is

$$G\left[\varepsilon\right] = \dot{\varepsilon} G_1 + \varepsilon G_0 \,, \tag{A.31}$$

then necessary and sufficient conditions that Hamilton equations (A.13) are invariant under (A.30) are

$$\begin{aligned} G_1 &= \text{PFC}\,, \\ G_0 + \{G_1, H\} &= \text{PFC}\,, \\ \{G_0, H\} &= \text{PFC}\,, \end{aligned} \tag{A.32}$$

where 'PFC' stands for a primary first class constraint. Thus, G_0 and G_1 are determined only up to primary first class constraints. This is Castellani's method and it can be generalized to the systems containing higher derivatives of a local parameter [101, 102].

Dirac conjecture. Dirac conjectured that all first class constraints are generators of the gauge symmetries [98]. From Castellani's method it can be seen that the generators contain, apart from the primary first class constraints G_1, also secondary constraints appearing in $\{G_1, H\}$. Since any higher power of constraints can be treated as strong equality, Castellani's algorithm stops if one obtains $\{G_1, H\} = \phi^K = 0$ $(K \geq 2)$. Therefore, the Dirac conjecture is replaced by the statement that all first class constraints generate gauge symmetry, apart from those appearing in the consistency conditions as a higher power of a constraint ϕ, and those following from the consistency conditions of ϕ [102].

d) Extended action

The total Hamiltonian (A.11) contains only primary constraints. Because the separation on primary and secondary constraints has no physical implications, it is natural to define a Hamiltonian which contains all constraints in a theory, or *extended Hamiltonian*

$$H_E = H_0 + u^r \phi_r \,. \tag{A.33}$$

This Hamiltonian contains more dynamical variables than H_T, thus the dynamics derived from the extended Hamiltonian is not equivalent to the Lagrangian one, but the difference is unphysical. The introduction of H_E is a new feature of the Hamiltonian scheme, which extends the Lagrangian formalism by making manifest all gauge freedom.

The action with the dynamics equivalent to that obtained from H_E is a canonical *extended action*

$$I_E\left[q, p, u\right] = \int_{t_0}^{t_1} dt \, \left(\dot{q}^i \, p_i - H_0 - u^r \phi_r\right) \,, \tag{A.34}$$

and it can always be reduced to the original action by gauge fixing of all extra multipliers.

In a theory with only first class constraints, the following PB are satisfied

$$\{\phi_r, \phi_s\} = C_{rs}{}^p \phi_p \,, \qquad \{\phi_r, H_0\} = C_r^{\ s} \phi_s \,, \tag{A.35}$$

where $C_{rs}{}^p\left(z\right)$ and $C_r^{\ s}\left(z\right)$ are structure functions. Then the action $I_E\left[q, p, u\right]$ is invariant under the following gauge transformations [99]:

$$\delta_\varepsilon z^n = \{z^n, \varepsilon^r \phi_r\} \, , \tag{A.36}$$

$$\delta_\varepsilon u^r = \dot{\varepsilon}^r + C_{ps}{}^r \, u^s \varepsilon^p + C_s{}^r \, \varepsilon^s \, . \tag{A.37}$$

Appendix B

Superspace notation in $D = 2$

Two-dimensional space-time manifold $\mathcal{M}$ with signature $(-,+)$ is parametrized by the local coordinates $x^\mu = (\tau, \sigma)$, where $\mu = 0, 1$.

In the tangent Minkowski space, the local coordinates x^m $(m = 0, 1)$ are exchanged by the *light-cone* coordinates x^a $(a = +, -)$, defined by $x^\pm \equiv \frac{1}{2}(x^0 \pm x^1)$. In the *light-cone* basis, the Minkowski metric $\eta_{mn} = \text{diag}\,(-1, 1)$, and its inverse η^{mn}, become

$$\eta_{ab} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix} , \qquad \eta^{ab} = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} , \tag{B.1}$$

and therefore the raising and lowering of the tangent space indices is performed as $V_\pm = -2V^\mp$. The Levi-Civita tensor ε^{mn} $(\varepsilon^{01} = 1)$ in the *light-cone* basis takes the form

$$\varepsilon^{ab} = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} , \qquad \varepsilon_{ab} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} . \tag{B.2}$$

Representation of γ-matrices. Dirac matrices, defined in the tangent space, satisfy the *Clifford algebra*

$$\{\gamma^m, \gamma^n\} = 2\eta^{mn} . \tag{B.3}$$

The following representation is used:

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \gamma \equiv \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{B.4}$$

so that there is the identity $\mathrm{Tr}\,(\gamma^m\gamma^n\gamma) = 2\varepsilon^{mn}$. The projective γ-matrices $\gamma^{\pm} = \frac{1}{2}(\gamma^0 \pm \gamma^1)$ are represented as:

$$\gamma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \gamma^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{B.5}$$

Spinors. A Majorana spinor is a Dirac spinor $\theta_\alpha \equiv \begin{pmatrix} \theta_+ \\ \theta_- \end{pmatrix}$ which obeys the Majorana condition $\theta = C\bar{\theta}^T$, with $\bar{\theta} \equiv \theta^\dagger\gamma^0$, and $C_{\alpha\beta}$ is the charge conjugation matrix $(C^{-1}\gamma_\mu C = -\gamma_\mu^T)$. In the representation (B.4) and with $C = \gamma^0$, the Majorana spinors are real, $\theta^*_\alpha = \theta_\alpha$. The inverse tensor $(C^{-1})^{\alpha\beta} \equiv C^{\alpha\beta}$ performs the raising of spinor indices $(\theta^\alpha = C^{\alpha\beta}\theta_\beta)$, while $C_{\alpha\beta}$ performs their lowering $(\theta_\alpha = C_{\alpha\beta}\theta^\beta)$. In components, it gives $\theta_\pm = \pm\theta^\mp$. Spinor contraction is denoted by $\theta\xi \equiv \theta^\alpha\xi_\alpha = -\theta_\alpha\xi^\alpha$.

Super covariant derivative. $(1,1)$ superspace is parametrized by four real coordinates $z^A = (x^a, \theta_\alpha)$, where x^a are the light-cone coordinates and θ_α is a Majorana spinor. The supercovariant derivative is

$$D_\alpha = \bar{\partial}_\alpha + i\,(\gamma^m\theta)_\alpha\,\partial_m\,, \tag{B.6}$$

where $\partial_m \equiv \partial/\partial x^m$ and $\bar{\partial}_\alpha \equiv \partial/\partial\bar{\theta}^\alpha$. More explicitly, the derivative (B.6) in the representation (B.4) is

$$D^\pm \equiv \partial_{\theta_\mp} - i\theta_\mp\partial_\pm\,, \tag{B.7}$$

where $D_\alpha \equiv \begin{pmatrix} -D^+ \\ D^- \end{pmatrix}$.

Super δ-function. A generalization of the δ-function to the super δ-function is

$$\delta_{\pm 12} \equiv \theta_{\mp 12}\,\delta(x_1^\pm - x_2^\pm)\,, \tag{B.8}$$

where $\theta_{12} = \theta_1 - \theta_2$. It has the following properties:

$$\begin{array}{rclcrcl} \int d^4z_1\,\delta_{\pm 12} & = & 1\,, & \qquad & \delta_{\pm 21} & = & -\delta_{\pm 12}\,, \\ F(z_1)\,\delta_{\pm 12} & = & F(z_2)\,\delta_{\pm 12}\,, & & D_1^\pm\delta_{\pm 12} & = & -D_2^\pm\delta_{\pm 12}\,, \end{array} \tag{B.9}$$

where $d^4z \equiv d^2x\,d^2\theta$ and basic integrals for Grassman odd numbers are $\int d\theta = 0$ and $\int d\theta\,\theta = 1$.

Appendix C

Components of the vielbein and metric in the *light-cone* basis

At each point of the curved space-time $\mathcal{M}$, there is a light-cone basis of 1-forms $e^a \equiv e^a{}_\mu \, dx^\mu$. The vielbein $e^a{}_\mu$ $(a = +,-; \quad \mu = 0,1)$ is expressed in terms of variables (h^-, h^+, F, f) as

$$e^\pm{}_\mu = e^{F \pm f} \hat{e}^\pm{}_\mu \,, \qquad \hat{e}^a{}_\mu = \frac{1}{2} \begin{pmatrix} -h^+ & 1 \\ h^- & -1 \end{pmatrix} . \tag{C.1}$$

The inverse vielbein $e^\mu{}_a$ $(e_a{}^\mu e_\mu{}^b = \delta^b_a$ and $e_\mu{}^a e_a{}^\nu = \delta^\nu_\mu)$ is

$$e^\mu{}_\pm = e^{-(F \pm f)} \hat{e}^\mu{}_\pm \,, \qquad \hat{e}^\mu{}_a = \frac{2}{h^- - h^+} \begin{pmatrix} 1 & 1 \\ h^- & h^+ \end{pmatrix} . \tag{C.2}$$

The related basis of tangent vectors $\partial_a \equiv e^\mu{}_a \partial_\mu$ can be written as

$$\partial_\pm = e^{-(F \pm f)} \hat{\partial}_\pm \,, \qquad \hat{\partial}_\pm = \frac{2}{h^- - h^+} \left(\partial_0 + h^\mp \partial_1 \right) . \tag{C.3}$$

It follows from (C.1) that the components of the metric tensor $g_{\mu\nu} = \eta_{ab} \, e^a{}_\mu e^b{}_\nu$ are

$$g_{\mu\nu} = e^{2F} \hat{g}_{\mu\nu} \,, \qquad \hat{g}_{\mu\nu} \equiv -\frac{1}{2} \begin{pmatrix} -2h^+h^- & h^+ + h^- \\ h^+ + h^- & -2 \end{pmatrix} , \tag{C.4}$$

while the inverse metric $g^{\mu\nu}$ has components

$$g^{\mu\nu} = e^{-2F} \hat{g}^{\mu\nu} \,, \qquad \hat{g}^{\mu\nu} \equiv -\frac{2}{(h^- - h^+)^2} \begin{pmatrix} 2 & h^+ + h^- \\ h^+ + h^- & 2h^-h^+ \end{pmatrix} . \tag{C.5}$$

Here $\sqrt{-g} = e^{2F}\sqrt{-\hat{g}} \equiv e^{2F}\frac{h^- - h^+}{2}$.

The Riemannian connection on $\mathcal{M}$ is defined by $T^a = 0$ as

$$\omega_a = \varepsilon^{bc}\, e^{\mu}{}_b\, \partial_c e_{a\mu}\,, \tag{C.6}$$

where ε^{mn} ($\varepsilon^{01} = 1$) is the constant totally antisymmetric tensor in the Minkowski space or, in light-cone basis, $\varepsilon^{-+} = -\varepsilon^{+-} = \frac{1}{2}$. Written in terms of variables (C.2) and (C.3), the connection becomes

$$\omega_\pm = e^{-(F\pm f)}\left[\hat{\omega}_\pm \mp \hat{\partial}_\pm (F \mp f)\right]\,, \qquad \hat{\omega}_\pm \equiv \mp\frac{2\partial_1 h^\mp}{h^- - h^+}\,. \tag{C.7}$$

Covariant derivative in the conformally flat (*light-cone*) coordinates, acting on a Weyl field with the weight n, is

$$\nabla_a = \partial_a + \frac{n}{2}\,\omega_a\,, \tag{C.8}$$

or explicitly

$$\nabla_\pm = e^{(\pm n - 1)F - (n\pm 1)f}\,\hat{\nabla}_\pm\, e^{\mp nF + nf}\,, \tag{C.9}$$

where $\hat{\nabla}_a = \hat{\partial}_a + \frac{n}{2}\,\hat{\omega}_a$.

Appendix D

Symplectic form in Chern-Simons theories

Consider a Chern-Simons (CS) theory in $D = 2n+1 \geq 5$ dimensions described by the action in Hamiltonian form (first order formalism)

$$\begin{aligned} I_{\mathrm{CS}}[A_i^a] &= \int\limits_{\mathbb{R}\times\sigma} L_{\mathrm{CS}}(A) = \int\limits_{t_0}^{t_1} dt \int\limits_{\sigma} d^{2n}x \left(\mathcal{L}_a^i \, \dot{A}_i^a - A_0^a \chi_a \right) , && \text{(D.1)} \\ dL_{\mathrm{CS}} &= k \; g_{a_1\cdots a_{n+1}} \, F^{a_1} \cdots F^{a_{n+1}} , && \text{(D.2)} \end{aligned}$$

where A_i^a define a space of gauge fields $\mathcal{A}$ and A_0^a is Lagrange multiplier. Similarly to the basis of 1-forms dx^μ on the space-time manifold $\mathcal{M}$, one can define a basis of 1-forms $\underline{\delta} A_i^a\,(x)$ on the manifold of gauge fields $\mathcal{A}$. The *symplectic form* $\hat{\Omega}$ is a quadratic 2-form defining the kinetic term in (D.1),

$$\begin{aligned} I_{\mathrm{CS}} &= \int dt \, \hat{\Omega} \, \underline{\delta} \dot{A} \underline{\delta} A \\ &= \int dt \int d^{2n}x \int d^{2n}x' \, \Omega_{ab}^{ij}\,(x,x') \; \underline{\delta} \dot{A}_i^a\,(x) \; \underline{\delta} A_j^b\,(x') \; , && \text{(D.3)} \end{aligned}$$

and it can be expressed as

$$\hat{\Omega}_{ab}^{ij}\,(x,x') = \frac{\delta \mathcal{L}_b^j\,(x')}{\delta A_i^a\,(x)} - \frac{\delta \mathcal{L}_a^i\,(x)}{\delta A_j^b\,(x')} \, . \tag{D.4}$$

The symplectic form can be calculated using

$$\frac{\delta^2 I_{\mathrm{CS}}}{\delta \dot{A}_i^a\,(x) \, \delta A_j^b\,(x')} = \frac{\delta \mathcal{L}_a^i\,(x)}{\delta A_j^b\,(x')} \, , \tag{D.5}$$

and varying dL_{CS} in (D.2), so that one obtains

$$\begin{aligned}
\delta L_{\text{CS}} &= k\,(n+1)\; g_{aa_1\cdots a_n}\, F^{a_1}\cdots F^{a_n}\delta A^a\,, \\
\delta^2 L_{\text{CS}} &= kn\,(n+1)\; g_{aba_1\cdots a_{n-1}}\, F^{a_1}\cdots F^{a_{n-1}} D\delta A^a \delta A^b \\
&= dt\, d^{2n}x\, \frac{kn}{2^{n-1}}\; (n+1)\; \varepsilon^{iji_1j_1\cdots i_{n-1}j_{n-1}}\, g_{aba_1\cdots a_{n-1}}\, F^{a_1}_{i_1j_1}\cdots F^{a_{n-1}}_{i_{n-1}j_{n-1}}\delta \dot{A}^a_i \delta A^b_j \\
&\quad + \quad \dot{A}\text{ - independent part}\,. \qquad \text{(D.6)}
\end{aligned}$$

Therefore, one finds that the symplectic form is diagonal in continual indices (without non-local operators),

$$\hat{\Omega}^{ij}_{ab}\,(x,x') = \Omega^{ij}_{ab}\,(x)\,\delta\,(x-x')\ , \tag{D.7}$$

with the symplectic matrix Ω^{ij}_{ab} given by

$$\Omega^{ij}_{ab} \equiv -\frac{kn}{2^{n-1}}\ (n+1)\ \varepsilon^{iji_2j_2\cdots i_nj_n} g_{aba_2\cdots a_n} F^{a_1}_{i_2j_2}\cdots F^{a_n}_{i_nj_n}\,. \tag{D.8}$$

In Hamiltonian approach, there is always a primary constraint $\phi^i_a \equiv \pi^i_a - \mathcal{L}^i_a \approx 0$ such that, by definition of PB and (D.4), gives

$$\left\{\phi^i_a, \phi^j_b\right\} = \Omega^{ij}_{ab}\delta\,. \tag{D.9}$$

Therefore, the explicit expression for $\mathcal{L}^i_a\,(A)$ is not necessary, since the symplectic matrix determines the dynamics of the theory.

The matrix Ω^{ij}_{ab} is degenerate because it always has at least $2n$ zero modes $\mathbf{V}_i$, solutions of the matrix equation $\Omega^{ik}_{ab}\,(V_j)^b_k = 0$. It follows from the identity

$$\Omega^{ik}_{ab} F^b_{kj} = -\delta^i_j \chi_a \approx 0\,, \tag{D.10}$$

where $(V_j)^b_k = F^b_{kj}$. The above identity can be shown using the fact that the tensor $g_{aa_1\cdots a_n} F^{a_1}_{i_1[j_1}\cdots F^{a_n}_{i_nj_n]}$ (the definition of antisymmetrization includes the factor $\frac{1}{(2n-1)!}$) is *totally antisymmetric* in indices $[i_1j_1\cdots i_nj_n]$, and it is therefore proportional to the Levi-Civita tensor,

$$g_{aa_1\cdots a_n} F^{a_1}_{i_1[j_1}\cdots F^{a_n}_{i_nj_n]} = C_a\,\varepsilon_{i_1j_1\cdots i_nj_n}\ , \tag{D.11}$$

with the factor of proportionality

$$C_a = \frac{1}{(2n)!}\, g_{aa_1\cdots a_n} \varepsilon^{i_1j_1\cdots i_nj_n} F^{a_1}_{i_1[j_1}\cdots F^{a_n}_{i_nj_n]}\,. \tag{D.12}$$

Then the identity

$$g_{aa_1\cdots a_n}\varepsilon^{sj_1i_2j_2\cdots i_nj_n}F^{a_1}_{k[j_1}\cdots F^{a_n}_{i_nj_n]} = (2n-1)!\,C_a\,\delta^s_k \tag{D.13}$$

is equivalent to (D.10).

Appendix E

Anti-de Sitter group, AdS_D

The D-dimensional AdS_D group, $SO(D-1,2)$, is the isometry group of the D-dimensional hyperboloid

$$H_D : -x_0^2 + x_1^2 + \cdots + x_{D-1}^2 - x_D^2 = -\ell^2 \tag{E.1}$$

defined in a $(D+1)$-dimensional space-time with signature

$$\eta_{AB} = (-,+,\ \cdots\ ,+,-)\,, \qquad (A,B = 0,\ldots,D)\ . \tag{E.2}$$

The group has $D\,(D+1)\,/2$ generators represented by $\mathbf{J}_{AB} = -\mathbf{J}_{BA}$, which satisfy the Lie-algebra

$$[\mathbf{J}_{AB}, \mathbf{J}_{CD}] = \eta_{AD}\,\mathbf{J}_{BC} - \eta_{BD}\,\mathbf{J}_{AC} - \eta_{AC}\,\mathbf{J}_{BD} + \eta_{BC}\,\mathbf{J}_{AD}\,. \tag{E.3}$$

The generators $\mathbf{J}_{AB}$ can be decomposed into

$$\mathbf{J}_{AB} : \quad \begin{cases} \mathbf{J}_a \equiv \mathbf{J}_{aD}\,, \\ \mathbf{J}_{ab}\,, \end{cases} \qquad (a,b = 0,\ldots,D-1)\ , \tag{E.4}$$

leading to the AdS_D algebra in the form

$$\begin{aligned} [\mathbf{J}_{ab}, \mathbf{J}_{cd}] &= \eta_{ad}\,\mathbf{J}_{bc} - \eta_{bd}\,\mathbf{J}_{ac} - \eta_{ac}\,\mathbf{J}_{bd} + \eta_{bc}\,\mathbf{J}_{ad}\,, \\ [\mathbf{J}_{ab}, \mathbf{J}_{c}] &= \eta_{bc}\,\mathbf{J}_{a} - \eta_{ac}\,\mathbf{J}_{b}\,, \\ [\mathbf{J}_{a}, \mathbf{J}_{b}] &= \mathbf{J}_{ab}\,, \end{aligned} \tag{E.5}$$

where the metric η_{ab} is $(-,+,\ldots,+)$.

The AdS_D group is related to the Poincaré group via the *Wigner-Inönü contraction* [132], as follows. After defining $\mathbf{P}_a = \frac{1}{\ell}\,\mathbf{J}_a$, the commutator $[\mathbf{P}_a, \mathbf{P}_b] = \frac{1}{\ell^2}\,\mathbf{J}_{ab}$ vanishes in the flat space limit, $\ell \to \infty$. Since $\mathbf{J}_{ab}$ become the generators of Lorentz transformations and $\mathbf{P}_a$ generators of translations in a-th direction, the AdS_D algebra reduces to the D-dimensional Poincaré group $ISO(D-1,1)$.

This motivates to construct a connection associated to AdS_D whose components are the *vielbein* e^a, and the *spin-connection* ω^{ab},

$$\mathbf{A} \equiv \frac{1}{2}\,W^{AB}\mathbf{J}_{AB} = \frac{1}{\ell}\,e^a\mathbf{J}_a + \frac{1}{2}\,\omega^{ab}\mathbf{J}_{ab}\,. \tag{E.6}$$

The corresponding field-strength, $\mathbf{F} = d\mathbf{A} + \mathbf{A}^2$, has the form

$$\begin{aligned} \mathbf{F} &= \tfrac{1}{\ell}\,T^a\mathbf{J}_a + \tfrac{1}{2}\,F^{ab}\mathbf{J}_{ab}\,, \\ F^{ab} &\equiv R^{ab} + \tfrac{1}{\ell^2}\,e^a e^b\,, \end{aligned} \tag{E.7}$$

where the *torsion* (T^a) and *Ricci curvature* (R^{ab}) are

$$\begin{aligned} R^{ab} &= d\omega^{ab} + \omega^a{}_c\omega^{cb}\,, \\ T^a &= de^a + \omega^a{}_b e^b. \end{aligned} \tag{E.8}$$

AdS space-time, which has a constant curvature $R^{ab} = -\frac{1}{\ell^2}\,e^a e^b$, can be interpreted as a *pure gauge* solution $F^{ab} = 0$, $T^a = 0$.

Appendix F

Supersymmetric extension of AdS_5, $SU(2,2\,|N)$

a) Generators

The supersymmetric extension of the AdS group in five dimensions is the super unitary group $SU(2,2\,|N\,)$ [58, 133, 134], containing supermatrices of unit superdeterminant which leave invariant the (real) quadratic form

$$q = \theta^{*\alpha} G_{\alpha\beta} \theta^{\beta} + z^{*r} g_{rs} z^{s}\,, \qquad (\alpha = 1, \dots, 4;\;\; r = 1, \dots, N)\,. \tag{F.1}$$

Here θ^{α} are complex Grassman numbers (with complex conjugation defined as $\left(\theta^{\alpha}\theta^{\beta}\right)^{*} = \theta^{*\beta}\,\theta^{*\alpha}$), and $G_{\alpha\beta}$ and g_{rs} are Hermitean matrices, antisymmetric and symmetric respectively, which can be chosen as

$$G_{\alpha\beta} = i\left(\Gamma_0\right)_{\alpha\beta}\,, \qquad\qquad g_{rs} = \delta_{rs}\,. \tag{F.2}$$

The bosonic sector of this supergroup is

$$SU(2,2) \otimes SU(N) \otimes U(1) \subset SU(2,2\,|N\,)\,, \tag{F.3}$$

where the AdS group is present on the basis of the isomorphism $SU(2,2) \simeq SO(2,4)$. Therefore, the generators of $su(2,2\,|N\,)$ algebra are

$$\begin{array}{lll} so(2,4): & \mathbf{J}_{AB} = (\mathbf{J}_{ab,}\mathbf{J}_{a}) \ , & (A,B = 0,\ldots,5) \ , \\ su(N): & \mathbf{T}_{\Lambda} \, , & (\Lambda = 1,\ldots,N^2-1) \ , \\ \text{SUSY}: & \mathbf{Q}^{\alpha}_{r} \, , \ \bar{\mathbf{Q}}^{r}_{\alpha} \, , & (\alpha = 1,\ldots,4; \ r = 1,\ldots,N) \ , \\ u(1): & \mathbf{G}_{1} \, , & \end{array} \tag{F.4}$$

where $\eta_{AB} = \text{diag}\,(-,+,+,+,+,-)$, and AdS rotations and translations are $\mathbf{J}_{ab}$ and $\mathbf{J}_{a} \equiv \mathbf{J}_{a5}$ $(a,b = 0,\ldots,4)$.

b) Representation of generators

A representation of the superalgebra acting in $(4+N)$-dimensional superspace (θ^{α}, y^{r}) is given by the $(4+N)\times(4+N)$ supermatrices as follows.

- ***AdS* generators**

$$\mathbf{J}_{AB} = \begin{pmatrix} \frac{1}{2}\,(\Gamma_{AB})_{\alpha}^{\ \beta} & 0 \\ 0 & 0 \end{pmatrix} , \tag{F.5}$$

with the 4×4 matrices Γ_{AB} defined by

$$\Gamma_{AB}: \qquad \begin{cases} \Gamma_{ab} = \frac{1}{2}\,[\Gamma_{a},\Gamma_{b}] \ , \\ \Gamma_{a5} = \Gamma_{a} \, , \end{cases} \tag{F.6}$$

where Γ_{a} are the Dirac matrices in five dimensions with the signature $(-,+,+,+,+)$;

- $su(N)$ **generators**

$$\mathbf{T}_{\Lambda} = \begin{pmatrix} 0 & 0 \\ 0 & (\tau_{\Lambda})_{r}^{\ s} \end{pmatrix} ,$$

where τ_{Λ} are anti-Hermitean generators of $su(N)$ acting in N-dimensional space y^{r},

$$[\tau_{\Lambda_1},\tau_{\Lambda_2}] = f_{\Lambda_1\Lambda_2}^{\quad\ \Lambda_3}\,\tau_{\Lambda_3} \, ; \tag{F.7}$$

- **Supersymmetry generators**

$$\mathbf{Q}_q^\gamma = \begin{pmatrix} 0 & 0 \\ -\delta_q^s \delta_\alpha^\gamma & 0 \end{pmatrix}, \qquad \bar{\mathbf{Q}}_\gamma^q = \begin{pmatrix} 0 & \delta_r^q \delta_\gamma^\beta \\ 0 & 0 \end{pmatrix}; \tag{F.8}$$

- $u(1)$ **generator**

$$\mathbf{G}_1 = \begin{pmatrix} \frac{i}{4}\,\delta_\alpha^\beta & 0 \\ 0 & \frac{i}{N}\,\delta_r^s \end{pmatrix}. \tag{F.9}$$

c) The algebra

From the given representation of the supermatrices, it is straightforward to find the explicit form of the corresponding Lie algebra. The commutators of the bosonic generators $\mathbf{J}_{AB}$, $\mathbf{T}_\Lambda$ and $\mathbf{G}_1$ close the algebra $su(2,2) \otimes su(N) \otimes u(1)$,

$$\begin{aligned} [\mathbf{J}_{AB}, \mathbf{J}_{CD}] &= \eta_{AD}\,\mathbf{J}_{BC} - \eta_{BD}\,\mathbf{J}_{AC} - \eta_{AC}\,\mathbf{J}_{BD} + \eta_{BC}\,\mathbf{J}_{AD}, \\ [\mathbf{T}_{\Lambda_1}, \mathbf{T}_{\Lambda_2}] &= f_{\Lambda_1\Lambda_2}{}^{\Lambda_3}\,\mathbf{T}_{\Lambda_3}. \end{aligned} \tag{F.10}$$

The supersymmetry generators transform as spinors under AdS and as vectors under $su(N)$,

$$\begin{aligned} [\mathbf{J}_{AB}, \mathbf{Q}_r^\alpha] &= -\tfrac{1}{2}\,(\Gamma_{AB})_\beta^\alpha\,\mathbf{Q}_r^\beta, & [\mathbf{T}_\Lambda, \mathbf{Q}_r^\alpha] &= (\tau_\Lambda)_r^s\,\mathbf{Q}_s^\alpha, \\ \left[\mathbf{J}_{AB}, \bar{\mathbf{Q}}_\alpha^r\right] &= \tfrac{1}{2}\,\bar{\mathbf{Q}}_\beta^r\,(\Gamma_{AB})_\alpha^\beta, & \left[\mathbf{T}_\Lambda, \bar{\mathbf{Q}}_\alpha^r\right] &= -\bar{\mathbf{Q}}_\alpha^s\,(\tau_\Lambda)_s^r, \end{aligned} \tag{F.11}$$

and they carry $u(1)$ charge:

$$\begin{aligned} [\mathbf{G}_1, \mathbf{Q}_r^\alpha] &= -i\left(\frac{1}{4} - \frac{1}{N}\right)\mathbf{Q}_r^\alpha, \\ \left[\mathbf{G}_1, \bar{\mathbf{Q}}_\alpha^r\right] &= i\left(\frac{1}{4} - \frac{1}{N}\right)\bar{\mathbf{Q}}_\alpha^r. \end{aligned} \tag{F.12}$$

The anticommutator of the supersymmetry generators has the following form:

$$\left\{\mathbf{Q}_r^\alpha, \bar{\mathbf{Q}}_\beta^s\right\} = \frac{1}{4}\,\delta_r^s\,\left(\Gamma^{AB}\right)_\beta^\alpha\,\mathbf{J}_{AB} - \delta_\beta^\alpha\,\left(\tau^\Lambda\right)_r^s\,\mathbf{T}_\Lambda + i\,\delta_\beta^\alpha\,\delta_r^s\,\mathbf{G}_1, \tag{F.13}$$

what can be shown using the orthogonality relations for Γ- and τ-matrices,

$$\begin{aligned} \tfrac{1}{2}\left(\Gamma^{AB}\right)_\beta^\alpha\,(\Gamma_{AB})_\lambda^\rho &= \delta_\beta^\alpha \delta_\lambda^\rho - 4\delta_\lambda^\alpha \delta_\beta^\rho, \\ \left(\tau^\Lambda\right)_s^r\,(\tau_\Lambda)_q^p &= \delta_q^r \delta_s^p - \tfrac{1}{N}\,\delta_s^r \delta_q^p. \end{aligned} \tag{F.14}$$

d) Killing metric

Denote all generators as $\mathbf{G}_M = (\mathbf{G}_{M'}, \mathbf{G}_1)$, where $\mathbf{G}_{M'}$ are the generators of $PSU(2,2\,|N)$ (closing the algebra without $U(1)$ generator). The components of the Killing metric of $SU(2,2\,|N)$ is an invariant tensor of rank two which is symmetric in the bosonic and antisymmetric in the fermionic indices, and has the form

$$g_{MN} = \langle \mathbf{G}_M \mathbf{G}_N \rangle = -\begin{pmatrix} \gamma_{M'N'} & 0 \\ 0 & 0 \end{pmatrix} , \tag{F.15}$$

where $\gamma_{M'N'}$ is the Killing metric of $PSU(2,2\,|N)$, and $\langle \cdots \rangle$ stands for the supertrace $\mathrm{Str}\,(\cdots)$, which is the difference between the trace of the upper and lower diagonal blocks. The components of the invertible Killing metric $\gamma_{M'N'}$ are

$$\begin{aligned} \gamma_{[AB][CD]} &= \eta_{[AB][CD]} \,, \\ \gamma_{\Lambda_1\Lambda_2} &= \mathrm{Tr}_N \left(\tau_{\Lambda_1}\tau_{\Lambda_2}\right) \,, \\ \gamma_{\binom{\alpha}{r}\binom{s}{\beta}} &= -\delta^{\alpha}_{\beta}\delta^{s}_{r} \,, \end{aligned} \tag{F.16}$$

and it raises and lowers $PSU(2,2\,|N)$ indices. Note that the metric g_{MN} is not invertible (the corresponding supergroup is not semi-simple). Here $\eta_{[AB][CD]} \equiv \eta_{AD}\,\eta_{BC} - \eta_{AC}\,\eta_{BD}$.

e) Symmetric invariant tensor

The invariant tensor of rank three, *completely* symmetric in bosonic and antisymmetric in fermionic indices, can be calculated from

$$g_{MNK} = \langle \mathbf{G}_M \mathbf{G}_N \mathbf{G}_K \rangle = \frac{1}{2}\,\mathrm{Str}\,\left[\left(\mathbf{G}_M \mathbf{G}_N + (-)^{\varepsilon_M \varepsilon_N}\, \mathbf{G}_N \mathbf{G}_M \right) \mathbf{G}_K \right] . \tag{F.17}$$

Note that, due to the cyclic property of the supertrace, (anti)symmetrization in first two indices in (F.17) leads to the completely (anti)symmetric tensor g_{MNK}. It has the following non-vanishing components:

$$\begin{aligned}
g_{[AB][CD][EF]} &= -\tfrac{i}{2}\,\varepsilon_{ABCDEF}\,,\\
g_{\Lambda_1\Lambda_2\Lambda_3} &= -\gamma_{\Lambda_1\Lambda_2\Lambda_3}\,,\\
g_{[AB]\binom{\alpha}{r}\binom{s}{\beta}} &= \tfrac{1}{4}\,(\Gamma_{AB})_\beta^\alpha\,\delta_r^s\,,\\
g_{\Lambda\binom{\alpha}{r}\binom{s}{\beta}} &= \tfrac{1}{2}\,\delta_\beta^\alpha\,(\tau_\Lambda)_r^s\,,\\
g_{1[AB][CD]} &= -\tfrac{i}{4}\,\eta_{[AB][CD]}\,,\\
g_{1\Lambda_1\Lambda_2} &= -\tfrac{i}{N}\,\gamma_{\Lambda_1\Lambda_2}\,,\\
g_{1\binom{\alpha}{r}\binom{s}{\beta}} &= \tfrac{i}{2}\,\left(\tfrac{1}{4}+\tfrac{1}{N}\right)\delta_\beta^\alpha\delta_r^s\,,\\
g_{111} &= -i\,\left(\tfrac{1}{4^2}-\tfrac{1}{N^2}\right)\,,
\end{aligned} \tag{F.18}$$

where $\gamma_{\Lambda_1\Lambda_2\Lambda_3} \equiv \frac{1}{2}\,\mathrm{Tr}_N\left(\{\tau_{\Lambda_1},\tau_{\Lambda_2}\}\,\tau_{\Lambda_3}\right)$ is the symmetric invariant tensor of rank three for $su(N)$ and Γ-matrices are normalized so that

$$\mathrm{Tr}_4\left(\Gamma_a\,\Gamma_b\,\Gamma_c\,\Gamma_d\,\Gamma_e\right) = -4i\,\varepsilon_{abcde}\,, \qquad \left(\varepsilon^{abcde5} \equiv \varepsilon^{abcde}, \quad \varepsilon^{012345} = 1\right)\,. \tag{F.19}$$

In the special case $N=4$, the invariant tensor g_{MNK} of $SU(2,2\,|4)$ simplifies to:

$$\begin{aligned}
g_{[AB][CD][EF]} &= -\tfrac{i}{2}\,\varepsilon_{ABCDEF}\,,\\
g_{\Lambda_1\Lambda_2\Lambda_3} &= -\gamma_{\Lambda_1\Lambda_2\Lambda_3}\,,\\
g_{[AB]\binom{\alpha}{r}\binom{s}{\beta}} &= \tfrac{1}{4}\,(\Gamma_{AB})_\beta^\alpha\,\delta_r^s\,,\\
g_{\Lambda\binom{\alpha}{r}\binom{s}{\beta}} &= \tfrac{1}{2}\,\delta_\beta^\alpha\,(\tau_\Lambda)_r^s\,,\\
g_{1M'N'} &= -\tfrac{i}{4}\,\gamma_{M'N'}\,,
\end{aligned} \tag{F.20}$$

with $g_{111}=0$ and the $PSU(2,2\,|4)$ Killing metric $\gamma_{M'N'}$ given by (F.16).

Appendix G

Supergroup conventions

Let $\mathbf{G}_M$ are supermatrices representing the generators of a Lie supergroup. They satisfy the superalgebra

$$[\mathbf{G}_M, \mathbf{G}_N] = f_{MN}{}^K \mathbf{G}_K \,, \tag{G.1}$$

where the commutators defined by $[\mathbf{G}_M, \mathbf{G}_N] \equiv \mathbf{G}_M \mathbf{G}_N - (-)^{\varepsilon_M \varepsilon_N} \mathbf{G}_N \mathbf{G}_M$ and the numbers $\varepsilon_M \equiv \varepsilon(\mathbf{G}_M)$ are 0 for bosonic and 1 for fermionic generators (modulo 2). Summation convention does not apply to $(-)^\varepsilon$ factors. The generators satisfy the generalized Jacobi identity

$$(-)^{\varepsilon_M \varepsilon_K} [[\mathbf{G}_M, \mathbf{G}_N], \mathbf{G}_K] + (-)^{\varepsilon_M \varepsilon_N} [[\mathbf{G}_N, \mathbf{G}_K], \mathbf{G}_M] + (-)^{\varepsilon_K \varepsilon_N} [[\mathbf{G}_K, \mathbf{G}_M], \mathbf{G}_N] = 0 \,, \tag{G.2}$$

what in terms of the structure constants stands for

$$(-)^{\varepsilon_M \varepsilon_K} f_{MN}{}^S f_{SK}{}^L + (-)^{\varepsilon_M \varepsilon_N} f_{NK}{}^S f_{SM}{}^L + (-)^{\varepsilon_K \varepsilon_N} f_{KM}{}^S f_{SN}{}^L = 0 \,. \tag{G.3}$$

The associated connection 1-form is $\mathbf{A} = A^M \mathbf{G}_M$, where the components A^M are Grassmann even fields (bosons) if $\varepsilon_M = 0$ and Grassmann odd fields (fermions) if $\varepsilon_M = 1$. Then one says that the corresponding generators are bosonic and fermionic as well. Covariant derivatives act on a Lie-valued form α as $D\alpha = d\alpha + [\mathbf{A}, \alpha]$, where the commutator of a p-form α and a q-form β is generalized to $[\alpha, \beta] \equiv \alpha\beta - (-)^{pq} (-)^{\varepsilon_\alpha \varepsilon_\beta} \beta\alpha$.

Denote the invariant multilinear form (supertrace) by $\langle \cdots \rangle$, which is antisymmetric for fermionic generators. Then the Killing metric, g_{MN}, and the invariant tensor of rank three, g_{MNK}, are

$$\begin{aligned} g_{MN} &= \langle \mathbf{G}_M \mathbf{G}_N \rangle = \mathrm{Str}\left(\mathbf{G}_M \mathbf{G}_N\right) , \\ g_{MNK} &= \langle \mathbf{G}_M \mathbf{G}_N \mathbf{G}_K \rangle = \tfrac{1}{2}\,\mathrm{Str}\left(\left[\mathbf{G}_M \mathbf{G}_N + (-)^{\varepsilon_M \varepsilon_N}\, \mathbf{G}_N \mathbf{G}_M\right] \mathbf{G}_K\right) . \end{aligned} \tag{G.4}$$

The invariant tensors ${f_{MN}}^K$, g_{MN} and g_{MNK} are Grassmann even variables, $\varepsilon\left(g_{MN}\right) = \varepsilon_M + \varepsilon_N = 0$, *etc.*, and they satisfy the identities

$$\begin{gathered} {f_{MK}}^S g_{SN} - g_{MS}\, {f_{KN}}^S = 0\,, \\ g_{MNS}\, {f_{LK}}^S - g_{MSK}\, {f_{NL}}^S - (-)^{\varepsilon_L \varepsilon_N}\, g_{SNK}\, {f_{ML}}^S = 0\,. \end{gathered} \tag{G.5}$$

The above identities follow from the definition of commutator and the cyclic property of supertrace.

A symmetric tensor $T_{M_1 \cdots M_n}$ of rank n is defined by an element of Lie algebra $\mathbf{T}$, as

$$T_{M_1 \cdots M_n} = \langle \mathbf{G}_{M_1} \cdots \mathbf{G}_{M_n} \mathbf{T} \rangle\ . \tag{G.6}$$

The identities (G.5) are equivalent to $D g_{MN} = 0$ and $D g_{MNK} = 0$.

Hamiltonian formalism. Hamiltonian formalism can be generalized to the systems containing fermions [99], [124]–[127]. If z^A are local coordinates on the phase space, then the Poisson bracket (PB) of functions $F\left(z\right)$ and $G\left(z\right)$ is

$$\left\{F, G\right\} = \frac{\partial^R F}{\partial z^A}\, \omega^{AB}\, \frac{\partial^L G}{\partial z^B}\,, \tag{G.7}$$

where $\partial^R / \partial z^A$ and $\partial^L / \partial z^A$ stand for right and left derivatives, respectively. The basic PB are $\left\{z^A, z^B\right\} = \omega^{AB}$. The convention that all derivatives are left is adopted ($\partial / \partial z \equiv \partial^L / \partial z$):

$$\delta F = \delta z^A\, \frac{\partial F}{\partial z^A}\,. \tag{G.8}$$

The PBs (G.7) are antisymmetric for bosons and symmetric for fermions, and they satisfy the generalized Jacobi identity (G.2).

Particularly, for canonical variables $z^A = \left(A_\mu^M(x), \pi_M^\mu(x)\right)$, the PB become

$$\left\{F(x), G\left(x'\right)\right\} = (-)^{\varepsilon_F \varepsilon_M} \int d^4 y\ \left[\frac{\partial F(x)}{\partial A_\mu^M\left(y\right)} \frac{\partial G(x')}{\partial \pi_M^\mu\left(y\right)} - (-)^{\varepsilon_M}\ \frac{\partial F(x)}{\partial \pi_M^\mu\left(y\right)} \frac{\partial G(x')}{\partial A_\mu^M\left(y\right)}\right]\ , \tag{G.9}$$

with the basic PBs

$$\left\{\pi^{\mu}_{M}(x), A^{N}_{\nu}\left(x'\right)\right\} = -\delta^{\mu}_{\nu}\delta^{N}_{M}\,\delta^{(4)}\left(x-x'\right) = -\left(-\right)^{\varepsilon_M}\left\{A^{M}_{\mu}(x), \pi^{\nu}_{N}\left(x'\right)\right\}\,. \tag{G.10}$$

The canonical Hamiltonian has the form

$$H = \int dx\,\left(\pi^{\mu}_{M}\dot{A}^{M}_{\mu} - \mathcal{L}\right)\,, \tag{G.11}$$

and the corresponding Hamilton equations are

$$\begin{aligned}
\dot{A}^{M}_{\mu} &= (-)^{\varepsilon_M}\frac{\delta H}{\delta\pi^{\mu}_{M}} \approx \left\{A^{M}_{\mu}, H\right\}\,,\\
\dot{\pi}^{\mu}_{M} &= -\frac{\delta H}{\delta A^{M}_{\mu}} \approx \left\{\pi^{\mu}_{M}, H\right\}\,.
\end{aligned} \tag{G.12}$$

Using definitions (G.9) and (G.11), it is straightforward to find the generalization of generators of local symmetries, as well as to introduce Dirac brackets defining a reduced phase space.

Appendix H

Killing spinors for the AdS_5 space-time

The AdS space-time can be given by the metric

$$ds^2_{\rm AdS} = \ell^2\,\left(dr^2 + e^{2r}\eta_{\bar n\bar m}\,dx^{\bar n}dx^{\bar m}\right)\,, \qquad \eta_{\bar n\bar m} = (-,+,+,+,+)\,, \tag{H.1}$$

with the local coordinates $x^\mu = (t, r, x^n)$, where $x^1 = r$ and $x^{\bar n} = (t, x^n)$. The corresponding vielbein (e^a) and the spin-connection (ω^{ab}) are given by

$$\begin{aligned} e^1 &= \ell dr\,, & \omega^{\bar n 1} &= \tfrac{1}{\ell}\,e^{\bar n} = e^r dx^{\bar n}\,, \\ e^{\bar n} &= \ell e^r dx^{\bar n}\,, & \omega^{\bar n\bar m} &= 0\,, \end{aligned} \tag{H.2}$$

so that the AdS connection takes the form

$$\mathbf{W} = \frac{1}{2\ell}\,e^a\Gamma_a + \frac{1}{4}\,\omega^{ab}\Gamma_{ab} = \frac{1}{2\ell}\,\left[e^1\Gamma_1 + e^{\bar n}\Gamma_{\bar n}\,(1+\Gamma_1)\right]\,. \tag{H.3}$$

The Killing spinors ε^α are solutions of the Killing equation

$$D\varepsilon = (d + \mathbf{W})\,\varepsilon = 0\,, \tag{H.4}$$

which for (H.3) splits to the system of the following partial differential equations

$$\left(\partial_r + \frac{1}{2}\,\Gamma_1\right)\varepsilon = 0\,, \tag{H.5}$$

$$\left[\partial_{\bar n} + e^r\,\Gamma_{\bar n}\,(1+\Gamma_1)\right]\varepsilon = 0\,. \tag{H.6}$$

In the ansatz[1]

$$\varepsilon = h^{-1}(r)\,\Theta^{-1}(x^{\bar{n}})\,\varepsilon_0\,, \tag{H.7}$$

where ε_0^α is a constant spinor and $h, \Theta \in SO(2,4)$, the equation (H.5) has the solution

$$h^{-1}(r) = e^{-\frac{r}{2}\Gamma_1}. \tag{H.8}$$

Then the equations (H.6) reduce to

$$\begin{aligned} &\left[\partial_{\bar{n}}\Theta^{-1} + C_{\bar{n}}\,\Theta^{-1}\right]\epsilon_0 = 0\,,\\ &C_{\bar{n}} \equiv e^r e^{\frac{r}{2}\Gamma_1}\Gamma_{\bar{n}}\left(1+\Gamma_1\right)\,e^{-\frac{r}{2}\Gamma_1}\,. \end{aligned} \tag{H.9}$$

Using the identity

$$e^{\frac{r}{2}(1+\Gamma_1)} = 1 + \frac{1}{2}\,(1+\Gamma_1)\,(e^r - 1)\;, \tag{H.10}$$

the coefficients $C_{\bar{n}}$ become

$$C_{\bar{n}} = \Gamma_{\bar{n}}\,(1+\Gamma_1)\;. \tag{H.11}$$

Therefore, the general solution of the equation (H.9) is

$$\Theta^{-1}\left(x^{\bar{n}}\right) = e^{-x^{\bar{n}}\Gamma_{\bar{n}}(1+\Gamma_1)} = 1 - x^{\bar{n}}\Gamma_{\bar{n}}\,(1+\Gamma_1)\;, \tag{H.12}$$

where it was used that the matrices $\Gamma_{\bar{n}}\,(1+\Gamma_1)$ are nilpotent. Therefore, from (H.7), (H.8) and (H.12), the *AdS* Killing spinor has the form [135]

$$\varepsilon = e^{-\frac{r}{2}\Gamma_1}\,\left[1 - x^{\bar{n}}\Gamma_{\bar{n}}\,(1+\Gamma_1)\right]\,\varepsilon_0\,. \tag{H.13}$$

The norm of ε is defined by $\|\varepsilon\|^2 \equiv \bar{\varepsilon}\varepsilon$, where $\bar{\varepsilon} = \varepsilon^\dagger\Gamma_0$. Using $\Gamma_{\bar{n}}^\dagger = \Gamma_0\Gamma_{\bar{n}}\Gamma_0$ and $\Gamma_1^\dagger = \Gamma_1^\dagger$, as well as the Clifford algebra of Γ-matrices, one obtains

$$\|\varepsilon\|^2 = \bar{\varepsilon}_0\,\left[1 + x^{\bar{n}}\,(1-\Gamma_1)\,\Gamma_{\bar{n}}\right]\left[1 - x^{\bar{n}}\Gamma_{\bar{n}}\,(1+\Gamma_1)\right]\varepsilon_0 = \|\varepsilon_0\|^2\,. \tag{H.14}$$

Therefore, the spinor ε given by (H.13) has a constant positive norm, $\|\varepsilon\| = \|\varepsilon_0\| > 0$.

[1] For $\mathbf{F} = 0$, the gauge field is a *pure gauge*, $\mathbf{A} = g^{-1}dg$, and the solution of the Killing equation $D\epsilon = (d+\mathbf{A})\,\epsilon = 0$ is $\epsilon = g^{-1}\epsilon_0$.

Bibliography

[1] J. Wess and B. Zumino, *Phys.Lett.* **B37** (1971) 95.

[2] S. P. Novikov, *Usp. Math. Nauk.* **37** (1982) 3.

[3] E. Witten, *Commun. Math. Phys.* **92** (1984) 455.

[4] J. Erlich and D. Z. Freedman, *Phys. Rev.* **D55** (1977) 6522.

[5] B. Zumino, *Nucl. Phys.* **B239** (1984) 477.

[6] J. Manes, R. Stora and B. Zumino, *Commun. Math. Phys.* **102** (1985) 157.

[7] A. Bogojević, O. Mišković and B. Sazdović, *Abelian Bosonization, the Wess-Zumino Functional and Conformal Symmetry.* E-Print Archive: `hep-th/9811001`.

[8] A. Bogojević, O. Mišković and B. Sazdović, *The Role of Conformal Symmetry in Abelian Bosonization of the Massive Thirring Model*, Talk given at the 11th Yugoslav Simposium on Nuclear and particle Physics, Studenica, Yugoslavia (1988). E-Print Archive: `hep-th/9810120`.

[9] E. Witten, *Nucl. Phys.* **B311** (1988) 46.

[10] E. Witten, *Commun. Math. Phys.* **121** (1989) 351.

[11] S. C. Zhang, T. H. Hansson and S. Kivelson, *Phys. Rev. Lett.* **62** (1989) 82.

[12] E. Witten, *Nucl. Phys.* **B223** (1983) 422.

[13] K. Gawedzki and A. Kupiainen, *Phys. Lett.* **B215** (1988) 119.

[14] K. Gawedzki and A. Kupiainen, *Nucl. Phys.* **B320** (1989) 625.

[15] D. Karabali and H. J. Schnitzer, *Nucl. Phys.* **B329** (1990) 649.

[16] G. Moore and N. Seiberg, *Phys. Lett.* **B220** (1989) 422.

[17] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Nucl. Phys.* **B326** (1989) 108.

[18] T. Regge and C. Teitelboim, *Ann. Phys. (N.Y.)* **88** (1974) 286.

[19] L. F. Abbott and S. Deser, *Nucl. Phys.* **B195** (1982) 76.

[20] L. F. Abbott and S. Deser, *Phys. Lett.* **B116** (1982) 259.

[21] A. M. Polyakov, *Mod. Phys. Lett.* **A2** (1987) 893.

[22] V. G. Knizhnik, A. M. Polyakov and A. B. Zamolodchikov, *Mod. Phys. Lett.* **A3** (1988) 819.

[23] E. Abdalla, M. C. B. Abdalla, J. Gamboa and A. Zadra, *Phys. Lett.* **B273** (1991) 222.

[24] P. Forgacs, A. Wipf, J. Balog, L. Feher and L. O'Raifeartaigh, *Phys. Lett.* **B227** (1989) 214.

[25] O. Coussaert, M. Henneaux and P. van Driel, *Class. Quant. Grav.* **12** (1995) 2961.

[26] A. Achúcarro and P. K. Townsend, *Phys. Lett.* **B180** (1986) 89.

[27] E. Witten, *Nucl. Phys.* **B311** (1988) 46.

[28] S. Carlip, *Quantum Gravity in 2+1 Dimensions* (Cambridge University Press, 1998).

[29] J. D. Brown and M. Henneaux, *Commun. Math. Phys.* **104** (1986) 207.

[30] M. T. Grisaru, *Phys. Lett.* **B66** (1977) 75.

[31] S. Deser and J. H. Kay, *Phys.Lett.* **B76** (1978) 189.

[32] M. F. Sohnius, *Phys. Rep.* **C128** (1985) 39.

[33] C. M. Hull and B. Spence, *Nucl. Phys.* **B345** (1990) 493.

[34] M. Bañados, K. Bautier, O. Coussaert, M. Henneaux and M. Ortiz, *Phys. Rev. Lett.* **D58**: 085020 (1998).

[35] M. Henneaux, L. Maoz and A. Schwimmer, *Ann. Phys.* **282** (2000) 31.

[36] E. B. Bogomol'nyi, *Sov. J. Nucl. Phys.* **24** (1976) 449.

[37] E. Witten, *Commun. Math. Phys.* **80** (1981) 381.

[38] S. Deser and C. Teitelboim, *Phys. Rev. Lett.* **39** (1977) 249.

[39] M. Grisaru, *Phys. Lett.* **B73** (1978) 207.

[40] D. Olive and E. Witten, *Phys. Lett* **B78** (1978) 97.

[41] E. Witten, *Nucl. Phys.* **B186** (1981) 412.

[42] K. Viswanathan, *Dimensional Reduction and Harmonic Expansions on Coset Spaces*, Proceeding of the Workshop on Kaluza-Klein Theories, Chalk River, Canada, ed. H. C. Lee, World Scientific, Singapore (1983).

[43] R. Floreanini and R. Percacci, *Phys. Lett.* **B224** (1989) 291.

[44] R. Floreanini, R. Percacci and R. Rajaraman, *Phys. Lett.* **B231** (1989) 119.

[45] A. Chamseddine, *Phys. Lett.* **B233** (1989) 291.

[46] V. P. Nair and J. Schiff, *Phys. Lett.* **B246** (1990) 423.

[47] V. P. Nair and J. Schiff, *Nucl. Phys.* **B371** (1992) 329.

[48] A. Chamseddine, *Nucl. Phys.* **B346** (1990) 213.

[49] M. Bañados, R. Troncoso and J. Zanelli, *Phys. Rev.* **D54** (1996) 2605.

[50] R. Troncoso and J. Zanelli, *Phys. Rev.* **D58:** R101703 (1998).

[51] R. Troncoso and J. Zanelli, *Int. J. Theor. Phys.* **38** (1999) 1181.

[52] M. Hassaine, R. Olea and R. Troncoso, *New $N = 2$ Supergravity With Local Poincaré Invariance in Nine Dimensions*, Report-No: CECS-PHY-02-06,

E-Print Archive: `hep-th/0210116`.

[53] R. Troncoso and J. Zanelli, *Class. Quant. Grav.* **17** (2000) 4451.

[54] M. Bañados, L. J. Garay and M. Henneaux, *Phys. Rev.* **D53** (1996) 593.

[55] M. Bañados, L. J. Garay and M. Henneaux, *Nucl. Phys.* **B476** (1996) 611.

[56] R. Troncoso and J. Zanelli, *Chern-Simons Supergravities with Off-Shell Local Superalgebras*, in "Black Holes and Structre of the Universe", editors C. Teitelboim and J. Zanelli (World Scientific, Singapore, 1999). E-Print Archive: `hep-th/9902003`.

[57] A. Losev, G. Moore, N. Nekrasov and S. Shatashvili, *Nucl. Phys. Proc. Suppl.* **46** (1996) 130.

[58] O. Chandía, R. Troncoso and J. Zanelli, *Dynamical Content of Chern-Simons Supergravity*, Talk given at The Second La Plata Meeting on Trends in Theoretical Physics, Buenos Aires, Argentina, November 1998. E-Print Archive: `hep-th/9903204`.

[59] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).

[60] C. Batlle, J. Gomis, J. M. Pons and N. Roman-Roy, *J. Math. Phys.* **27** (1986) 2953.

[61] J. A. García and J. M. Pons, *Int. J. Mod. Phys.* **A13** (1998) 3691.

[62] R. Di Stefano, *Phys. Rev.* **D27** (1983) 1752.

[63] J. M. Pons, D. C. Salisbury and L. C. Shepley, *Phys. Rev.* **D62** (2000) 064024.

[64] O. Mišković and B. Sazdović, *Mod. Phys. Lett.* **A17** (2002) 1923.

[65] O. Mišković and J. Zanelli, *Irregular Hamiltonian Systems*, Proceedings of the XIII Chilean Symposium of Physics, Concepción, Chile, November 2002. E-Print Archive: `hep-th/0301256`.

[66] O. Mišković and J. Zanelli, *J. Math. Phys.* **44** (2003) 3876.

[67] J. Saavedra , R. Troncoso and J. Zanelli, *J. Math. Phys.* **42** (2001) 4383.

[68] J. Saavedra , R. Troncoso and J. Zanelli, *Rev. Mex. Fis.* **48** (2002) SUPPL 387-89.

[69] O. Mišković, R. Troncoso and J. Zanelli, *Dynamics and Charge Algebra for Chern-Simons-AdS_5 Supergravity*. Report-No: CECS-PHY-04/02. (in preparation)

[70] B. Sazdović, *Phys. Lett.* **B352** (1995) 64.

[71] H. Sugawara, *Phys. Rev.* **170** (1968) 2019.

[72] C. Sommerfield, *Phys. Rev.* **176** (1968) 2019.

[73] P. Goddard and D. Olive, *Int. J. Mod. Phys.* **A1** (1986) 303.

[74] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, *Nucl. Phys.* **B241** (1984) 333.

[75] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, New York, 1997).

[76] P. West, *Introduction to Supersymmetry and Supergravity* (World Scientific, Singapore, extended second edition, 1990).

[77] A. M. Polyakov and P. B. Wiegmann, *Phys. Lett.* **B131** (1983) 121.

[78] B. Sazdović, *The WZNW Term in the Quantization of Theories with Anomalies*, Proceedings of Danube Workshop '93, p.34 (Editors A. R. Bogojević, T. Vukašinac, Belgrade, Yugoslavia, 1993).

[79] P. Di Vecchia, V. G. Knizhnik, J. L. Petersen and P. Rossi, *Nucl. Phys.* **B253** (1985) 701.

[80] E. Abdalla and M. C. B. Abdalla, *Phys. Lett.* **B152** (1985) 59.

[81] P. Ramond, *Phys. Rev.* **D3** (1971) 2415.

[82] A. Neveu and J. H. Schwarz, *Nucl. Phys.* **B31** (1971) 86.

[83] A. Neveu and J. H. Schwarz, *Phys. Rev.* **D4** (1971) 1109.

[84] E. Bergshoeff, S. Randjbar-Daemi, A. Salam, H. Sarmadi and E. Sezgin, *Nucl. Phys.* **B269** (1986) 77.

[85] S. James Gates Jr. and H. Nishino, *Class. Quantum. Grav.* **3** (1986) 391.

[86] S. V. Ketov, *Class. Quantum. Grav.* **4** (1987) 1163.

[87] L. Brink, S. Deser, B. Zumino, P. Di Vecchia and P. S. Howe, *Phys. Lett.* **B64** (1976) 435.

[88] L. Brink , P. Di Vecchia and P. S. Howe, *Phys. Lett.* **B65** (1976) 471.

[89] S. Deser and B. Zumino, *Phys. Lett.* **B65** (1976) 369.

[90] M. Blagojević, D. S. Popović and B. Sazdović, *Mod. Phys. Lett.* **A13** (1998) 911.

[91] M. Blagojević, D. S. Popović and B. Sazdović, *Phys. Rev.* **D59:** 044021 (1999).

[92] A. Miković and B. Sazdović, *Mod. Phys. Lett.* **A10** (1995) 1041.

[93] A. Miković and B. Sazdović, *Mod. Phys. Lett.* **A12** (1997) 501.

[94] D. S. Popović and B. Sazdović, *Mod. Phys. Lett.* **A16** (2001) 589.

[95] B. Sazdović, *Phys. Rev.* **D62:** 045011 (2000).

[96] B. Sazdović, *Mod. Phys. Lett.* **A17** (2002) 1923.

[97] E. D'Hoker and D. H. Phong, *Rev. Mod. Phys.* **60** (1988) 91.

[98] P. A. M. Dirac, *Can. J. Math.* **3** (1951) 1.

[99] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton University Press, Princeton, 1992).

[100] C. Leiva and M. S. Plyushchay, *Ann. Phys.* (2003) 307.

[101] L. Castellani, *Ann. Phys. (N. Y.)* **143** (1982) 357.

[102] M. Blagojević, *Gravity and Gauge Symmetries* (Institute of Physics Publishing, London, 2001).

[103] W. Siegel, *Nucl. Phys.* **B238** (1984) 307.

[104] M. Henneaux and C. Teitelboim, *Consistent Quantum Mechanics of Chiral p-forms*, in "Quantum Mechanics of Fundamental Systems 2", editors C. Teitelboim and J. Zanelli (Plenum Press, New York, 1989).

[105] M. S. Plyushchay and A. V. Razumov, *Dirac versus reduced phase space quantization*, Proceedings of Geometry of Constrained Dynamical Systems, p. 239 (Cambridge, 1994). E-Print Archive: `hep-th/9412137`.

[106] M. S. Plyushchay and A. V. Razumov, *Int. J. Mod. Phys.* **A11** (1996) 1427.

[107] G. Grignani, M. S. Plyushchay and P. Sodano, *Nucl. Phys.* **B464** (1996) 189.

[108] K. S. Nirov and M. S. Plyushchay, *Nucl. Phys.* **B512** (1998) 295.

[109] T. R. Allcock, *Phil. Trans. R. Soc. (Lond.)* **279** (1975) 33.

[110] R. Cawley, *Phys. Rev. Lett.* **42** (1979) 413.

[111] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, *Phys. Rev.* **D48** (1993) 1506.

[112] J. D. Brown and M. Henneaux, *J. Math. Phys.* **27** (1986) 489.

[113] M. Bañados, *Black Holes of Constant Curvature*, in "Recent developments in theoretical and experimental general relativity, gravitation, and relativistic field theories, Pt.B" (Jerusalem, 1997). E-Print Archive: `gr-qc/9710052`.

[114] M. Bañados, C. Teitelboim and J. Zanelli, *Phys. Rev.* **D49** (1994) 975.

[115] Rong-Gen Cai and Kwang-Sup Soh, *Phys. Rev.* **D59**: 044013 (1999).

[116] J. Crisostomo, R. Troncoso and J. Zanelli, *Phys. Rev.* **D62**: 084013 (2000).

[117] R. Aros, R. Troncoso and J. Zanelli, *Phys. Rev.* **D63**: 084015 (2001).

[118] O. Coussaert and M. Henneaux, *Phys. Rev. Lett.* **72** (1994) 183.

[119] P. A. M. Dirac, *Can. J. Math.* **2** (1950) 129.

[120] P. A. M. Dirac, *Proc. R. Soc. (Lond.)* **A246** (1958) 326.

[121] J. L. Anderson and P. G. Bergman, *Phys. Rev.* **83** (1951) 1018.

[122] P. G. Bergman, I. Goldberg, A. Janis and E. Newman, *Phys. Rev.* **103** (1956) 807.

[123] P. G. Bergman, *Rev. Mod. Phys.* **33** (1961) 510.

[124] F. A. Berezin and M. S. Marinov, *Ann. Phys.* **104** (1977) 336.

[125] R. Casalbuoni, *Nuovo Cimento* **A33** (1976) 115.

[126] R. Casalbuoni, *Nuovo Cimento* **A33** (1976) 389.

[127] J. L. Gervais and B. Sakita, *Nucl. Phys.* **B64** (1971) 632.

[128] A. J. Hanson, T. Regge and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).

[129] K. Sundermeyer, *Constrained Dynamics* (Springer Lecture Notes 169, Berlin, 1982).

[130] M. Henneaux, *Phys. Rept.* **126** (1985) 1.

[131] A. W. Wipf, *Hamilton's Formalism for Systems with Constraints*, Lectures given at the Seminar "The Canonical Formalism in Classical and General relativity" (Bad Honnef, Germany, 1993). E-Print Archive: `hep-th/9312078`.

[132] E. Inönü, *Contractions of Lie Groups and Their Representations,* “Group Theoretical Concepts and Methods in Elementary Particle Physics” (Ed F. Gürsey; Gordon and Breach, New York, 1964, p.391).

[133] W. Nahm, *Nucl. Phys.* **B135** (1978) 149.

[134] J. Strathdee, *Int. J. Mod. Phys.* **A2** (1987) 273.

[135] H. Lü, C. N. Pope and P. K. Townsend, *Phys. Lett.* **B391** (1997) 39.

[136] R. Aros, C. Martinez, R. Troncoso and J. Zanelli, *JHEP* 0205 (2002) 020.