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Conformal Symmetry, Non-commutative Landau Problem and Vortex Dynamics

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Abstract

The dynamics of the non-commutative Landau problem (NCLP) and the system of two point vortices are studied, identifying three different phases in both systems. A correspondence in the phases, sub-critical (non-chiral), super-critical (chiral) and critical between the two systems is established. As a result, a trivial permutation symmetry of the point vortices induces a weak-strong coupling duality in the NCLP.

We introduce a linear combination of angular momentum integral and Hamiltonian together with a two component object obtained by space reflection. These elements allowed us to generate quantum sub-critical and super-critical phases of both systems from a two-dimensional quantum free particle or a quantum vortex-antivortex system by applying conformal bridge transformation to latter systems. The composition of the inverse and direct transformations of the conformal bridge also makes it possible to link the non-chiral and chiral phases in each of these two systems.

Resumen

Se estudia la dinámica del problema de Landau no conmutativo (NCLP) y el sistema de dos vórtices puntuales, identificándose tres fases diferentes en ambos sistemas. Se establece una correspondencia en las fases, subcrítica (no quiral), supercrítica (quiral) y crítica entre los sistemas mencionados. Como resultado, una simetria de permutación trivial de los vórtices puntuales induce una dualidad de acoplamiento débil-fuerte en el NCLP.

Introducimos una combinación lineal de integral de momento angular y Hamiltoniano junto con un objeto de dos componentes obtenido por reflexión espacial. Estos elementos nos permitieron generar fases cuánticas subcríticas y supercríticas de ambos sistemas a partir de una partícula libre cuántica bidimensional o un sistema cuántico vórtice-antivórtice aplicando transformación de puente conforme a estos últimos sistemas. La composición de transformaciones inversas y directas del puente conforme también hace posible vincular las fases no quiral y quiral en cada uno de estos dos sistemas.

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1. Introduction

Symmetries play an essential role in physics. They are useful tools to describe particular configurations or special properties of a system. They can also be used as guiding tools for building theories. General relativity and the Standard model are two examples.

An important symmetry to be cited is the conformal symmetry which throughout history acquired more importance and significance. Conformal invariance was first introduced into physics by Cunningham and Batman. They showed that Maxwell's equations are covariant not only under 10-parameter Lorentz group but under the larger 15-parameter conformal group [1].

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho} - i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\rho}M_{\mu\sigma}\,,\\
[M_{\mu\nu}, P_\rho] &= i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu\,,\\
[P_\mu, P_\nu] &= 0\,,\\
[P_\mu, D] &= iP_\mu\,,\\
[M_{\mu\nu}, D] &= 0\,,\\
[K_\mu, K_\nu] &= 0\,,\\
[M_{\mu\nu}, K_\rho] &= i\eta_{\mu\rho}K_\nu - i\eta_{\nu\rho}K_\mu\,,\\
[K_\mu, D] &= -iK_\mu\,,\\
[K_\mu, P_\nu] &= 2i\eta_{\mu\nu}D - 2iM_{\mu\nu}\,,
\end{aligned}
\tag{1.0.1}
$$

After Weyl's theory of gravitation and electrodynamics was proposed in 1918, interest appeared in extending the general relativity by removing the restriction of invariance of proper differential line elements $ds \neq 0$. (See Appendix A). After the triumph of Einstein relativistic theory of gravitation (confirmed by the measurements done during the solar eclipse of 1919), these ideas were relegated and treated as non-physical. Nevertheless, they revealed to be very fruitful in the development of local gauge theories. Instead of considering space-time transformations, imaginary rescales or complex phase transformations belonging to an internal space can be considered. These new ideas, started a revolution that lead to the $U(1)$ gauge theory and further generalizations as embodied by Yang-Mills non-commutative (non-abelian) theories. Nowadays, conformal symmetry, as well as conformal theories, play a pivot role in many different fields in physics, such as AdS/CFT correspondence and Gauge/Gravity duality, black hole physics and cosmology [2, 3, 4, 5, 6, 7, 8, 9, 10], just to mention a few.

The conception of symmetry can be applied in a large number of areas both at a classical-quantum levels with relativistic and non-relativistic symmetry. In addition to space-time symmetries, one can seek for *symmetries in the evolution of a system.* (See Appendix B). At both the quantum and classical levels, the evolution of a system is determined by a Hamiltonian. However, there is an arbitrariness in the choice of the evolution parameter of the system. Consider for example a given system with relativistic symmetry. As considered by Dirac [11], ten quantities P_μ and $M_{\mu\nu}$ are characteristic for the dynamical system. They are called the *ten fundamental quantities.* They determine how all dynamical variables are affected by a change in the coordinate system of the kind that occurs in special relativity. Each of them is associated with a type of infinitesimal transformation of the inhomogeneous Lorentz group.

$$
\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho} - i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\rho}M_{\mu\sigma}\,, \\
[M_{\mu\nu}, P_\rho] &= i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu\,, \\
[P_\mu, P_\nu] &= 0\,.
\end{aligned} \tag{1.0.2}
$$

To construct a theory of a dynamical system one must obtain expressions for the ten fundamental quantities that satisfy these Poisson brackets (PB) relations. The problem of finding a new dynamical system reduces to the problem of finding a new solution of these equations. In this sense, Dirac finds three forms of dynamics associated with different surfaces in space-time, an instantaneous, an hyperboloid and a conic surface. Each of these forms of dynamics brings with it a different explicit form of the energetic parameters i.e Hamiltonian as linear combinations of the inhomogeneous Lorentz group algebra generators.

In the series of the recent works, interesting connections between some conformal mechanical systems are found employing conformal bridge transformation (CBT) technique, which is a non-unitary, non-local transformation inspired in Dirac forms of dynamics. This method made possible to connect various quantum mechanical systems with a free particle in spaces of various dimensions and geometric backgrounds, including those of a magnetic monopole and a cosmic string [12, 13, 14, 15, 16, 17, 18]. Thus, explicit and hidden symmetries as well as super-symmetries of various systems can be derived from a free particle. This transformation maps asymptotically free systems to harmonically confined systems, relating their non-compact and compact symmetry generators of the conformal algebra $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ and their eingenstates. The CBT construction is analogous to the Weierstrass transformation and turns out to be related to the unitary Bargmann-Segal transformation, which connects the Hilbert space in the coordinate representation with the Fock-Bargmann space of the holomorphic representation [14]. At the same time, CBT corresponds to a Dyson map [19], to which the $\mathcal{PT}$ symmetry is intimately related [12, 13]. The non-local CBT generator has the nature of the eighth-order root of the identity operator in the quantum phase space.

An interesting problem covered by CBT [14, 18], is the usual Landau problem, which is a charged particle subjected to a constant uniform magnetic field. The motion in the plane perpendicular to the magnetic field is quantized and the energy levels are strongly degenerate in the quantum number corresponding to the center of the Larmor circle in classical mechanics. This theory constitutes the base for explaining interesting phenomena such as the Magnetic oscillations: Shubnikov-de Haas and de Haas-van Alphen effects and the Hall effect [20].

On the other hand, hidden connections and correspondences between, in principle, different systems models are of great importance in physics. For example, the construction of (multi)soliton solutions of the classical KdV equation and equations of its hierarchy is related to the stationary Schrodinger theory via the inverse scattering problem and allows to obtain them from a free particle by employing the Darboux covariance of the corresponding Lax formulation [21, 22]. Similarly, Bäcklund transformations connect various integrable systems and generate more complex solutions for them starting from simpler ones [23]. In the same vein hidden correspondences between some integrable systems are provided by the Newton-Hooke duality, based on conformal mapping [24, 25, 26], and their generalizations in the form of the coupling constant metamorphosis phenomenon [27, 28]. Another interesting system for which many strands of classical mathematical physics come together is the point vortex system, which can be viewed as discrete or localized solutions of the Euler equations in two dimensions (See Appendix C). Of course, there one encounters the theory of dynamical systems, of systems of ordinary differential equations (ODEs), Hamiltonian dynamics, and several other topics that one might think of as expected. But there are also unexpected or less expected connections to subjects such as projective geometry, to aspects of the theory of polynomials, to elliptic functions when the vortices are in periodic or bounded domains, and to pole decompositions of some of the integrable partial differential equations such as Burgers equation and the KdV equation. Applications of even more exotic objects such as the Schwarzian function and the Schottky-Klein prime function have appeared [29]. In this vein, this Thesis addresses the following problems:

• i) Investigation of the correspondence between the non-commutative Landau problem and the two-vortex dynamics.

The integral generators for the N-point vortex systems generate the centrally extended Lie algebra $\mathfrak{e}_\Gamma(2)$ of the two-dimensional Euclidean group, where the dynamics is dictated by the $\mathfrak{e}_\Gamma(2) \oplus \mathfrak{u}(1)$ algebra. For the particular case of two-vortex, the trajectories are circular and the Hamiltonian function can be written in terms of the Casimir element of the $\mathfrak{e}_\Gamma(2)$ algebra, analogously to the (non-commutative) Landau problem, where it is well known that their *dynamical integrals of motion* generate the $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ conformal algebra. We address the problem of establishing a correspondence between the non-commutative Landau problem and the two-vortex dynamics

• ii) The application of the conformal bridge transformation (CBT) to relate the quantum two-vortex dynamics and non-commutative Landau problem to the quantum free particle dynamics.

We address the problem of establishing a procedure to obtain the two-vortex system through CBT, exploiting the conformal symmetry of the system. Through this method we also relate a free particle in non-commutative plane with non-commmutative Landau problem.

This Thesis is based on the results obtained in one year research project framework and presented in [arXive:2304.06677 [hep-th]] [30]. It is organized as follows. In Section 2 we discussed the realization of $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ algebra in terms of one-dimensional free particle symmetry and displayed the flows produced by the algebra generators. Complex canonical transformation is introduced at classical level and extended to its quantum analog as similarity transformation forwarding to the conformal bridge transformation. Also, we have considered the outer $\mathbb{Z}_2$ automorphism of the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra, which has the nature of a $\mathcal{PT}$-inversion and provide an explanation for the difference between non-chiral and chiral phases for the related systems. In Section 3 we have examined the non-commutative Landau Problem in sub-(non-chiral), super-(chiral) and critical phases. In Section 4 we reviewed the symplectic structure and integrals of motion for the general case of a system of N point vortices. Then, we focused on the case of maximally superintegrable two-vortex systems. In Section 5 we established the correspondence of the non-commutative Landau Problem with the two-vortex systems. We have shown that trivial permutation symmetry of vortices induces a weak-strong coupling duality in the NCLP. In Section 6 we provided an alternative study of the non-chiral and chiral phases for both systems, introducing an integral of motion, allowing us to find two different canonical sets of variables in the non-chiral and chiral phases, respectively. In Section 7 we have obtained the NCLP and two-vortex systems in their different phases by means of CBT, employing the canonical pairs introduced in the previous section. Section 8 is devoted to the Conclusions and Outlook. In Appendix A, we introduced the generators and particularities of the conformal group. In Appendix B we briefly discuss the symplectic Hamiltonian formalism. Finally, in Appendix C we discuss the evolution equations of point vortices from the Euler equation in two dimensions.

2. Conformal symmetry

Consider a free particle in $d = 1$ dimension, where we set, for the sake of simplicity, the mass parameter $m = 1$. The phase space coordinates (q, p) can be combined into a two-component object $\xi^T_\alpha = (q, p)$ with Poisson brackets $\{\xi_\alpha, \xi_\beta\} = \epsilon_{\alpha\beta}$.

We can generate $\mathfrak{so}(2,1)$ algebra by quadratic functions of ξ^T_α

$$\begin{gathered} H_0 = \tfrac{1}{2}p^2\,, \qquad D = qp\,, \qquad K = \tfrac{1}{2}q^2\,, \\ \{D, H_0\} = 2H_0\,, \qquad \{D, K\} = -2K\,, \qquad \{K, H_0\} = D\,. \end{gathered} \tag{2.0.1}$$

With linear combinations of these quadratic functions, the isomorphic $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$ algebra generators are introduced by

$$\begin{gathered} H_+ = \tfrac{1}{2}(p^2 + q^2) = H + K := 2J_0\,, \qquad H_- = \tfrac{1}{2}(p^2 - q^2) = H - K := -2J_1\,, \\ D := -2J_2\,, \end{gathered} \tag{2.0.2}$$

that according to (2.0.1) satisfy the relations

$$\{J_0, J_j\} = \epsilon_{jk}J_k\,, \quad j,k = 1,2\,, \qquad \{J_j, J_k\} = -\epsilon_{jk}J_0\,, \tag{2.0.3}$$

which can be presented in a compact (2+1)D form

$$\{J_\mu, J_\nu\} = -\epsilon_{\mu\nu\lambda}J^\lambda\,, \tag{2.0.4}$$

where $\mu,\nu,\lambda = 0,1,2$, $\epsilon_{\mu\nu\lambda}$ is an antisymmetric tensor, $\epsilon^{012} = 1$, and the metric tensor is defined as $\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1)$.

The H_+ generator of (2.0.2) is compact and can be thought as the harmonic oscillator Hamiltonian, with frequency $\omega = 1$. The H_- generator is non-compact and can be thought as the inverted harmonic oscillator Hamiltonian.

2.1. Conformal Flows

Canonical phase space transformations of ξ_α can be produced as

$$F\,:\; \xi_\alpha \to \xi'_\alpha = \exp(\tau F) \star \xi_\alpha = \xi_\alpha + \sum_{n=1}^{\infty} \frac{\tau^n}{n!}\{F, \{\ldots, \{F, \xi_\alpha \underbrace{\}\ldots\}}_{n}\,, \tag{2.1.1}$$

where $F(\xi_\alpha)$ is an arbitrary phase space function.

Thereby, considering the generators (2.0.2) and $\tau \in \mathbb{R}$ we produce the following transformations

$$H_+\,:\; \xi_\alpha \to \xi'_\alpha = R_{\alpha\beta}\xi_\beta\,, \qquad R_{\alpha\beta} = \begin{pmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{pmatrix}\,, \tag{2.1.2}$$

$$D\,:\; q \to q' = e^{-\tau}q\,, \qquad p \to p' = e^{\tau}p\,, \tag{2.1.3}$$

$$H_- \, : \, \xi_\alpha \to \xi'_\alpha = L_{\alpha\beta}\xi_\beta \, , \qquad L_{\alpha\beta} = \begin{pmatrix} \cosh\tau & -\sinh\tau \\ -\sinh\tau & \cosh\tau \end{pmatrix} . \tag{2.1.4}$$

If we consider transformations generated by J_μ, we produce the same transformations (2.1.2)-(2.1.4) changing τ for $\tau/2$. This means ξ_α is an $\mathfrak{so}(2,1) \cong \mathfrak{sl}(2,\mathbb{R})$ spinor [31].

Classical analogs of the creation-annihilation operators are defined

$$a^\mp = \frac{1}{\sqrt{2}}(q \pm ip). \tag{2.1.5}$$

Their phase space transformations produced with (2.0.2) generators are

$$H_+ \, : \, a^\pm \to a'^\pm = e^{\pm i\tau} a^\pm \, , \tag{2.1.6}$$

$$D \, : \, a^- \to a'^- = a^- \cosh\tau - a^+ \sinh\tau \, , \quad a^+ \to a'^+ = a^+ \cosh\tau - a^- \sinh\tau \, , \tag{2.1.7}$$

$$H_- \, : \, a^- \to a'^- = a^- \cosh\tau - i\, a^+ \sinh\tau \, , \quad a^+ \to a'^+ = a^+ \cosh\tau + i\, a^- \sinh\tau \, , \tag{2.1.8}$$

where the action of D produces a Bogoliubov transformation.

In particular, fixing the parameter $\tau = \pi/4$, and considering the transformations (2.1.2) and (2.1.6) we obtain

$$q' = \tfrac{1}{\sqrt{2}}(q - p) \, , \qquad p' = \tfrac{1}{\sqrt{2}}(q + p) \, , \tag{2.1.9}$$

$$a'^{\,\mp} = e^{\pm i\pi/4} a^\mp \, , \tag{2.1.10}$$

correspondingly. In terms of variables (2.1.9) the (2.0.2) generators are expressed as

$$H_+ \to H'_+ = H_+ \, , \qquad D \to D' = -H_- \, , \qquad H_- \to H'_- = D \, , \tag{2.1.11}$$

where, as expected, H_+ remains invariant. Thus, the D generator take the form of the inverted harmonic oscillator Hamiltonian.

Using relationships

$$J_1 = -\frac{1}{2}H_- \, , \qquad J_2 = -\frac{1}{2}D \, , \tag{2.1.12}$$

we conclude that transformation (2.1.9) correspond to a rotation of the non-compact generators (2.0.4) in $\pi/2$ such

$$J_i \to J'_i = \epsilon_{ij} J_j \, , \qquad J_0 \to J_0 \, . \tag{2.1.13}$$

2.2. Conformal Bridge Transformation

The transformations considered above, are obtained by taking real values of the parameter τ, which do not change the compact or non-compact nature of the $\mathfrak{sl}(2,\mathbb{R})$ generators. Now, we allow the parameter τ to take complex values, obtaining complex canonical transformations.

We set the parameter $\tau = i\pi/4$, considering the $H_- = -2J_1$ generator. The phase space coordinates (q,p), and complex linear combinations $a^\pm$, are transformed under these considerations as

$$\exp\Big(i\frac{\pi}{4}H_-\Big) \star (q, p, a^-, a^+) = (a^+, -ia^-, q, -ip) \, . \tag{2.2.1}$$

These transformations are illustrated in Figure 2.1.

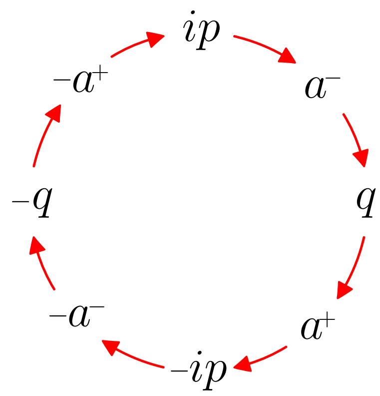

Figure 2.1.: Action of conformal bridge transformation (2.2.1).

The generators (2.0.1) are transformed as

$$\exp\left(i\frac{\pi}{4}H_-\right)\star(H_+, H_-, iD) = (-iD, H_-, H_+)\,, \tag{2.2.2}$$

or, equivalently,

$$\exp\left(-i\frac{\pi}{2}J_1\right)\star(J_0, J_1, J_2) = (iJ_2, J_1, iJ_0) = (J_0', J_1', J_2')\,. \tag{2.2.3}$$

In this sense, the Wick rotated non-compact generator D of $\mathfrak{sl}(2,\mathbb{R})$ algebra (multiplied by i) is transformed into the compact generator $H_+ = 2J_0$.

This complex canonical transformation is of eighth-order root nature of the identity transformation for the linear phase space variables, while, it constitutes a fourth-order root of the identity transformation for $\mathfrak{sl}(2,\mathbb{R})$ algebra generators.

The analog quantum transformation of (2.1.1) corresponds to a similarity transformation

$$\hat{O} \to \hat{O}' = \exp\left(i\tau\hat{F}\right)\hat{O}\exp\left(-i\tau\hat{F}\right)\,. \tag{2.2.4}$$

In case of (2.2.1)-(2.2.2), we have

$$\hat{O}' = \hat{\mathfrak{S}}\hat{O}\hat{\mathfrak{S}}^{-1}\,, \qquad \hat{\mathfrak{S}} = \exp\left(\frac{\pi}{4}\hat{H}_-\right)\,, \tag{2.2.5}$$

where the $\hat{\mathfrak{S}}$ generator has the evolution operator form of the inverted harmonic oscillator for complex time $t = i\pi/4$.

The action of (2.2.5) over the canonical operators $\hat{q}$ and $\hat{p}$ correspond to the transformation from the Schrödinger representation of the Heisenberg algebra to its Fock-Bargmann representation in accordance with the Stone-von Neumann theorem [14].

Consider the monomials

$$\phi_n = q^n\,, \qquad n = 0, 1, \dots\,, \tag{2.2.6}$$

which are formal eigenfunctions of the Wick rotated dilatation operator,

$$i\hat{D}\phi_n = \left(n + \frac{1}{2}\right)\phi_n \,. \tag{2.2.7}$$

At the same time, ϕ_n constitute Jordan states [32] of the free particle corresponding to its zero energy eigenvalue

$$(\hat{H}_0)^{\lfloor \frac{n}{2} \rfloor}\phi_n = 0 \,, \tag{2.2.8}$$

where $\lfloor \frac{n}{2} \rfloor$ is the integer part of $n/2$.

Consider the inverse Weierstrass transformation [33, 34] of the monomials

$$\exp\left(-\frac{1}{4}\frac{d^2}{dq^2}\right)\phi_n = \exp\left(\frac{1}{4}\hat{H}_0\right)\phi_n = 2^{-n}H_n(q)\,, \tag{2.2.9}$$

where $H_n(q)$ are the Hermite polynomials. By means of this relation, the application of CBT over ϕ_n transforms them, up to a normalization, into eigenstates of the quantum harmonic oscillator,

$$\hat{\mathfrak{S}}\phi_n(q) \propto \psi_n(q) = \frac{1}{2^n \pi^{1/2} n!}H_n(q)e^{-q^2/2} \,. \tag{2.2.10}$$

At the same time, it can be verified that the free particle planar wave eigenstates and the Gaussian packet are transformed into coherent states and single-mode squeezed coherent states of the quantum harmonic oscillator respectively [14].

CBT constitute a non-unitary, non-local transformation, which transmute the anti-hermitian Wick rotated Dilatation operator

$$i\hat{D} = \frac{i}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) = \frac{i}{2}(\hat{p}'^{\,2} - \hat{q}'^{\,2}) = i\hat{H}'_{-} \tag{2.2.11}$$

into Hermitian Hamiltonian operator

$$\hat{H}_{+} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2) \tag{2.2.12}$$

of the quantum harmonic oscillator, and corresponds to the Dyson map to which the $\mathcal{PT}$ symmetry is intimately related [12, 13].

CBT transformation can be generalized for $d > 1$ dimensions. By choosing d-dimensional Cartesian coordinates in such a way that the generator (2.2.4) acts only on the coordinates which it is built. Therefore, by adding an index $j = 1, \ldots, d$ to the phase space coordinates q and p we obtain generators of $\mathfrak{sl}(2,\mathbb{R})$ algebra, which are invariant under $so(d)$ rotations.

2.3. $\mathbb{Z}_2$ Automorphism

It will be useful to study the outer $\mathbb{Z}_2$ automorphism of $\mathfrak{sl}(2,\mathbb{R})$ algebra. Let us take two copies of $\mathfrak{sl}(2,\mathbb{R})$ algebra generated by $J^{(a)}_\mu$, $a = 1, 2$ (2.0.2) that satisfy the relationship $\{J^{(1)}_\mu, J^{(2)}_\nu\} = 0$. From them, we can construct three sets of the $\mathfrak{sl}(2,\mathbb{R})$ generators as

$$\mathcal{J}_\mu = J^{(1)}_\mu + J^{(2)}_\mu \,, \qquad \mathcal{J}'_\mu = J'^{\,(1)}_\mu + J'^{\,(2)}_\mu \,, \qquad \widetilde{\mathcal{J}}_\mu = J^{(1)}_\mu + J'^{\,(2)}_\mu \,, \tag{2.3.1}$$

where J'_μ is produced by a Z_2 outer automorphism

$$J_\mu \to J'_\mu \,, \qquad (J'_0, J'_1, J'_2) = (-J_0, J_2, J_1) \,. \tag{2.3.2}$$

Note that (2.3.2) represents a kind of the $\mathcal{PT}$-inversion applied to J_μ, which is a composition of the time, $J_0 \to -J_0$, and space, $J_i = (J_1, J_2) \to -\epsilon_{ij}\check{J}_j = (J_2, J_1)$, inversions acting on the (2+1)D space with coordinates J_μ.

Now, supposing that $J_\mu^{(a)}$ generators are realized in terms of the one-dimensional canonical variables $(q^{(a)}, p^{(a)})$, particularly, $J_0^{(a)} = \frac{1}{2}H_+^{(a)}$. Consequently, it can be found that $\mathcal{J}_0$ takes only non-negative values, $\mathcal{J}_0'$ takes only non-positive values, and $\widetilde{\mathcal{J}}_0$ takes values on all the real line.

At the quantum level, this automorphism acts in the space that corresponds to the direct sum of the two infinite-dimensional (reducible) unitary representations of the sets $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.3.1), each of which is, in turn, a direct sum of the two irreducible representations with eigenvalues of the corresponding compact generators shifted mutually in 1/2.

In these representations, the Casimirs take the same value

$$\hat{J}_\mu^{(1)}\hat{J}^{(1)\mu} = \hat{J}_\mu'^{(2)}\hat{J}'^{(2)\mu} = -\alpha(\alpha-1) = 3/16\,, \tag{2.3.3}$$

while $\hat{J}_0^{(1)}$ and $\hat{J}_0'^{(2)}$ take eigenvalues $\alpha + n_1$ and $-(\alpha + n_2)$ with $\alpha = 1/4$, $3/4$, and n_1,n_2 take integer values $0, 1, \ldots$. The direct sums of the irreducible $\mathfrak{sl}(2,\mathbb{R}) \cong \mathfrak{so}(2,1)$ representations with $\alpha = 1/4$ and $\alpha = 1/4 + 1/2 = 3/4$ constitute irreducible representations of the two copies of $\mathfrak{osp}(2,1)$ superalgebra in which $\hat{J}_\mu^{(1)}$ and $\hat{J}_\mu'^{(2)}$ are treated as its even generators, while $\hat{\xi}_\alpha^{(1)}$ and $\hat{\xi}_\alpha^{(2)}$ are its odd generators that mutually transform the corresponding states from the $\alpha = 1/4$ and $\alpha = 3/4$ subspaces [31].

3. Non-commutative Landau Problem

To understand the dynamics and main properties of the two-vortex system, it is initially convenient to study the non-commutative Landau problem (NCLP). The generalization of the usual Landau problem to the case of noncommutative quantum mechanics has been actively examined in the context of physics associated with noncommutative geometry [35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46]. Recently, the Landau problem has attracted attention in the study of the non-relativistic conformally invariant Schwartzian mechanical system associated with the low energy limit of the Sachdev-Ye-Kitaev model [47, 48, 49].

The non-commutative Landau problem (NCLP) can be described by Lagrangian in symmetric gauge is

$$\mathbb{L} = \mathbb{P}_i\dot{\mathbb{X}}_i + \frac{1}{2}\theta\epsilon_{ij}\mathbb{P}_i\dot{\mathbb{P}}_j + \frac{1}{2}B\epsilon_{ij}\mathbb{X}_i\dot{\mathbb{X}}_j - \frac{1}{2m}\mathbb{P}_i^2\,, \tag{3.0.1}$$

where $\mathbb{X}_i$ are the coordinates of the charged particle of mass m subjected to the constant magnetic field B, while θ is defined as the non-commutative parameter.

We use units in which the speed of light $c = 1$ and charge $e = 1$. It is important to recall, that we do not distinguish between uppercase and lowercase index, unless otherwise are specified, and imply summation over repeated index.

Due to the non-quadratic velocity terms in the 'kinetic term', Lagrangian (3.0.1) is singular, and has degenerate Hessian

$$Det|W_{ab}| = 0\,, \qquad W_{ab} = \frac{\partial^2\mathbb{L}}{\partial q_a \partial q_b}\,, \tag{3.0.2}$$

where q_a correspond to the local coordinates $\mathbb{X}_i$ and $\mathbb{P}_i$. In this sense, the canonical momenta $\pi_i^{(\mathbb{X})}$ of the $\mathbb{X}_i$ coordinate and canonical momenta $\pi_i^{(\mathbb{P})}$ of the $\mathbb{P}_i$ coordinates

$$\pi_i^{(\mathbb{X})} = \mathbb{P}_i - \tfrac{1}{2}B\epsilon_{ij}\mathbb{X}_j, \qquad \pi_i^{(\mathbb{P})} = -\tfrac{1}{2}\theta\epsilon_{ij}\mathbb{P}_j, \tag{3.0.3}$$

are not invertible with respect to the velocities, and consequently constrained with the coordinates.

To study the model, we proceed to build a reduced phase space, with its corresponding symplectic two-form. To this end, we employ the Dirac analysis [50] and introduce the weakly zero primary constraints between the canonical momenta and coordinates

$$\phi_i^{(\mathbb{X})} = \pi_i^{(\mathbb{X})} - \mathbb{P}_i + \tfrac{1}{2}B\epsilon_{ij}\mathbb{X}_j \approx 0\,, \qquad \phi_i^{(\mathbb{P})} = \pi_i^{(\mathbb{P})} + \tfrac{1}{2}\theta\epsilon_{ij}\mathbb{P}_j \approx 0\,, \tag{3.0.4}$$

and the primary Hamiltonian

$$\mathbb{H} = \frac{1}{2m}\mathbb{P}_i^2 + u_i\phi_i^{(\mathbb{X})} + v_i\phi_i^{(\mathbb{P})}\,, \tag{3.0.5}$$

where $u_i(t)$, $v_i(t)$ are arbitrary functions.

Our aim is to reduce the defined $\sigma = d\pi_i^{(\mathbb{X})} \wedge d\mathbb{X}_i + d\pi_i^{(\mathbb{P})} \wedge d\mathbb{P}_i$ to a subspace defined by the set of constraints (3.0.4).

The Poisson brackets of the constraints (3.0.4) are

$$\{\phi_i^{(\mathbb{X})}, \phi_j^{(\mathbb{X})}\} = B\epsilon_{ij}\,, \qquad \{\phi_i^{(\mathbb{P})}, \phi_j^{(\mathbb{P})}\} = \theta\epsilon_{ij}\,, \qquad \{\phi_i^{(\mathbb{X})}, \phi_j^{(\mathbb{P})}\} = -\delta_{ij}\,, \tag{3.0.6}$$

and their matrix is

$$\Delta = \begin{pmatrix} 0 & B & -1 & 0 \\ -B & 0 & 0 & -1 \\ 1 & 0 & 0 & \theta \\ 0 & 1 & -\theta & 0 \end{pmatrix}, \qquad \det(\Delta) = (1-\beta)^2\,, \tag{3.0.7}$$

where $\beta := B\theta$.

When $\beta \neq 1$, the determinant of the matrix is non-zero and the requirement of conservation of the primary constraints (3.0.4) $\dot{\phi}_i^{(\mathbb{X})} \approx 0$, $\dot{\phi}_i^{(\mathbb{P})} \approx 0$, can always be done by choosing appropriately the u_i and v_i functions.

3.1. NCLP Critical case

First, we consider the case where the magnetic field and non-commutative parameter are set in such a way that $\beta = 1$, which implies $\det(\Delta) = 0$. In this particular case, the primary constraints (3.0.4) are no longer linear independent, and we must use a linear combination of them to build a set of two first class constraints

$$\tilde{\phi}_i = \phi_i^{(\mathbb{X})} - \frac{1}{\theta}\epsilon_{ij}\phi_j^{(\mathbb{P})} \approx 0\,, \qquad \phi_j^{(\mathbb{P})} \approx 0, \tag{3.1.1}$$

these constraints have Poisson brackets

$$\{\tilde{\phi}_i, \tilde{\phi}_j\} = \{\tilde{\phi}_i, \phi_j^{(\mathbb{P})}\} = \{\tilde{\phi}_i^{(\mathbb{P})}, \phi_j^{(\mathbb{P})}\} = 0\,, \tag{3.1.2}$$

and the primary Hamiltonian takes the form

$$\mathbb{H} = \frac{1}{2m}\mathbb{P}_i^2 + \tilde{u}_i\tilde{\phi}_i + v_i\phi_i^{(\mathbb{P})}\,. \tag{3.1.3}$$

The requirement of conservation of primary constraints (3.1.1), $\dot{\tilde{\phi}}_i \approx 0$, $\dot{\phi}_i^{(\mathbb{P})} \approx 0$, produces secondary constraints $\psi_i = \mathbb{P}_i \approx 0$.

These six constraints form a set of the second class constraints and the reduction to the subspace defined by them, provide the two-dimensional reduced phase space described by coordinates $\mathbb{X}_i$ which have non-commuting components with respect to the Dirac-Poisson brackets (DPBs) $\{\mathbb{X}_i, \mathbb{X}_j\} = \theta\epsilon_{ij}$.

The Hamiltonian, space translations and rotation generators become

$$\mathbb{H} = 0, \qquad \mathcal{P}_i = \frac{1}{\theta}\epsilon_{ij}\mathbb{X}_j, \qquad \mathbb{M} = -\frac{1}{2\theta}\mathbb{X}_i^2 \tag{3.1.4}$$

respectively.

This model corresponds to a particle in non-commutative plane which has no dynamics at all $\dot{\mathbb{X}}_i = 0$.

3.2. NCLP Non-critical case

Now, considering $\beta \neq 1$, relations (3.0.4) constitute a set of the second class constraints, and the subspace defined by them produces the symplectic two-form

$$\sigma = \frac{1}{2}\epsilon_{ij}(B\, d\mathbb{X}_i \wedge d\mathbb{X}_j + \theta\, d\mathbb{P}_i \wedge d\mathbb{P}_j) + d\mathbb{P}_i \wedge d\mathbb{X}_i, \tag{3.2.1}$$

obtaining a four-dimensional reduced phase space described by the independent coordinates $\mathbb{X}_i$ and $\mathbb{P}_i$ with DPBs

$$\{\mathbb{X}_i, \mathbb{X}_j\} = \frac{\theta}{1-\beta}\epsilon_{ij}\,, \qquad \{\mathbb{X}_i, \mathbb{P}_j\} = \frac{1}{1-\beta}\delta_{ij}\,, \qquad \{\mathbb{P}_i, \mathbb{P}_j\} = \frac{B}{1-\beta}\epsilon_{ij}\,. \tag{3.2.2}$$

The constraint terms (3.0.4) become strongly zero and the reduced Hamiltonian takes the form

$$\mathbb{H} = \frac{1}{2m}\mathbb{P}_i^2\,. \tag{3.2.3}$$

The Noetherian integrals of motion are vector space translation generator of $\mathbb{X}_i$ coordinate

$$\mathcal{P}_i = \mathbb{P}_i - B\epsilon_{ij}\mathbb{X}_j\,, \tag{3.2.4}$$

$$\{\mathcal{P}_j, \mathbb{X}_i\} = -\delta_{ij}\,, \qquad \{\mathbb{P}_i, \mathcal{P}_j\} = 0\,, \tag{3.2.5}$$

and angular momentum generator

$$\mathbb{M} = \epsilon_{ij}\mathbb{X}_i\mathbb{P}_j + \frac{1}{2}\left(\theta\mathbb{P}_i^2 + B\mathbb{X}_i^2\right)\,, \tag{3.2.6}$$

$$\{\mathbb{M}, \mathbb{X}_i\} = \epsilon_{ij}\mathbb{X}_i\,, \qquad \{\mathbb{M}, \mathbb{P}_i\} = \epsilon_{ij}\mathbb{P}_i\,. \tag{3.2.7}$$

The vector space translation generator has non-commuting components in the sense of the DPBs, and together with the rotation generator, set up a centrally extended Lie algebra $\mathfrak{e}_B(2)$ of the two-dimensional Euclidean group

$$\{\mathcal{P}_i, \mathcal{P}_j\} = -B\,\epsilon_{ij}\,, \qquad \{\mathbb{M}, \mathcal{P}_i\} = \epsilon_{ij}\mathcal{P}_j\,, \tag{3.2.8}$$

with $-B$ playing the role of the central charge. The Casimir element of this $\mathfrak{e}_B(2)$ algebra is

$$\mathcal{C} = \frac{1}{2}\mathcal{P}_i^2 - B\mathbb{M}\,. \tag{3.2.9}$$

Therefore, we can present the Hamiltonian in terms of Casimir element as $\mathbb{H} = (m(1-\beta))^{-1}\mathcal{C}$. Integrals (3.2.4) and (3.2.6) together with Hamiltonian (3.2.3) generate $\mathfrak{e}_B(2) \oplus \mathfrak{u}(1)$ algebra.

Equations of motion (EOM) generated by Hamiltonian (3.2.3) through DPBs (3.2.2) are

$$\dot{\mathbb{X}}_i = \frac{1}{m(1-\beta)}\mathbb{P}_i\,, \qquad \dot{\mathbb{P}}_i = \frac{B}{m(1-\beta)}\epsilon_{ij}\mathbb{P}_j\,. \tag{3.2.10}$$

We can employ the vector space translation integral to solve them, obtaining

$$\mathbb{X}^i(t) = \mathbb{X}_0^i + \mathcal{R}\, n^i(t)\,, \qquad \mathbb{X}_0^i = \tfrac{1}{B}\epsilon^{ij}\mathcal{P}_j\,, \qquad \mathcal{R} = \tfrac{\sqrt{2m\mathbb{H}}}{B}\,, \tag{3.2.11}$$

$$n^i(t) = (\cos(\omega(t-t_0)), \sin(\omega(t-t_0)))\,, \qquad \omega = \tfrac{B}{m(1-\beta)}\,. \tag{3.2.12}$$

This dynamics, corresponds to circular trajectories of the charged particle with radius $\mathcal{R}$ centered at $\mathbb{X}_0^i$, where the sign of ω parameter fixes the rotation direction. This dynamics is illustrated in Figure 3.1, for the particular case when the centered vector is smaller in magnitude with respect to the radio.

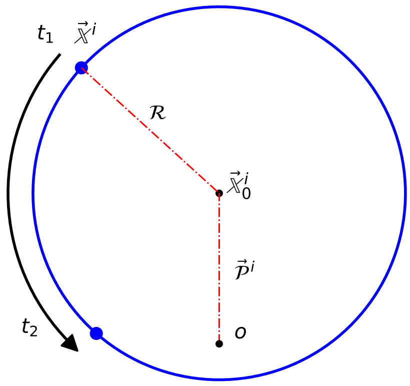

Figure 3.1.: NCLP particle trajectory $(B \neq 0)$ $(\theta \neq 0)$.

Now, we define a new coordinate vector as

$$\mathbb{Y}_i = \mathbb{X}_i + \theta\epsilon_{ij}\mathbb{P}_j = (1-\beta)\mathbb{X}_i + \theta\epsilon_{ij}\mathcal{P}_j \,, \tag{3.2.13}$$

where, by means of the second equality above, it can be thought as an "imaginary mirror particle" vector coordinate, where its trajectory is determined by the β parameter. The trajectory can be circumscribed with respect to the real particle trajectory, at closer points of their respective circles for $0 < \beta < 1$ or at the farthest points of their respective circles for $1 < \beta < 2$. These trajectories are illustrated in Figures 3.2 and 3.4 respectively. Also, the trajectory can be subscribed with respect to the real particle trajectory, at closer points of their respective circles for $\beta < 0$ or at the farthest points of their respective circles for $\beta > 2$. These trajectories are illustrated in Figures 3.3 and 3.5 respectively. Besides, when $\beta = 2$ they share the same trajectory at opposite points of the same circle. For case $\beta = 0, B \neq 0$, the imaginary and real coordinates match.

Vector $\mathbb{Y}_i$ has zero DPBs with $\mathbb{X}_i$, additionally is translated by $\mathcal{P}_i$, and its DPBs components depend only on the θ parameter

$$\{\mathbb{Y}_i, \mathbb{X}_j\} = 0\,, \qquad \{\mathbb{Y}_i, \mathcal{P}_j\} = \delta_{ij}\,, \qquad \{\mathbb{Y}_i, \mathbb{Y}_j\} = -\theta\epsilon_{ij}\,. \tag{3.2.14}$$

The phase space of the NCLP can be described, up to a canonical transformation, in terms of any of the six pairs of the vector variables, $(\mathbb{X}_i, \mathbb{P}_i)$, $(\mathbb{X}_i, \mathbb{Y}_i)$, $(\mathbb{X}_i, \mathcal{P}_i)$, $(\mathbb{P}_i, \mathcal{P}_i)$, $(\mathbb{P}_i, \mathbb{Y}_i)$ and $(\mathbb{Y}_i, \mathcal{P}_i)$. For simplicity, it is convenient to represent the integrals of motion, symplectic two-form and DPBs in terms of the pairs $(\mathbb{P}_i, \mathcal{P}_i)$ and $(\mathbb{X}_i, \mathbb{Y}_i)$.

To this purpose, we express $\mathbb{P}_i$ and $\mathcal{P}_i$ in terms of the real and imaginary particle coordinates as

$$\mathcal{P}_i = \frac{1}{\theta}\epsilon_{ij}\left((1-\beta)\mathbb{X}_j - \mathbb{Y}_j\right)\,, \qquad \mathbb{P}_i = \frac{1}{\theta}\epsilon_{ij}\left(\mathbb{X}_j - \mathbb{Y}_j\right)\,, \tag{3.2.15}$$

with inverse relations

$$\mathbb{X}_i = \frac{1}{B}\epsilon_{ij}\left(\mathcal{P}_j - \mathbb{P}_j\right) , \qquad \mathbb{Y}_i = \frac{1}{B}\epsilon_{ij}\left(\mathcal{P}_j - (1-\beta)\mathbb{P}_j\right) . \tag{3.2.16}$$

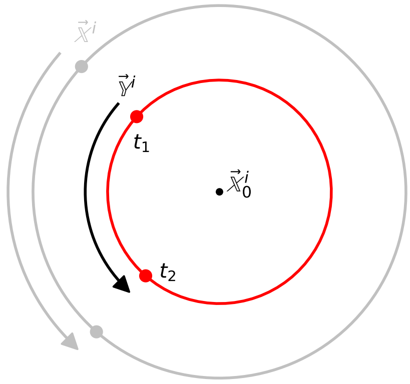

Figure 3.2.: NCLP imaginary particle trajectory for $(0 < \beta < 1)$.

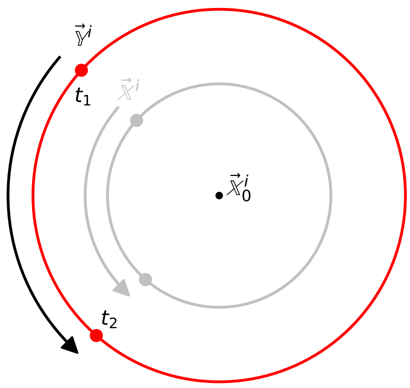

Figure 3.3.: NCLP imaginary particle trajectory for $(\beta < 0)$.

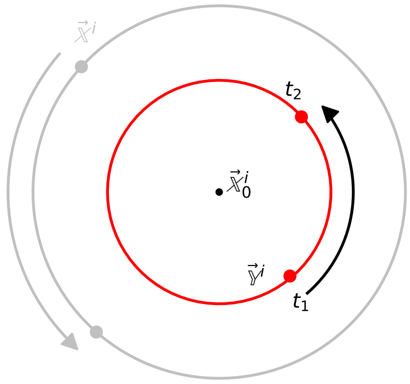

Figure 3.4.: NCLP imaginary particle trajectory for $(1 < \beta < 2)$.

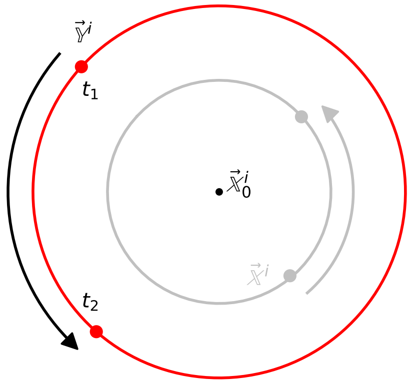

Figure 3.5.: NCLP imaginary particle trajectory for $(2 < \beta)$.

Now, we summarize the Hamiltonian, integrals of motion, two-symplectic form and DPBs for the chosen coordinates pairs

$$
\begin{aligned}
(\mathbb{P}_i, \mathcal{P}_i) \rightarrow \qquad & \mathbb{H} = \tfrac{1}{2m}\mathbb{P}_i^2\,, \qquad \mathcal{P}_i\,, \\
& \mathbb{M} = \tfrac{1}{2B}\left(\mathcal{P}_i^2 - (1-\beta)\mathbb{P}_i^2\right)\,, \\
& \sigma = \tfrac{1}{2B}\left(1-\beta\right)\epsilon_{ij} d\mathbb{P}_i \wedge d\mathbb{P}_j - \tfrac{1}{2B}\epsilon_{ij} d\mathcal{P}_i \wedge d\mathcal{P}_j\,, \\
& \{\mathbb{P}_i, \mathbb{P}_j\} = \tfrac{B}{1-\beta}\epsilon_{ij}\,, \qquad \{\mathcal{P}_i, \mathcal{P}_j\} = -B\epsilon_{ij}\,, \qquad \{\mathbb{P}_i, \mathcal{P}_j\} = 0\,.
\end{aligned} \tag{3.2.17}
$$

$$
\begin{aligned}
(\mathbb{Y}_i,\mathbb{X}_i)\rightarrow \qquad & \mathbb{H}=\tfrac{1}{2m\theta^2}\left(\mathbb{X}_i-\mathbb{Y}\right)^2\,, \qquad \mathcal{P}_i=\tfrac{1}{\theta}\epsilon_{ij}\left((1-\beta)\,\mathbb{X}_i-\mathbb{Y}_i\right)\,,\\
& \mathbb{M}=\tfrac{1}{2\theta}\left(\mathbb{Y}_i^2-(1-\beta)\mathbb{X}_i^2\right)\,,\\
& \sigma=\tfrac{1}{2\theta}\left(1-\beta\right)\epsilon_{ij}d\mathbb{X}_i\wedge d\mathbb{X}_j-\tfrac{1}{2\theta}\epsilon_{ij}d\mathbb{Y}_i\wedge d\mathbb{Y}_j\,,\\
& \{\mathbb{X}_i,\mathbb{X}_j\}=\tfrac{\theta}{1-\beta}\epsilon_{ij}\,, \qquad \{\mathbb{Y}_i,\mathbb{Y}_j\}=-\theta\epsilon_{ij}\,, \qquad \{\mathbb{X}_i,\mathbb{Y}_j\}=0\,.
\end{aligned}
\tag{3.2.18}
$$

We distinguish two phases for the system. The super-critical phase $\beta>1$, where the angular momentum takes nonzero values of the sign of magnetic field B, in $(\mathbb{P},\mathcal{P})$ coordinates, or sign of the non-commutative parameter θ in $(\mathbb{Y},\mathbb{X})$ coordinates and the sub-critical phase $\beta<1$ where the angular momentum takes values of both signs, including zero value.

Using the set of coordinates (3.2.17) we can define a new set of complex variables $\mathfrak{a}$ and $\mathfrak{b}$

$$
\mathfrak{a}^{\varepsilon_{\mathfrak{a}}}=i\,\mathrm{sgn}\,\theta\cdot\sqrt{\tfrac{|1-\beta|}{2|B|}}(\mathbb{P}_1+i\mathbb{P}_2)\,, \qquad \mathfrak{b}^{\varepsilon_{\mathfrak{b}}}=\frac{1}{\sqrt{2|B|}}(\mathcal{P}_1+i\mathcal{P}_2)\,, \tag{3.2.19}
$$

$$
\varepsilon_{\mathfrak{a}}=\mathrm{sgn}\left(B(\beta-1)\right)\,, \qquad \varepsilon_{\mathfrak{b}}=\mathrm{sgn}\,B\,. \tag{3.2.20}
$$

with DPBs

$$
\{\mathfrak{a}^-,\mathfrak{a}^+\}=-i\,, \qquad \{\mathfrak{b}^-,\mathfrak{b}^+\}=-i\,, \qquad \{\mathfrak{a}^\pm,\mathfrak{b}^\pm\}=0\,, \tag{3.2.21}
$$

where $\mathrm{sgn}(A)$ stands for the parameter A sign. Thereby, we encompass all the sign possibilities for B and θ parameters.

Expressing the Hamiltonian and integrals of motion in variable terms (3.2.19), we obtain

$$
\mathbb{H}=|\omega|\,\mathfrak{a}^+\mathfrak{a}^-\,, \qquad \mathbb{M}=\varepsilon_{\mathfrak{a}}\,\mathfrak{a}^+\mathfrak{a}^-+\varepsilon_{\mathfrak{b}}\,\mathfrak{b}^+\mathfrak{b}^-\,, \qquad |\omega|=\frac{|B|}{m|1-\beta|}\,. \tag{3.2.22}
$$

Coordinates (3.2.19) have simple evolution in time given by

$$
\dot{\mathfrak{a}}^\pm=\{\mathfrak{a}^\pm,\mathbb{H}\}=\pm|\omega|i\mathfrak{a}^\pm\,, \qquad \dot{\mathfrak{b}}^\pm=\{\mathfrak{b}^\pm,\mathbb{H}\}=0\,. \tag{3.2.23}
$$

Only the $\mathfrak{a}^\pm$ variables have nontrivial dynamics and their solution is given by

$$
\mathfrak{a}^\pm(t)=e^{\pm i|\omega|t}\mathcal{A}^\pm\,, \qquad \mathcal{A}^\pm:=e^{\mp i|\omega|t}\mathfrak{a}^\pm\,, \tag{3.2.24}
$$

where we obtain the *dynamical*, explicitly depending on time, integrals of motion $\frac{d}{dt}\mathcal{A}^\pm=\partial\mathcal{A}^\pm/\partial t+\{\mathcal{A}^\pm,\mathcal{H}\}=0$.

We can use these dynamical integrals (3.2.24) and the *true*, not depending explicitly on time, integrals $\mathcal{P}_i$ to define quadratic functions. Changing the notation $\mathfrak{b}^\pm\rightarrow\mathcal{B}^\pm$ we define

$$
\mathcal{J}_0=\frac{1}{2}\mathcal{A}^+\mathcal{A}^-\,, \quad \mathcal{J}_\pm=\frac{1}{2}\left(\mathcal{A}^\pm\right)^2\,, \qquad \mathcal{L}_0=\frac{1}{2}\mathcal{B}^+\mathcal{B}^-\,, \quad \mathcal{L}_\pm=\frac{1}{2}\left(\mathcal{B}^\pm\right)^2\,, \tag{3.2.25}
$$

that generate two copies of $\mathfrak{sl}(2,\mathbb{R})$ algebra, $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R})\cong AdS_2\oplus AdS_2\cong\mathfrak{so}(2,2)\cong AdS_3$,

$$
\{\mathcal{J}_0,\mathcal{J}_\pm\}=\mp i\mathcal{J}_\pm\,, \qquad \{\mathcal{J}_-,\mathcal{J}_+\}=-2i\mathcal{J}_0\,, \tag{3.2.26}
$$

$$
\{\mathcal{L}_0,\mathcal{L}_\pm\}=\mp i\mathcal{L}_\pm\,, \qquad \{\mathcal{L}_-,\mathcal{L}_+\}=-2i\mathcal{L}_0\,, \tag{3.2.27}
$$

$$
\{\mathcal{J}_{0,\pm},\mathcal{L}_{0,\pm}\}=0\,. \tag{3.2.28}
$$

The Hamiltonian $\mathbb{H}$ and angular momentum $\mathbb{M}$ generators of the system are functions of the compact generators $\mathcal{J}_0$ and $\mathcal{L}_0$.

$$
\mathbb{H}=2|\omega|\mathcal{J}_0\,, \qquad \mathbb{M}=2\left(\varepsilon_{\mathfrak{a}}\mathcal{J}_0+\varepsilon_{\mathfrak{b}}\mathcal{L}_0\right)\,. \tag{3.2.29}
$$

Setting explicitly $t = 0$ in the time-dependent functions $\mathcal{A}^{\pm}$, we do not modify the algebraic structure in (3.2.21), (3.2.26)–(3.2.28), and (3.2.29). Then, the canonical quantization of the system entails the commutation relations

$$[\hat{\mathcal{A}}^-, \hat{\mathcal{A}}^+] = 1\,, \quad [\hat{\mathcal{B}}^-, \hat{\mathcal{B}}^+] = 1\,, \quad [\hat{\mathcal{A}}^\pm, \hat{\mathcal{A}}^\pm] = 0\,, \quad [\hat{\mathcal{B}}^\pm, \hat{\mathcal{B}}^\pm] = 0\,, \tag{3.2.30}$$

$$[\hat{\mathcal{A}}^\pm, \hat{\mathcal{B}}^\pm] = 0\,, \quad [\hat{\mathcal{A}}^\pm, \hat{\mathcal{B}}^\mp] = 0\,, \tag{3.2.31}$$

Due to $\hat{\mathcal{A}}^\mp$ commute with $\hat{\mathcal{B}}^\mp$ we can construct simultaneous eingenstates of these operators

$$\begin{aligned} \hat{\mathcal{A}}^+\hat{\mathcal{A}}^- \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle &= n_\mathcal{B} \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle \,, \\ \hat{\mathcal{B}}^+\hat{\mathcal{B}}^- \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle &= n_\mathcal{B} \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle \,, \end{aligned} \tag{3.2.32}$$

with $|n_\mathcal{A}, n_\mathcal{B}\rangle = |n_\mathcal{A}\rangle \otimes |n_\mathcal{A}\rangle$. The operators $\hat{\mathcal{A}}^\mp$ and $\hat{\mathcal{B}}^\mp$ satisfy

$$\begin{aligned} \hat{\mathcal{A}}^+ \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle &= \sqrt{n_\mathcal{A}+1}\left|n_\mathcal{A}+1, n_\mathcal{B}\right\rangle \,, \qquad & \hat{\mathcal{A}}^- \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle &= \sqrt{n_\mathcal{A}}\left|n_\mathcal{A}-1, n_\mathcal{B}\right\rangle \,, \\ \hat{\mathcal{B}}^+ \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle &= \sqrt{n_\mathcal{B}+1}\left|n_\mathcal{A}, n_\mathcal{B}+1\right\rangle \,, \qquad & \hat{\mathcal{B}}^- \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle &= \sqrt{n_\mathcal{B}}\left|n_\mathcal{A}, n_\mathcal{B}-1\right\rangle \,. \end{aligned}$$

Thus, we can express any normalized state with the Fock space representation spanned by the states

$$|n_\mathcal{A}, n_\mathcal{B}\rangle = |n_\mathcal{A}\rangle \otimes |n_\mathcal{B}\rangle = \frac{1}{\sqrt{n_\mathcal{A}!\, n_\mathcal{B}!}}\left(\hat{\mathcal{A}}^+\right)^{n_\mathcal{A}}\left(\hat{\mathcal{B}}^+\right)^{n_\mathcal{B}}|0,0\rangle \,, \tag{3.2.33}$$

with eigenvalues

$$\hat{\mathcal{A}}^+\hat{\mathcal{A}}^- \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle = n_\mathcal{A}\left|n_\mathcal{A}, n_\mathcal{B}\right\rangle \,, \qquad \hat{\mathcal{B}}^+\hat{\mathcal{B}}^- \left|n_\mathcal{A}, n_\mathcal{B}\right\rangle = n_\mathcal{B}\left|n_\mathcal{A}, n_\mathcal{B}\right\rangle \,, \quad n_\mathcal{A}, n_\mathcal{A} = 0, 1, \ldots \tag{3.2.34}$$

where $\langle 0,0|0,0\rangle = 1$, and ket $|0,0\rangle$ represent the ground state.

The quantum analogs of (3.2.25) with antisymmetrized ordering in compact generators

$$\hat{\mathcal{J}}_0 = \frac{1}{4}(\hat{\mathcal{A}}^+\hat{\mathcal{A}}^- + \hat{\mathcal{A}}^-\hat{\mathcal{A}}^+)\,, \qquad \hat{\mathcal{L}}_0 = \frac{1}{4}(\hat{\mathcal{B}}^+\hat{\mathcal{B}}^- + \hat{\mathcal{B}}^-\hat{\mathcal{B}}^+)\,, \tag{3.2.35}$$

yields the quantum version of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ algebra (3.2.26)–(3.2.28). Through relationships (3.2.29) quantum states (3.2.33) are eigenstates of the Hamiltonian and angular momentum operators with eigenvalues

$$\hat{\mathcal{H}}\left|n_\mathcal{A}, n_\mathcal{B}\right\rangle = E_{n_\mathcal{A}}\left|n_\mathcal{A}, n_\mathcal{A}\right\rangle \,, \qquad E_{n_\mathcal{A}} = \left(n_\mathcal{A} + \tfrac{1}{2}\right) \,, \tag{3.2.36}$$

$$\hat{\mathbb{M}}\left|n_\mathcal{A}, n_\mathcal{B}\right\rangle = M_{n_\mathcal{A}, n_\mathcal{B}}\left|n_\mathcal{A}, n_\mathcal{B}\right\rangle \,, \qquad M_{n_\mathcal{A}, n_\mathcal{B}} = \varepsilon_\mathcal{A}\left(n_\mathcal{A} + \tfrac{1}{2}\right) + \varepsilon_\mathcal{B}\left(n_\mathcal{B} + \tfrac{1}{2}\right) \,. \tag{3.2.37}$$

We distinguish two different phases of the system. The super-criticial phase $\beta > 1$, where the angular momentum takes nonzero integer values of the magnetic field B sign, $\text{sgn}\, B \cdot M_{n_a, n_b} \in \mathbb{Z}_{>0}$, and the subcritical phase $\beta < 1$, where the angular momentum takes integer values of both signs, including zero value, $M_{n_a, n_b} \in \mathbb{Z}$.

This picture is intimately related to the unitary reducible representation of $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$ explained in Sec.2.3.

3.2.1. Landau Problem

The usual Landau problem can be obtained by fixing $\theta = 0$, $B \neq 0$. The translation generator and Hamiltonian do not suffer any change in their form, while the angular momentum generator turns into

$$\mathbb{M} \to M = \frac{1}{2B}\left(\mathcal{P}_i^2 - \mathbb{P}_i^2\right) . \tag{3.2.38}$$

On the other hand, the coordinate of imaginary particle coincides with the coordinate of the real particle $\mathbb{Y} = \mathbb{X}$. This is just a particular case of NCLP in the sub-critical phase. Hence, its dynamics and canonical quantization are covered by the previous section where once again the ladder coordinates can be used to construct the AdS_3 algebra (3.2.26) and with them, the Hilbert space corresponding to the system.

3.3. NC Free Particle

The case of a free particle in non-commutative plane (NCFP) can be reached by setting $B = 0$, $\theta \neq 0$. In this limit, the vector $\mathbb{P}_i$ coincides with the space translation generator $\mathbb{P}_i = \mathcal{P}_i$.

For this case, it is convenient to express the system in terms of coordinates set (3.2.18) obtaining vector space translation generator

$$\mathcal{P}_i = \frac{1}{\theta}\epsilon_{ij}\left(\mathbb{X}_i - \mathbb{Y}_i\right) . \tag{3.3.1}$$

The Hamiltonian and angular momentum turn into

$$\mathbb{H} \to \mathbb{H}_0 = \frac{1}{2m\theta^2}\left(\mathbb{X}_i - \mathbb{Y}\right)^2 , \qquad \mathbb{M} \to \mathbb{M}_0 = \frac{1}{2\theta}\left(\mathbb{Y}_i^2 - \mathbb{X}_i^2\right) . \tag{3.3.2}$$

Based on the angular momentum form, is convenient to define a new vector coordinate as

$$\mathcal{X}_i = \frac{1}{2}(\mathbb{X}_i + \mathbb{Y}_i) = \mathbb{X}_i + \frac{1}{2}\theta\epsilon_{ij}\mathcal{P}_j . \tag{3.3.3}$$

This vector coordinate, along with the vector translation integral $\mathcal{P}_i$ form a canonical set of variables,

$$\{\mathcal{X}_i, \mathcal{X}_j\} = \{\mathcal{P}_i, \mathcal{P}_j\} = 0 , \qquad \{\mathcal{X}_i, \mathcal{P}_j\} = \delta_{ij} . \tag{3.3.4}$$

From the canonical structure viewpoint, the form of the Hamiltonian and angular momentum are

$$\mathbb{H}_0 = \frac{1}{2m}\mathcal{P}_i^2 , \qquad \mathbb{M}_0 = \epsilon_{ij}\mathcal{X}_i\mathcal{P}_j . \tag{3.3.5}$$

So, the system looks like a free scalar particle in two-dimensional commuting space.

Nonetheless, it is necessary to keep in mind that vector $\mathcal{X}_i$ is an auxiliary coordinate, and the particle coordinate vector $\mathbb{X}_i$ with the non-commuting components is a linear combination of this auxiliary vector $\mathcal{X}_i$ and vector momentum coordinates $\mathbb{X}_i = \mathcal{X}_i - \frac{1}{2}\theta\epsilon_{ij}\mathcal{P}_j$.

Vector space translation generator $\mathcal{P}_i$ and angular momentum $\mathbb{M}_0$ generate the Lie algebra $\mathfrak{e}(2)$ of the two-dimensional Euclidean group

$$\{\mathcal{P}_i, \mathcal{P}_j\} = 0 , \qquad \{\mathbb{M}_0, \mathbb{P}_i\} = \epsilon_{ij}\mathbb{P}_j , \tag{3.3.6}$$

where, anew, we can present the Hamiltonian in terms of the Casimir element

$$\mathcal{C}_0 = \frac{1}{2}\mathcal{P}_i^2 , \qquad \mathbb{H}_0 = \frac{1}{m}\mathcal{C}_0 \tag{3.3.7}$$

Integrals (3.3.5) together with the vector space translation $\mathcal{P}_i$ generate the $\mathfrak{e}(2)\oplus\mathfrak{u}(1)$ algebra.

If we take into account $\mathbb{X}_i$ as the vector particle coordinate, we can construct an exotic Galilean boosts, which transform this vector in a covariant form,

$$\mathbb{G}_i = m\mathbb{X}_i - t\mathcal{P}_i + m\theta\epsilon_{ij}\mathcal{P}_j\,, \qquad \{\mathbb{G}_i,\mathbb{G}_j\} = m^2\theta\epsilon_{ij}\,, \qquad \{\mathbb{X}_i,\mathbb{G}_j\} = -\delta_{ij}t\,. \tag{3.3.8}$$

The $\mathbb{G}_i$ generators, are NCPF dynamical integral, $\frac{d}{dt}\mathbb{G}_i = \frac{\partial\mathbb{G}_i}{\partial t} + \{\mathbb{G}_i,\mathbb{H}_0\} = 0$.

Furthermore, coordinate $\mathcal{X}_i$ does not transform covariantly under this exotic Galilean boosts, $\{\mathcal{X}_i,\mathbb{G}_j\} = -\delta_{ij}t - m\theta\epsilon_{ij}$ and hence plays a role analogous to the Newton-Wigner coordinate for a Dirac particle [37, 41, 51, 52]. The vector $\mathbb{Y}_i$, that can be express as $\mathbb{Y}_i = \mathcal{X}_i + \frac{1}{2}\theta\epsilon_{ij}\mathcal{P}_j = \mathbb{X}_i + \theta\epsilon_{ij}\mathcal{P}_j$, also does not transform covariantly under the action of the exotic Galilean boosts, $\{\mathbb{Y}_i,\mathbb{G}_j\} = -\delta_{ij}t - m\theta\epsilon_{ij}$.

We can employ the auxiliary vector $\mathcal{X}_i$ to solve the EOM obtaining solution $\mathcal{X}^i(t) = \mathcal{X}^i_0 + \frac{1}{m}\mathcal{P}^i t$. This solution provides the vector dynamical integral $\mathcal{X}^i_0 = \mathcal{X}^i(t) - \frac{1}{m}\mathcal{P}^i t$ that can be used to construct quadratic dynamical integrals

$$\mathbb{D} = \mathcal{X}^i_0\mathcal{P}^i = \mathcal{X}^i\mathcal{P}^i - 2\mathbb{H}_0 t\,, \qquad \mathbb{K} = \frac{m}{2}\mathcal{X}^i_0\mathcal{X}^i_0 = \frac{m}{2}\mathcal{X}^i\mathcal{X}^i - \mathbb{D}t - \mathbb{H}_0 t^2\,, \tag{3.3.9}$$

which combined with Hamiltonian $\mathbb{H}_0$, generate the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$\{\mathbb{D},\mathbb{H}_0\} = 2\mathbb{H}_0\,, \qquad \{\mathbb{D},\mathbb{K}\} = -2\mathbb{K}\,, \qquad \{\mathbb{K},\mathbb{H}_0\} = \mathbb{D}\,, \tag{3.3.10}$$

whose Casimir is proportional to the angular momentum (3.3.2)

$$\mathbb{C} = \mathbb{H}_0\mathbb{K} - \frac{1}{2}\mathbb{D}^2 = \frac{1}{4}\mathbb{M}_0^2\,. \tag{3.3.11}$$

These quadratic integrals, defined in (3.3.9), constitute the dilatation and conformal special transformation respectively and are Noetherian integrals of motion, which can be provided from the quasi-invariance of the first order Lagrangian. Setting $t = 0$, relationships (3.3.10) remain, and particularly, we can rewrite the dilatation generator in terms of the “imaginary mirror particle" $\mathbb{Y}_i$ obtaining

$$\tilde{\mathbb{D}} = \mathcal{X}_i\mathcal{P}_i = -\frac{1}{\theta}\epsilon_{ij}\mathbb{X}_i\mathbb{Y}_j\,. \tag{3.3.12}$$

The $\mathbb{X}^i$ and $\mathbb{Y}^i$ coordinates can be combined into the “isospace" vector

$$\mathbb{X}^i_a = (\mathbb{X}^i,\mathbb{Y}^i)\,, \tag{3.3.13}$$

finding that in index $a = 1, 2$, this coordinate behaves like a (1+1)-dimensional Lorentz vector with respect to the global transformations generated by $\tilde{\mathbb{D}}$

$$\mathbb{X}'^i = \cosh\alpha\,\mathbb{X}^i - \sinh\alpha\,\mathbb{Y}^i\,, \qquad \mathbb{Y}'^i = \cosh\alpha\,\mathbb{Y}^i - \sinh\alpha\,\mathbb{X}^i\,. \tag{3.3.14}$$

The application of this generator over the phase space canonical variables $\mathcal{P}_i$ and $\mathcal{X}_i$ produces rescaling transformations $\mathcal{P}'_i = e^{\alpha}\,\mathcal{P}_i$, $\mathcal{X}'_i = e^{-\alpha}\,\mathcal{X}_i$.

This particularly interesting property can be reached noticing the Lagrangian (3.0.1) with $B = 0$ and $\theta \neq 0$ can be rewritten as

$$\mathbb{L}_0 = \frac{1}{\theta}\epsilon_{ij}\left(\mathbb{Y}^i\dot{\mathbb{Y}}^j - \mathbb{X}^i\dot{\mathbb{X}}^j\right) - \frac{1}{2m\theta^2}(\mathbb{X}^i - \mathbb{Y}^i)^2\,, \tag{3.3.15}$$

where the total time derivative term $\frac{d}{dt}\left(-\frac{1}{2\theta}\epsilon_{ij}\mathbb{X}_i\mathbb{Y}_j\right)$ has been omitted.

The first order in time derivative Lagrangian term can be presented equivalently as $\frac{1}{2\theta}\epsilon_{ij}\eta^{ab}\mathbb{X}^i_a\dot{\mathbb{X}}^j_b$, where $\eta_{ab} = \mathrm{diag}\,(-1,1)$ is the (1+1)D Minkowski metric. This term is invariant under global (1+1)D Lorentz boosts transformations of $\mathbb{X}^i_a$ in the "isospace" corresponding to index a.

With the "isospace" Minkowski metric η_{ab}, the angular momentum integral (3.3.2) takes the form $\mathbb{M}_0 = \frac{1}{2}\gamma\eta^{ab}\delta_{ij}\mathbb{X}^i_a\mathbb{X}^j_b$, and the non-commutative free particle can be described by the $\mathfrak{so}(1,1)\oplus\mathfrak{so}(2)$ algebra, which is generated by $\tilde{\mathbb{D}}$ and $\mathbb{M}_0$.

These two generators, along with $\mathbb{H}_0$ and $\mathbb{K}$, will play a key role in the Conformal Bridge Transformation (Sec.7).

This picture of emergent "isospace" is explained in Sec.2.1 by means of a canonical transformation generated by the compact generator flow of the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra.

Finally, at the quantum level the non-covariant coordinate with respect to the exotic Galilean boosts vector, $\mathcal{X}_i$, allows us to work in a diagonal representation in the operator $\hat{\mathcal{X}}_i$, in which $\hat{\mathcal{P}}_j = -i\partial/\partial\mathcal{X}_j$, $\hat{\mathbb{X}}_j = \mathcal{X}_j + i\frac{1}{2}\theta\,\epsilon_{jk}\partial/\partial\mathcal{X}_k$, and $\hat{\mathbb{Y}}_j = \mathcal{X}_j - i\frac{1}{2}\theta\,\epsilon_{jk}\partial/\partial\mathcal{X}_k$. Then, we find the wave eigenfunctions

$$\psi_{\mathcal{P}}(\mathcal{X}) = \frac{1}{2\pi}\exp(i\mathcal{X}_j\mathcal{P}_j)\,, \qquad E_{\mathcal{P}} = \frac{1}{2m}\mathcal{P}_i^2\,, \tag{3.3.16}$$

of the Hamiltonian operator $\hat{\mathbb{H}}_0$, where $\mathcal{P}_i$ is eigenvalue of the momentum operator.

4. Vortex dynamics

The properties of those portions of fluid in which vorticity occurs were illustrated by Helmholtz in his seminal paper of 1858. Picturing the vorticity confined to a set of infinitely thin, straight, parallel filaments, each of which carries an invariant amount of “strength", he derived a set of equations of motion for the intersection interacting points between these filaments and the perpendicular plane to all of them. These intersection were defined as point vortices.

This dynamical model can be obtained in different ways, for example, the discretization of the continuum equations for 2D inviscid flow, referred as Euler equations, lead to the point vortices equations (See Appendix C). Properties of the fractional quantum Hall effect can be explained in terms of fractional statistics of the corresponding quasiparticle (anyon) excitations [53, 54]. These anyons can be realized in the form of point vortices through a mechanism of statistics transmutation which employs the Chern-Simons gauge theory [55, 56]. The idealization of a two dimensional ideal flow as a collection of point vortex yields a wide variety of studies in superconductivity, superfluidity, and Bose-Einstein condensate physics [57, 58, 59, 60, 61], just to mention a few.

The $N \geq 2$ point vortex dynamics on a plane can be described by Lagrangian [62, 63]

$$L = -\frac{1}{2}\sum_{n=1}^{N} \gamma_n \epsilon_{ij} x_n^i \dot{x}_m^j + \frac{1}{2}\sum_{n<m}^{N} \gamma_n \gamma_m \log r_{nm}^2 \,, \tag{4.0.1}$$

constituted by first order in time derivative (0+1)-dimensional Chern-Simons term and logarithm ‘potential term’, where ϵ_{ij} is an antisymmetric tensor ($\epsilon_{12} := 1$) and the n-th vortex coordinates are given by $\vec{x}_n = (x_n^1, x_n^2)$. Each vortex has a strength γ_n. If $n \neq m \to \vec{x}_n \neq \vec{x}_m$, and consequently, $r_{nm}^2 = (\vec{x}_n - \vec{x}_m)^2 > 0$.

For the sake of simplicity, the vortex coordinates and their strengths γ_n are assumed to be dimensionless.

Analogously to NCLP, we deal with a singular Lagrangian whose canonical momenta $p_n^i = \partial L / \partial \dot{x}_n^i$ and coordinates x_n^i, with Poisson brackets

$$\{x_n^i, p_m^j\} = \delta_{nm}\delta^{ij} \,, \qquad \{x_n^i, x_m^j\} = \{p_n^i, p_m^j\} = 0 \,, \tag{4.0.2}$$

are subject to the set of primary second class constraints

$$\phi_n^i = p_n^i - \frac{1}{2}\delta_{nm}\gamma_m \epsilon_{ij} x_m^j \approx 0 \,, \tag{4.0.3}$$

that satisfy relationships

$$\{\phi_n^i, \phi_m^j\} = -\delta_{nm}\gamma_m \epsilon_{ij} \,. \tag{4.0.4}$$

Equations above, indicate the $N \times N$ matrix Δ, obtained from them has non zero determinant $det|\Delta| \neq 0$ for $\gamma_n \neq 0$, and we can always fix the arbitrary functions $u_n^i(t)$ that are imposed in the primary Hamiltonian

$$H = -\frac{1}{2}\sum_{n<m} \gamma_n \gamma_m \log r_{nm}^2 + \sum_{n=1} u_n^i \phi_n^i \,, \tag{4.0.5}$$

in such way the constrains (4.0.3) hold with the time evolution. Then, we arrive the subspace described by the symplectic two-form $\sigma = \frac{1}{2}\gamma_n \epsilon_{ij} dx_n^i \wedge dx_n^j$ and corresponding Dirac-Poisson brackets

$$\{x_n^i, x_m^j\} = \gamma_n^{-1} \delta_{nm} \epsilon^{ij} \,. \tag{4.0.6}$$

The Hamiltonian turns into

$$H = -\frac{1}{2} \sum_{n<m} \gamma_n \gamma_m \log r_{nm}^2 \,, \tag{4.0.7}$$

and together with (4.0.6) produces the EOM

$$\dot{x}_n^i = -\epsilon^{ij} \sum_{n \neq m} \gamma_m (x_n^j - x_m^j) r_{nm}^{-2} \,, \tag{4.0.8}$$

that can also be obtained following directly from Lagrangian (4.0.1). Let us note that Kirchhoff was the first who showed that equations 4.0.8 have a Hamiltonian nature given by Eqs. 4.0.6 and 4.0.7, see [62, 63]. Symplectic structure 4.0.6 can also be obtained by setting $m = 0$ in Landau problem, see [70].

The Noetherian integrals of motion of the system (4.0.1) are

$$P_i = \sum_{n=1} \epsilon_{ij} \gamma_n x_n^j \,, \qquad M = -\frac{1}{2} \sum_{n=1} \gamma_n \vec{x}_n^{\,2} \,, \tag{4.0.9}$$

which, generate translations, $\{P_i, x_a^j\} = -\delta_{ij}$, and rotations, $\{M, x_i^a\} = \epsilon_{ij} x_a^j$. Analogously to the NCLP, these generators form a centrally extended Lie algebra $\mathfrak{e}_\Gamma(2)$ of the two-dimensional Euclidean group,

$$\{P_i, P_j\} = \Gamma \epsilon_{ij} \,, \qquad \{M, P_i\} = \epsilon_{ij} P_j \,, \qquad \Gamma = \sum_{n=1} \gamma_n \,, \tag{4.0.10}$$

where the total vorticity of the system Γ plays the role of the central charge. Casimir element of the algebra $\mathfrak{e}_\Gamma(2)$ is

$$\mathcal{C}_\Gamma = P_i^2 + 2\Gamma M = -\sum_{n<m} \gamma_n \gamma_m r_{nm}^2 \,. \tag{4.0.11}$$

Integrals (4.0.9) together with Hamiltonian (4.0.7) generate the $\mathfrak{e}_\Gamma(2) \oplus \mathfrak{u}(1)$ algebra.

System (4.0.6), (4.0.7) is maximally superintegrable in the case of $N = 2$ vortices, and completely integrable for $N = 3$ in the sense of Liouville. For $N \geq 4$ it is not integrable [63].

In the general case of N vortices, equations of motion (4.0.8) are invariant under rescaling $x_a^i \to e^\alpha x_a^i$, $t \to e^{2\alpha} t$. With respect to these transformations, Lagrangian is quasi-invariant $L \to L + \frac{d}{dt}(Ct)$, $C = \alpha \sum_{n<m} \gamma_n \gamma_m$, but the corresponding action $S = \int L dt$ is rescaled, $S \to e^{2\alpha} S$. In this sense, this symmetry in the equations of motion is not Noetherian.

4.1. Two-vortex with nonzero total vorticity

Now, we focus our attention in the $N = 2$ superintegrable system given by Hamiltonian

$$H_\Gamma = -\frac{1}{2} \kappa \log r_{12}^2 \,, \qquad \kappa := \gamma_1 \gamma_2 \,. \tag{4.1.1}$$

From (4.0.11), we recognize the Hamiltonian as a function of the Casimir element of (4.0.10) algebra

$$H_\Gamma = -\frac{1}{2}\kappa \log\left(-\kappa^{-1}\mathcal{C}_\Gamma\right) . \tag{4.1.2}$$

Looking at (4.0.10), it is intuitive to suppose there are differences in the dynamics whether we fix the total vorticity parameter equal zero or not. In this section, we will consider the two-vortex system for which the total vorticity $\Gamma \neq 0$, while the next section will be guided to the study of the system for which the total vorticity $\Gamma = 0$.

When $\Gamma \neq 0$ it is convenient to describe the four-dimensional symplectic manifold by the vector integral $\vec{P}$ and the relative coordinate vector $\vec{r}$

$$P_i = \epsilon_{ij}\left(\gamma_1 x_1^j + \gamma_2 x_2^j\right) , \qquad r_{12}^i = x_1^i - x_2^i := r^i , \qquad r^i r^i > 0 . \tag{4.1.3}$$

Their DPBs are

$$\{r_i, r_j\} = \varrho^{-1}\epsilon_{ij} , \qquad \{P_i, P_j\} = \Gamma\epsilon_{ij} , \qquad \{r_i, P_j\} = 0 , \qquad \varrho = \frac{1}{\frac{1}{\gamma_1}+\frac{1}{\gamma_2}} = \frac{\kappa}{\Gamma} , \tag{4.1.4}$$

where we define ϱ as a "reduced vorticity". In these coordinates, the EOM of the system are given in simple terms

$$\dot{P}_i = 0 , \qquad \dot{r}_i = -\omega\epsilon_{ij} r_j , \qquad \omega = \Gamma R^{-2} , \qquad R^2 = r_i r_i , \tag{4.1.5}$$

and their solution lead to the circular motion dynamics

$$r_1(t) = R\cos\omega(t-t_0) , \quad r_2(t) = R\sin\omega(t-t_0) , \qquad R = \exp\left(-\kappa^{-1}H_\Gamma\right) , \tag{4.1.6}$$

which can also be expressed in terms of the vortex pair coordinates as

$$x_a^i(t) = X_0^i + \tfrac{1}{\Gamma}\epsilon_{ab}\gamma_b r^i(t) , \qquad X_0^i = -\tfrac{1}{\Gamma}\epsilon^{ij}P_j , \tag{4.1.7}$$

Following Eq. (4.1.7) we find the trajectories correspond to circular motion given by the radii $R_1 = |\gamma_2\Gamma^{-1}|R$ and $R_2 = |\gamma_1\Gamma^{-1}|R$ sharing the same guiding center $\vec{X}_0$, however, we face two types of dynamics depending on the sign of κ. In case of different sign strengths $\kappa < 0$, the vortices perform their circular dynamics at the closest points in their respective circles, while for vortices of same strengths sign $\kappa > 0$ remain at the farthest points of their respective circles. In the case of vortices with the same strengths $\gamma_1 = \gamma_2$, they move on the same circle of radius $R/2$. These three cases are illustrated by Figures 4.1, 4.2 and 4.3 respectively.

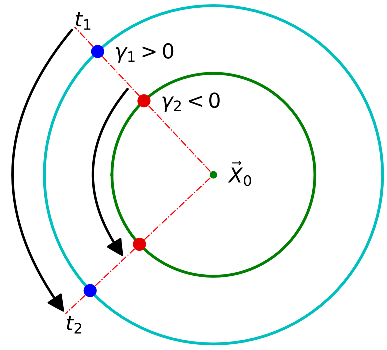

Figure 4.1.: Two-vortex with $\kappa = \gamma_1\gamma_2 < 0$.

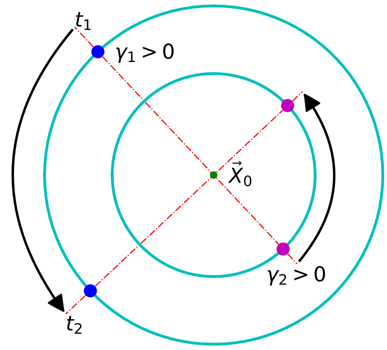

Figure 4.2.: Two-vortex with $\kappa = \gamma_1\gamma_2 > 0$.

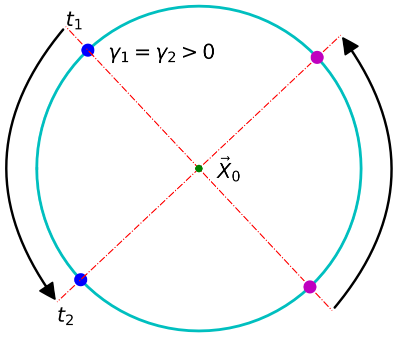

Figure 4.3.: Two-vortex with $\gamma_1 = \gamma_2$.

Using coordinates (4.1.3) we can express the angular momentum integral as

$$M_\Gamma = -\frac{1}{2\Gamma}\left(P_i^2 + \kappa R^2\right) . \tag{4.1.8}$$

It may take arbitrary values, $M_\Gamma \in \mathbb{R}$ when $\kappa < 0$ and nonzero values of the sign $-\varepsilon_\Gamma = -\mathrm{sgn}\,\Gamma$, in case of $\kappa > 0$. For the last case, the angular momentum can take zero value, this occur when the modulus of the vector corresponding to the guiding center equals the geometric mean of the radii of the circles $|\vec{X}_0| = \sqrt{R_1 R_2} = \sqrt{|\varrho|}\,R$.

Analogously to the NCLP, we define complex valued variables

$$\mathcal{A}^{-\varepsilon_\varrho} = \sqrt{\frac{|\varrho|}{2}}\left(r_1 + i r_2\right) , \qquad \mathcal{B}^{-\varepsilon_\Gamma} = \frac{1}{\sqrt{2|\Gamma|}}\left(P_1 + i P_2\right), \qquad \varepsilon_\varrho = \mathrm{sgn}\,\varrho\,, \quad \varepsilon_\Gamma = \mathrm{sgn}\,\Gamma\,, \tag{4.1.9}$$

with DPBs

$$\{\mathcal{A}^-, \mathcal{A}^+\} = -i\,, \qquad \{\mathcal{B}^-, \mathcal{B}^+\} = -i\,, \qquad \{\mathcal{A}^\pm, \mathcal{B}^\pm\} = 0\,. \tag{4.1.10}$$

Quadratic integrals similar to (3.2.26) can be constructed from these complex variables leading to $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \cong AdS_3$ algebra. Thus, we obtain

$$H_\Gamma = -\frac{1}{2}\kappa \log\left(4|\varrho|^{-1}\mathcal{J}_0\right) , \qquad M_\Gamma = -2\left(\varepsilon_\varrho \mathcal{J}_0 + \varepsilon_\Gamma \mathcal{L}_0\right) , \tag{4.1.11}$$

where the integral $\mathcal{J}_0$ is related to the Casimir element (4.0.11) of the $\mathfrak{e}_\Gamma(2)$ algebra, $\mathcal{J}_0 = \frac{1}{4}|\Gamma^{-1}\mathcal{C}_\Gamma|$.

Hamiltonian, together with the dynamical integrals $\mathcal{J}_\pm$, generate a nonlinear deformation of the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra,

$$\{\mathcal{J}_-, \mathcal{J}_+\} = \frac{1}{2} i |\varrho| R^2\,, \qquad \{H_\Gamma, \mathcal{J}_\pm\} = \pm 2\, i\, \mathrm{sng}\kappa\, R^{-2}\, \mathcal{J}_\pm\,, \qquad R^2 = \exp\!\left(-2H_\Gamma\kappa^{-1}\right). \tag{4.1.12}$$

In accordance with (4.1.11), the quantum states (3.2.33) are eigenstates of the Hamiltonian and angular momentum operators, with eigenvalues

$$\hat{H}_\Gamma\,|n_a, n_b\rangle = E_{n_a}\,|n_a, n_b\rangle \ , \qquad E_{n_a} = -\tfrac{1}{2}\kappa \log\left(2|\varrho|^{-1}\left(n_a + \tfrac{1}{2}\right)\right) , \tag{4.1.13}$$

$$\hat{M}_\Gamma\,|n_a, n_b\rangle = M_{n_a,n_b}\,|n_a, n_b\rangle \ , \qquad M_{n_a,n_b} = -\varepsilon_\varrho\left(n_a + \tfrac{1}{2}\right) - \varepsilon_\Gamma\left(n_b + \tfrac{1}{2}\right) . \tag{4.1.14}$$

Considering $\kappa < 0$, the energy eigenvalues are bounded from below, and the angular momentum eigenvalues take integer values $M_{n_a,n_b} \in \mathbb{Z}$. Instead, considering $\kappa > 0$ the energy eigenvalues are bounded from above, and the angular momentum takes nonzero integer values of the sign of $(-\Gamma)$, $-\varepsilon_\Gamma M_{n_a,n_b} \in \mathbb{Z}_{>0}$, We find that supercritical (chiral) phase of the NCLP is analogous to $\kappa > 0$ case, while subcritical (non-chiral) phase is analogous to $\kappa < 0$.

4.2. Two-vortex with zero total vorticity

The case of zero total vorticity of the system $\Gamma = 0$, correspond to the vortex-antivortex dynamics. If we denote $\gamma_1 = \gamma$, $\gamma_2 = -\gamma + \Gamma$, taking the limit $\Gamma \to 0$ we obtain

$$\lim_{\Gamma\to 0} P_i = \gamma\epsilon_{ij} r_j := \Pi_i\,, \qquad r^i = x_1^i - x_2^i\,. \tag{4.2.1}$$

We introduce a new set of linear independent of Π_i coordinates set of variables, in the four-dimensional symplectic manifold, as

$$\chi^i = \frac{1}{2}(x_1^i + x_2^i)\,. \tag{4.2.2}$$

Vector (4.2.2) is in accordance with the limit $\Gamma \to 0$ applied to the arithmetic mean of the vortex coordinates given by Eq. (4.1.7). In this sense, χ_i and Π_i form the canonical set of variables with DPBs

$$\{\chi_i, \Pi_j\} = \delta_{ij}\,, \qquad \{\chi_i, \chi_j\} = \{\Pi_i, \Pi_j\} = 0\,. \tag{4.2.3}$$

The Hamiltonian (4.1.1) and angular momentum reduce to

$$H_\Gamma \to H_0 = \frac{1}{2}\gamma^2 \log\left(\gamma^{-2}\Pi_i^2\right), \qquad M_\Gamma \to M_0 = \epsilon_{ij}\chi_i\Pi_j\,, \tag{4.2.4}$$

correspondingly. These integrals, analogously to the NCFP, generate the $\mathfrak{e}(2)\oplus\mathfrak{u}(1)$ algebra, while the Casimir of $\mathfrak{e}(2)$ subalgebra becomes $\mathcal{C}_0 = \Pi_i^2 > 0$.

The solution of EOM $\dot{\Pi}_i = 0\,,\ \dot{\chi}_i = \gamma^2\Pi_i/\vec{\Pi}^2$, are given by

$$\chi^i(t) = \chi_0^i + \gamma^2\frac{\Pi^i}{\vec{\Pi}^2}t\,, \qquad \Pi^i = const\,, \tag{4.2.5}$$

and terms of the vortices coordinates are

$$x_1^i(t) = \chi_0^i - \frac{1}{2\gamma}\epsilon^{ij}\Pi_j + \gamma^2\frac{\Pi^i}{\vec{\Pi}^2}t\,, \qquad x_2^i(t) = \chi_0^i + \frac{1}{2\gamma}\epsilon^{ij}\Pi_j + \gamma^2\frac{\Pi^i}{\vec{\Pi}^2}t\,. \tag{4.2.6}$$

These equations describe rectilinear motion of the vortex-antivortex pair. They move maintaining a fixed distance between them given by $|\vec{r}| = |\vec{\Pi}/\gamma| = R$ and fixed speed inverse to this distance $|\dot{\vec{\chi}}| = |\gamma|/R = |\gamma|\exp\left(-H_0/\gamma^2\right)$.

These trajectories are illustrated in Figure 4.4

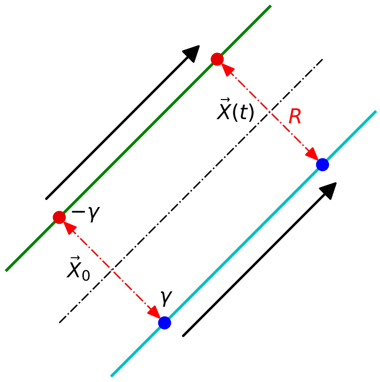

Figure 4.4.: Two-vortex with $\gamma_1 = -\gamma_2$.

The rectilinear orbits of the vortex-antivortex system can be obtained by taking the limit $\Gamma \to 0$ into the EOM solutions (4.1.7) when $\kappa < 0$, illustrated on Figure 4.1. For example, we may consider

$$t_0 = 0\,, \qquad \gamma_1 = \gamma\,, \qquad \gamma_2 = -\gamma + \Gamma\,, \qquad P_1 = \mu \Gamma R\,, \qquad P_2 = \nu \Gamma R - \gamma\,, \tag{4.2.7}$$

where μ, ν are arbitrary constants in $\mathbb{R}$. Now, by taking $\Gamma \to 0$ in the circular trajectories given by Eqs. (4.1.6), (4.1.7), we obtain rectilinear trajectories of the vortex-antivortex system given by Eq. (4.2.6) with

$$\chi_0^1 = (\frac{1}{2} - \nu)R\,, \qquad \chi_0^2 = \mu R\,, \qquad \Pi_1 = 0,, \qquad \Pi_2 = -\gamma R\,. \tag{4.2.8}$$

The EOM solution provide the dynamical integrals of motion

$$\chi_0^i = \chi^i - \gamma^2 (\Pi^i / \vec{\Pi}^2) t\,, \tag{4.2.9}$$

that can be employed to construct two dynamical integrals of motion

$$D = \chi_0^i \Pi^i = \chi^i \Pi^i - \gamma^2 t\,, \tag{4.2.10}$$

$$K = \tfrac{1}{2} \vec{\chi}_0^{\,2} = \tfrac{1}{2} \vec{\chi}^{\,2} - \left(\gamma^{-1} D t + \tfrac{1}{2} t^2\right) \exp\left(-2\gamma^{-2} H_0\right)\,. \tag{4.2.11}$$

These integrals along with the integral

$$\mathcal{H}_0 = \tfrac{1}{2} \vec{\Pi}^2 = \tfrac{1}{2} \gamma^2 \exp\bigl(2H_0/\gamma^2\bigr)\,, \tag{4.2.12}$$

generate the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$\{D, \mathcal{H}_0\} = 2\mathcal{H}_0\,, \qquad \{D, K\} = -2K\,, \qquad \{K, \mathcal{H}_0\} = D\,, \tag{4.2.13}$$

whose Casimir is proportional to the angular momentum

$$\mathcal{C}_0 = \mathcal{H}_0 K - \frac{1}{2} D^2 = \frac{1}{4} M_0^2. \tag{4.2.14}$$

It is important to recall this algebra is obtained from $\mathcal{H}_0$ integral. If we employ Hamiltonian generator H_0 instead of $\mathcal{H}_0$ integral we produce the deformed algebra

$$\{D, H_0\} = \gamma^2 \,, \qquad \{K, H_0\} = D \exp\left(-2\gamma^{-2} H_0\right) \,. \tag{4.2.15}$$

The obtained dynamical integrals of motion, χ_0^i, D and K are Noetherian, which can be obtained from the quasi-invariance of the first order Lagrangian

$$L = \dot{\chi}^i \Pi^i - \frac{1}{2}\gamma^2 \log\left(\Pi_i^2/\gamma^2\right) , \tag{4.2.16}$$

under corresponding transformations generated by them.

Completely analogous to NCFP, the first order in time derivative Lagrangian term can be rewritten equivalently as $\frac{1}{2}\gamma\epsilon_{ij}\eta^{ab}x_a^i\dot{x}_b^j$, where $\eta_{ab} = \text{diag}\,(-1, 1)$ is the (1+1)D Minkowski metric in indexes $a, b = 1, 2$. The dynamical integral D generate Lorentz boosts transformations of x_a^i in the “isospace" corresponding to index a,

$$x_1'^i = \cosh\alpha\, x_1^i - \sinh\alpha\, x_2^i \,, \qquad x_2'^i = \cosh\alpha\, x_2^i - \sinh\alpha\, x_1^i \,, \tag{4.2.17}$$

The set of canonical variables Π_i and χ_i rescales $\Pi_i' = e^{\alpha}\,\Pi_i$, $\chi_i' = e^{-\alpha}\,\chi_i$ under these transformations.

Again, this ”isospace" emerges analogously to the NCFP (3.3.13).

The canonical quantization of the system entails DPBs relations (4.2.3), so we can work in the representation where the operators $\hat{\chi}_j$ are diagonal and $\hat{\Pi}_j = -i\,\partial/\partial\chi_j$. Therefore, we find that the plane wave functions

$$\psi_{\Pi}(\xi) = \frac{1}{2\pi}\exp(i\chi_j\Pi_j) \tag{4.2.18}$$

are the normalized for delta function eigenstates of the vector space translation operator integral $\hat{\Pi}_j\psi_{\Pi}(\chi) = \Pi_j\psi_{\Pi}(\chi)$, and Hamiltonian $\hat{H}_0$ operator integral takes on them real eigenvalues $E_{\Pi} = \frac{1}{2}\gamma^2\log\left(\gamma^{-2}\Pi_i^2\right)$.

It is important to remark due to the commutation in components $[\hat{x}_a^j, \hat{x}_b^k] = i(-1)^a\gamma^{-1}\delta_{ab}\epsilon^{jk}$, the Heisenberg uncertainty relation prevents to know simultaneously both coordinates of each vortex.

This is also true in the two-vortex system of nonzero total vorticity and it is a characteristic of the non-commutative space systems.

5. NCLP & Two-Vortex correspondence

Now, we are in position to compare the studied systems. One can establish, up to the permutation $1 \leftrightarrow 2$ of the vortices, the correspondence between the NCLP and the two-vortex system as

$$\mathbb{X}^i \sim x_1^i\,, \quad \mathbb{Y}^i \sim x_2^i\,, \qquad \mathbb{P}_i \sim -\gamma_2 \epsilon_{ij} r^j\,, \qquad \mathcal{P}_i \sim P_i\,, \qquad \mathbb{M} \sim M_\Gamma\,, \tag{5.0.1}$$

$$\gamma_1 \sim \tfrac{1-\beta}{\theta}\,, \qquad \gamma_2 \sim -\tfrac{1}{\theta}\,, \qquad \Gamma \sim -B\,, \qquad \varrho^{-1} \sim \theta \tfrac{\beta}{1-\beta}\,, \qquad \tfrac{\gamma_1}{\gamma_2} \sim \beta - 1\,. \tag{5.0.2}$$

Nevertheless, there is an important difference between these two systems reflected in linear dependence of $\mathbb{H}$ on $(\mathbb{X}_i - \mathbb{Y}_i)^2$ (3.2.18), but logarithmic dependence of H_Γ on $(x_1^i - x_2^i)^2 = R^2$ (4.1.1). Because of this essential difference their rotational frequency, on one hand, for the NCLP, is given in terms of the parameters θ, B and m, $\omega = \frac{B}{m(1-\beta)}$, while on the other hand, for the two-vortex system, is given in terms of the energy, $\omega = \Gamma \exp(2H_\Gamma/\gamma_1\gamma_2)$.

From the last relation in (5.0.2) we obtain that the sub-critical (non-chiral) phase $\beta < 1$ corresponds to the vortex pair with the opposite sign case $\kappa < 0$, while the super-critical (chiral) phase $\beta > 1$ corresponds to the vortex pair of the same sign case $\kappa > 0$.

The correspondence with the usual Landau problem, $\theta = 0$, $B \neq 0$, is also covered by relations (5.0.1), (5.0.2). By performing a change of variables, $(x_1^i, x_2^i) \to (x^i, \pi^i)$ in the $\Gamma \neq 0$ system we obtain

$$x^i = x_1^i \quad x_2^i = x^i - \frac{1}{\gamma_2}\epsilon_{ij}\pi_j. \tag{5.0.3}$$

Now, we redefine the Hamiltonian by shifting and rescaling

$$H_\Gamma \to h_\Gamma = \frac{\gamma^2}{\gamma_2^2}(H_\Gamma + \log \gamma_2^2)\,, \tag{5.0.4}$$

where γ belongs to $\mathbb{R}$. According to relations (5.0.1), (5.0.2), the ordinary Landau problem ($\theta = 0$) corresponds to take the limit $\gamma_2 \to \infty$ obtaining

$$\begin{gathered} x_2^i = x_1^i = x^i \sim \mathbb{X}^i \qquad \pi_i \sim \mathbb{P}_i \\ P_i = \pi_i + \Gamma \epsilon_{ij} x_j \sim \mathcal{P}_i \\ M_\Gamma = \epsilon_{ij} x_i \pi_j - \tfrac{1}{2}\Gamma x_i^2 \sim \mathbb{M}_0 \\ \{x_i, x_j\} = 0 \qquad \{x_i \pi_j\} = \delta_{ij} \qquad \{\pi_i, \pi_j\} = -\Gamma \epsilon_{ij} \end{gathered} \tag{5.0.5}$$

where the time evolution of the system is given by the “regularized" Hamiltonian

$$h_\Gamma = \frac{1}{2}\gamma^2 \log \pi_i^2\,. \tag{5.0.6}$$

The NCFP ($B = 0\,, \theta \neq 0$) corresponds to the vortex-antivortex system, $\Gamma = 0$.

Finally, the critical case $\beta = 1$ corresponds to the two-vortex system with one of the strengths set equal to zero. By putting $\gamma_2 = 0$, we obtain the two second-class constraints (4.0.3) $\phi_1^i \approx 0$ and two first-class constraints $\phi_2^i = p_2^i \approx 0$. The latter pair of constraints means the coordinates associated with 2 index are pure gauge, consequently, the reduction of the system to the surface of the second class constraints $\phi_1^i \approx 0$ results in the two-dimensional symplectic manifold described by coordinates x_1^i with DPBs $\{x_1^i, x_1^j\} = \gamma_1^{-1}\epsilon_{ij}$ and zero Hamiltonian $H = 0$. c.f. (3.1)

5.1. Weak-strong duallity

For the two-vortex system, the permutation of pair vortex constitutes a symmetry of the nature of inversion, $\mathcal{I}^2 = id$.

$$\mathcal{I}:\ x_1^i \leftrightarrow x_2^i\,, \qquad \gamma_1 \leftrightarrow \gamma_2\,, \tag{5.1.1}$$

It acts as a reflection on the relative coordinate $r^i = x_1^i - x_2^i$, $\mathcal{I}:\, r^i \to -r^i$ where the energy, the momentum vector and the angular momentum integral remain invariant.

In the context of the NCLP, the correspondence (5.0.1), (5.0.2) leads to the analog of transformation (5.1.1),

$$\mathcal{I}:\ \mathbb{X}_i \leftrightarrow \mathbb{Y}_i\,, \quad B \to B\,, \quad \theta \to \frac{\theta}{\beta - 1}\ \Rightarrow\ (\beta-1) \to (\beta-1)^{-1}\,, \qquad \mathcal{I}^2 = id\,. \tag{5.1.2}$$

Under this transformation the first DPBs in (3.2.2) and (3.2.14) mutually transform one into another, mantaining the symplectic two-form $\sigma = -\frac{1}{2\theta}\epsilon_{ij}((1-\beta)d\mathbb{X}_i \wedge d\mathbb{X}_j + d\mathbb{Y}_i \wedge d\mathbb{Y}_j)$ invariant. Otherwise, the translation vector integral $\mathcal{P}_i$ along with the angular momentum $\mathbb{M}$ are invariant under this transformation, while $\mathbb{P}_i$ coordinate and Hamiltonian $\mathbb{H}$ are rescaled as

$$\mathcal{I}:\ \mathbb{P}_i \to (1-\beta)^{-1}\,\mathbb{P}_i\,, \qquad \mathbb{H} \to (1-\beta)^{-2}\,\mathbb{H}\,. \tag{5.1.3}$$

In this sense, Hamiltonian of NCLP is invariant under transformation (5.1.2) for fixed values $\beta = 0$ and $\beta = 2$. The case $\beta = 0 \Rightarrow B = 0$ correspond to NCFP system, which is analogous to the vortex-antivortex system, $\Gamma = 0$ or the two vortex system related with the usual Landau problem. And the case $\beta = 2 \Rightarrow B = 2/\theta$ correspond to the NCLP system, that is analogous to the same vortex strength $\gamma_1 = \gamma_2$ system.

Because of the rescaling of the Hamiltonian $\mathbb{H}$ for values distinct from $\beta = 0$ or $\beta = 2$, this transformation relates the regims of a weak and strong coupling with respect to the critical value $\beta = 1$ within the same sub- or super- critical phase,

$$\mathcal{I}:\ (\beta-1) \to 0^{\pm}\ \longleftrightarrow\ (\beta-1)^{-1} \to \pm\infty\,, \tag{5.1.4}$$

where $\beta = 0$ and $\beta = 2$ correspond to the unique stable points. Therefore, $\mathcal{I}^2$ transformation (5.1.2) has a nature of a weak-strong coupling duality.

6. Canonical coordinates for CBT

The relations showed in Section 5 allow us to treat the chiral and non chiral phases of the NCLP and the two-vortex system, respectively, in a quite similar way. As we explore in previous sections, the sub- ($\beta < 1$) and super- ($\beta > 1$) critical phases possess different symplectic structures. Even though, the phase space coordinate $\mathbb{X}_i$ and momentum $\mathbb{P}_i$, with non-commutative components, form two-dimensional vectors with respect to the angular momentum $\mathbb{M}$ generator, for both phases, all their brackets have different signs due to their proportionality to $(1-\beta)^{-1}$, while the signs of the magnetic field B and the non-commutativity parameter θ are maintained fixed. When they reach the critical value $\beta = 1$, the system decreases the dimension of the phase space from four to two as we already shown in Sec.3.1.

The main difference observed between these phases is that for supercritical phase the system acquires chirality and the angular momentum integral $\mathbb{M}$ takes one sign value that coincides with the sign of the magnetic field B. In this way, generation of the two phases of NCLP from a free particle, as we will see in Sec.6, requires different treatment.

As a result of chirality, we define a new integral as a linear combination of the angular momentum $\mathbb{M} = \frac{1}{2B}(\mathcal{P}_i^2 - (1-\beta)\mathbb{P}_i^2)$ and the Hamiltonian $\mathbb{H}$ as

$$\breve{\mathbb{M}} := \mathbb{M} + \tfrac{2m(1-\beta)}{B}\mathbb{H} = \tfrac{1}{2B}(\mathcal{P}_i^2 + (1-\beta)\mathbb{P}_i^2)\,, \tag{6.0.1}$$

which is different in sign before the second term in comparison with the structure of the angular momentum. In terms of integral (6.0.1) the Hamiltonian can be rewritten as

$$\mathbb{H} = \tfrac{1}{2m}\mathbb{P}_i^2 = \tfrac{B}{2m(1-\beta)}(\mathbb{M} - \breve{\mathbb{M}})\,. \tag{6.0.2}$$

We also introduce the two-component object $\breve{\mathbb{P}}_i$ performing a spatial reflection on $\mathbb{P}_i$ as

$$\breve{\mathbb{P}}_i := (-\mathbb{P}_1, \mathbb{P}_2)\,, \qquad \breve{\mathbb{P}}_i{}^2 = \mathbb{P}_i^2, \tag{6.0.3}$$

cf. (2.3.2).

The DPBs of integral (6.0.1) and the angular momentum integral over $(\mathcal{P}_i, \mathbb{P}_i, \breve{\mathbb{P}}_i)$ can be summarized as

$$\{\mathbb{M}, \mathcal{V}_i\} = s\epsilon_{ij}\mathcal{V}_j\,, \qquad s = (+,+,-) \text{ for } \mathcal{V}_i = (\mathcal{P}_i, \mathbb{P}_i, \breve{\mathbb{P}}_i)\,, \tag{6.0.4}$$

$$\{\breve{\mathbb{M}}, \mathcal{V}_i\} = \breve{s}\epsilon_{ij}\mathcal{V}_j\,, \qquad \breve{s} = (+,-,+) \text{ for } \mathcal{V}_i = (\mathcal{P}_i, \mathbb{P}_i, \breve{\mathbb{P}}_i)\,. \tag{6.0.5}$$

Integral $\mathcal{P}_i$ is transformed as a vector by both integrals $\mathbb{M}$ and $\breve{\mathbb{M}}$, while $\mathbb{P}_i$ and $\breve{\mathbb{P}}_i$ are transformed as 2D vectors only by $\mathbb{M}$ and $\breve{\mathbb{M}}$ respectively.

6.1. Non-chiral phase

We start considering the non-chiral phase $\beta < 1$ of NCLP.

Based on relations (3.2.17) we can define a set of canonical variables $(\mathfrak{q}_i, \mathfrak{p}_i)$ as

$$\mathfrak{q}_i = \tfrac{1}{B}\epsilon_{ij}(\mathcal{P}_j - \lambda\mathbb{P}_j)\,, \qquad \mathfrak{p}_i = \tfrac{1}{2}(\mathcal{P}_i + \lambda\mathbb{P}_i)\,, \qquad \lambda = \sqrt{1-\beta}\,. \tag{6.1.1}$$

with DPBs

$$\{\mathfrak{q}_i, \mathfrak{q}_j\} = \{\mathfrak{p}_i, \mathfrak{p}_j\} = 0\,, \qquad \{\mathfrak{q}_i, \mathfrak{p}_j\} = \delta_{ij}\,, \tag{6.1.2}$$

The non-commutative vector phase space variables $(\mathbb{X}_i, \mathbb{Y}_i)$ and $(\mathbb{P}_i, \mathcal{P}_i)$ are expressed in terms of $(\mathfrak{q}_i, \mathfrak{p}_i)$ as

$$\mathcal{P}_i = \mathfrak{p}_i - \tfrac{1}{2}B\epsilon_{ij}\mathfrak{q}_j\,, \qquad \lambda\mathbb{P}_i = \mathfrak{p}_i + \tfrac{1}{2}B\epsilon_{ij}\mathfrak{q}_j\,, \tag{6.1.3}$$

$$\mathbb{X}_i = \tfrac{1}{2}(1+\lambda^{-1})\mathfrak{q}_i + \tfrac{1}{B}(1-\lambda^{-1})\epsilon_{ij}\mathfrak{p}_j\,, \qquad \mathbb{Y}_i = \tfrac{1}{2}(1+\lambda)\mathfrak{q}_i + \tfrac{1}{B}(1-\lambda)\epsilon_{ij}\mathfrak{p}_j\,. \tag{6.1.4}$$

These canonical variables (6.1.1) are defined such as

$$\mathcal{P}^i{}_{|B=0} = \mathbb{P}^i{}_{|B=0} = \mathfrak{p}^i\,, \tag{6.1.5}$$

$$\mathbb{X}^i{}_{|B\to 0} = \mathfrak{q}^i - \tfrac{1}{2}\theta\epsilon^{ij}\mathfrak{p}_j\,, \qquad \mathbb{Y}^i{}_{|B\to 0} = \mathfrak{q}^i + \tfrac{1}{2}\theta\epsilon^{ij}\mathfrak{p}_j\,, \tag{6.1.6}$$

obtaining

$$\mathfrak{q}_{i|B\to 0} = \mathcal{X}_i\,, \tag{6.1.7}$$

corresponding to NCFP set coordinates shown in (3.3.3), (3.3.4).

Conversely, in the limit $\theta = 0$ we obtain

$$\mathbb{X}^i{}_{|\theta=0} = \mathbb{Y}^i{}_{|\theta=0} = \mathfrak{q}^i\,, \tag{6.1.8}$$

$$\mathcal{P}^i{}_{|\theta=0} = \mathfrak{p}^i - \tfrac{1}{2}B\epsilon^{ij}\mathfrak{q}_j\,, \qquad \mathbb{P}^i{}_{|\theta=0} = \mathfrak{p}^i + \tfrac{1}{2}B\epsilon^{ij}\mathfrak{q}_j \tag{6.1.9}$$

that corresponds to ordinary Landau problem in commutative plane.

Finally, case $B = \theta = 0$ corresponds to a free particle in commutative plane

$$\mathbb{X}^i{}_{|B=\theta=0} = \mathbb{Y}^i{}_{|B=\theta=0} = \mathfrak{q}^i\,, \qquad \mathcal{P}^i{}_{|B=\theta=0} = \mathbb{P}^i{}_{|B=\theta=0} = \mathfrak{p}^i\,. \tag{6.1.10}$$

The integrals $\mathbb{M}$ and $\breve{\mathbb{M}}$ are rewritten in terms of the canonical variables (6.1.1) as

$$\mathbb{M} = \epsilon_{ij}\mathfrak{q}_i\mathfrak{p}_j := \mathfrak{M}, \qquad \breve{\mathbb{M}} = \Omega^{-1}\mathfrak{H}_{\text{osc}}\,, \qquad \mathfrak{H}_{\text{osc}} = \tfrac{1}{2m}\mathfrak{p}_i^2 + \tfrac{1}{2}m\Omega^2\mathfrak{q}_i^2\,, \tag{6.1.11}$$

and the Hamiltonian (3.2.3) takes the form

$$\mathbb{H} = \tfrac{1}{1-\beta}\left(\mathfrak{H}_{\text{osc}} - \Omega\mathfrak{M}\right)\,, \qquad \Omega = \tfrac{B}{2m}\,. \tag{6.1.12}$$

Here $\mathfrak{H}_{\text{osc}}$ may be thought as the planar isotropic harmonic oscillator Hamiltonian of mass m and frequency Ω. The DPBs of integrals (6.1.11),(6.1.12) over the phase $(\mathfrak{q}_i, \mathfrak{p}_i)$ coordinates are

$$\{\mathbb{M}, \mathfrak{q}_i\} = \epsilon_{ij}\mathfrak{q}_j\,, \quad \{\mathbb{M}, \mathfrak{p}_i\} = \epsilon_{ij}\mathfrak{p}_j\,, \qquad \{\breve{\mathbb{M}}, \mathfrak{q}_i\} = -\tfrac{2}{B}\mathfrak{p}_i\,, \quad \{\breve{\mathbb{M}}, \mathfrak{p}_i\} = \tfrac{B}{2}\mathfrak{q}_i\,, \tag{6.1.13}$$

$$\{\mathbb{H}, \mathfrak{q}_i\} = -m^{-1}\lambda^{-1}\,\mathbb{P}_i\,, \qquad \{\mathbb{H}, \mathfrak{p}_i\} = -\Omega\lambda^{-1}\,\epsilon_{ij}\mathbb{P}_j\,. \tag{6.1.14}$$

In terms of canonical variables, we define "linearly polarized" creation-annihilation operators, restoring the explicit dependence on $\hbar$

$$\hat{\mathfrak{a}}_j^{\mp} = \sqrt{\tfrac{m|\Omega|}{2\hbar}}\left(\hat{\mathfrak{q}}_j \pm i\tfrac{1}{m|\Omega|}\,\hat{\mathfrak{p}}_j\right)\,, \tag{6.1.15}$$

with commutation relations

$$[\hat{\mathfrak{a}}_j^-, \hat{\mathfrak{a}}_k^+] = \delta_{jk}\,, \qquad [\hat{\mathfrak{a}}_j^+, \hat{\mathfrak{a}}_k^+] = [\hat{\mathfrak{a}}_j^-, \hat{\mathfrak{a}}_k^-] = 0\,, \tag{6.1.16}$$

In terms of these coordinates, we also define the "ciricularly polarized" creation-annihilation operators

$$\hat{\mathfrak{b}}_{\varepsilon}^{-}=\tfrac{1}{\sqrt{2}}(\hat{\mathfrak{a}}_1^{-}+i\varepsilon\hat{\mathfrak{a}}_2^{-})\,,\qquad \hat{\mathfrak{b}}_{\varepsilon}^{+}=(\hat{\mathfrak{b}}_{\varepsilon}^{-})^{\dagger}\,,\qquad \varepsilon=\pm\,, \tag{6.1.17}$$

with commutation relations

$$[\hat{\mathfrak{b}}_{\varepsilon}^{-},\hat{\mathfrak{b}}_{\varepsilon}^{+}]=1\,,\qquad [\hat{\mathfrak{b}}_{-}^{\pm},\hat{\mathfrak{b}}_{+}^{\pm}]=0\,. \tag{6.1.18}$$

where we identify the parameter ε with the sign of magnetic field $\varepsilon=\operatorname{sgn}B$

The quantum analogs of the angular momentum and integral $\breve{\mathbb{M}}$ written in terms of these quantum set of coordinates are

$$\hat{\mathfrak{M}}=\varepsilon\hbar(\hat{\mathfrak{N}}_{-\varepsilon}-\hat{\mathfrak{N}}_{\varepsilon})\,,\quad \hat{\breve{\mathbb{M}}}=\varepsilon\hbar(\hat{\mathfrak{N}}_{\varepsilon}+\hat{\mathfrak{N}}_{-\varepsilon}+1)\,, \tag{6.1.19}$$

where $\hat{\mathfrak{N}}_{\varepsilon}=\hat{\mathfrak{b}}_{\varepsilon}^{+}\hat{\mathfrak{b}}_{\varepsilon}^{-}$, $\hat{\mathfrak{N}}_{-\varepsilon}=\hat{\mathfrak{b}}_{-\varepsilon}^{+}\hat{\mathfrak{b}}_{-\varepsilon}^{-}$.

The quantum Hamiltonian becomes

$$\hat{\mathbb{H}}=\hbar\tfrac{|\Omega|}{1-\beta}(2\hat{\mathfrak{N}}_{\varepsilon}+1)\,. \tag{6.1.20}$$

In this alternative way, we reproduce the results of Sec.3.2 for NCLP in sub-critical phase.

6.2. Chiral phase

Now, we consider the chiral phase $\beta>1$ of NCLP.

Analogously to the non-chiral phase, we can define canonical variables $(\widetilde{\mathfrak{q}}_i,\widetilde{\mathfrak{p}}_i)$ as

$$\widetilde{\mathfrak{q}}_i=\tfrac{1}{B}\epsilon_{ij}(\mathcal{P}_j-\tilde{\lambda}\breve{\mathbb{P}}_j)\,,\qquad \widetilde{\mathfrak{p}}_i=\tfrac{1}{2}(\mathcal{P}_i+\tilde{\lambda}\breve{\mathbb{P}}_i)\,,\qquad \tilde{\lambda}=\sqrt{\beta-1}\,, \tag{6.2.1}$$

with DPBs

$$\{\widetilde{\mathfrak{q}}_i,\widetilde{\mathfrak{q}}_j\}=\{\widetilde{\mathfrak{p}}_i,\widetilde{\mathfrak{p}}_j\}=0\,,\qquad \{\widetilde{\mathfrak{q}}_i,\widetilde{\mathfrak{p}}_j\}=\delta_{ij}\,. \tag{6.2.2}$$

These coordinates are obtained by changing $\lambda=\sqrt{1-\beta}\to\tilde{\lambda}=\sqrt{\beta-1}$ and $\mathbb{P}_i\to\breve{\mathbb{P}}_i$ into the canonical variables for non-chirial phase.

The inverse relations of (6.2.1) are given by

$$\mathcal{P}_i=\widetilde{\mathfrak{p}}_i-\tfrac{1}{2}B\epsilon_{ij}\widetilde{\mathfrak{q}}_j\,,\qquad \tilde{\lambda}\breve{\mathbb{P}}_i=\widetilde{\mathfrak{p}}_i+\tfrac{1}{2}B\epsilon_{ij}\widetilde{\mathfrak{q}}_j\,. \tag{6.2.3}$$

The integrals $\mathbb{M}$ and $\breve{\mathbb{M}}$ are rewritten in terms of the canonical variables (6.2.1) as

$$\breve{\mathbb{M}}=\epsilon_{ij}\widetilde{\mathfrak{q}}_i\widetilde{\mathfrak{p}}_j:=\breve{\mathfrak{M}}\,,\qquad \mathbb{M}=\Omega^{-1}\widetilde{\mathfrak{H}}_{\rm osc}\,,\qquad \widetilde{\mathfrak{H}}_{\rm osc}=\tfrac{1}{2m}\widetilde{\mathfrak{p}}_i^2+\tfrac{1}{2}m\Omega^2\widetilde{\mathfrak{q}}_i^2\,, \tag{6.2.4}$$

and Hamiltonian (3.2.3) takes the form

$$\mathbb{H}=\tfrac{1}{\beta-1}\left(\widetilde{\mathfrak{H}}_{\rm osc}-\Omega\breve{\mathfrak{M}}\right)\,, \tag{6.2.5}$$

Since the set of canonical coordinates $(\widetilde{\mathfrak{q}}_i,\widetilde{\mathfrak{p}}_i)$ are in terms of the two component object $\breve{\mathbb{P}}_i$ and integrals $\mathcal{P}_i$ they are no longer two-dimensional vectors with respect to the angular momentum $\mathbb{M}$, acording to Eq. (6.0.4). We emphasize this particularity by supplying their characters with a tilde. However, these coordinates transform like two-dimensional vector under the action of the integral $\breve{\mathbb{M}}$

$$\{\breve{\mathbb{M}},\widetilde{\mathfrak{q}}_i\}=\epsilon_{ij}\widetilde{\mathfrak{q}}_j\,,\quad \{\breve{\mathbb{M}},\widetilde{\mathfrak{p}}_i\}=\epsilon_{ij}\widetilde{\mathfrak{p}}_j\,,\qquad \{\mathbb{M},\widetilde{\mathfrak{q}}_i\}=-\tfrac{2}{B}\widetilde{\mathfrak{p}}_i\,,\quad \{\mathbb{M},\widetilde{\mathfrak{p}}_i\}=\tfrac{B}{2}\widetilde{\mathfrak{q}}_i\,, \tag{6.2.6}$$

$$\{\mathbb{H}, \widetilde{\mathfrak{q}}_i\} = -m^{-1}\tilde{\lambda}^{-1}\,\check{\mathbb{P}}_i\,, \qquad \{\mathbb{H}, \mathfrak{p}_i\} = -\Omega\tilde{\lambda}^{-1}\epsilon_{ij}\check{\mathbb{P}}_j\,. \tag{6.2.7}$$

Analogously to (6.1.15) and (6.1.17), we define the "linearly polarized", $\hat{\widetilde{\mathfrak{a}}}{}^{\pm}_j$ starting from these new set of coordinates $(\widetilde{\mathfrak{q}}_i, \widetilde{\mathfrak{p}}_i)$, and using them, the "circularly polarized", $\hat{\widetilde{\mathfrak{b}}}{}^{\pm}_{\varepsilon}$, $\hat{\widetilde{\mathfrak{b}}}{}^{\pm}_{-\varepsilon}$, creation-annihilation operators.

Then, the quantum analogs of $\mathbb{M}$, $\breve{\mathfrak{M}}$ and $\mathbb{H}$ integrals are presented in terms of number operators $\hat{\widetilde{\mathcal{N}}}_{\varepsilon} = \hat{\widetilde{\mathfrak{b}}}{}^{+}_{\varepsilon}\hat{\widetilde{\mathfrak{b}}}{}^{-}_{\varepsilon}$ and $\hat{\widetilde{\mathcal{N}}}_{-\varepsilon} = \hat{\widetilde{\mathfrak{b}}}{}^{+}_{-\varepsilon}\hat{\widetilde{\mathfrak{b}}}{}^{-}_{-\varepsilon}$ as

$$\hat{\mathbb{M}} = \varepsilon\hbar(\hat{\widetilde{\mathfrak{N}}}_{\varepsilon} + \hat{\widetilde{\mathfrak{N}}}_{-\varepsilon} + 1)\,, \qquad \hat{\breve{\mathfrak{M}}} = \varepsilon\hbar(\hat{\widetilde{\mathfrak{N}}}_{-\varepsilon} - \hat{\widetilde{\mathfrak{N}}}_{\varepsilon})\,, \qquad \hat{\mathbb{H}} = \hbar\tfrac{|\Omega|}{\beta-1}(2\hat{\widetilde{\mathfrak{N}}}_{\varepsilon} + 1)\,. \tag{6.2.8}$$

In this alternative way, we reproduce the result of Sec.3.2 for NCLP in super-critical phase.

7. CBT for NCLP and Two-vortex system

We establish the method to obtain the quantum Landau problem in non-commutative plane from the free particle system by means of CBT [14, 12, 13], and then, employing correspondences (5.0.1) and (5.0.2), extrapolate this method to obtain in a similar way the non-zero total vorticity two-vortex system.

Considering a quantum free particle system in two-dimensional Euclidean space described by the set of canonical coordinates operators, $\hat{q}_i$ and $\hat{p}_i$. We define the signed frequency parameter as $\Omega = \frac{B}{2m}$ introduced in (6.1.12). Then, generators of the conformal $\mathfrak{sl}(2,\mathbb{R})$ symmetry of the free particle, which are the two-dimensional analogs of those considered in Sec.2.2 with the restored parameters m, Ω and constant $\hbar$ are

$$\hat{J}_0 = \tfrac{1}{2\hbar|\Omega|}\hat{H}_+ \,, \qquad \hat{J}_1 = -\tfrac{1}{2\hbar|\Omega|}\hat{H}_- \,, \qquad \hat{J}_2 = -\tfrac{1}{2\hbar}\hat{D} \,, \tag{7.0.1}$$

where

$$\hat{H}_\pm = \tfrac{1}{2m}\hat{p}_i^2 \pm \tfrac{1}{2}m\Omega^2\hat{q}_i^2 \,, \qquad \hat{D} = \tfrac{1}{2}(\hat{q}_i\hat{p}_i + \hat{p}_i\hat{q}_i) \,. \tag{7.0.2}$$

The two-dimensional analog of the similarity transformation (2.2.5) is

$$\hat{O}' = \hat{\mathfrak{S}}\hat{O}\hat{\mathfrak{S}}^{-1} \,, \qquad \hat{\mathfrak{S}} = \exp\Big(-\tfrac{\pi}{2}\hat{J}_1\Big) \,, \tag{7.0.3}$$

Application of the transformation (7.0.3) yields a mapping

$$\begin{aligned}\Bigg(i\sqrt{\tfrac{2}{|B|\hbar}}\,\hat{p}_j,\, \sqrt{\tfrac{|B|}{2\hbar}}\,\hat{q}_j,\, i\sqrt{|B|\hbar}\,\hat{a}_j^+,\, \sqrt{\tfrac{\hbar}{|B|}}\,\hat{a}_j^-,\, \hat{H}_+,\, \hat{H}_-,\, i\hat{D},\, \hat{M}\Bigg) \rightarrow \\ \Big(\hat{a}_j^-,\, \hat{a}_j^+,\, \hat{p}_j,\, \hat{q}_j,\, -i\hat{D},\, \hat{H}_-,\, \hat{H}_+,\, \hat{M}\Big) \,,\end{aligned} \tag{7.0.4}$$

where $\hat{a}_j^\pm$ are the “linearly polarized" creation-annihilation operators of the form (6.1.15), and

$$\hat{M} = \epsilon_{ik}\hat{q}_j\hat{p}_k = i\hbar\epsilon_{jk}\hat{a}_j^-\hat{a}_k^+ \,. \tag{7.0.5}$$

We can define the complex coordinates and canonically conjugate momenta,

$$w_\varepsilon = \tfrac{1}{\sqrt{2}}(q_1 + i\varepsilon q_2) \,, \qquad \bar{w}_\varepsilon = w_{-\varepsilon} \,, \qquad p_\varepsilon = \tfrac{1}{\sqrt{2}}(p_1 - i\varepsilon p_2) \,, \qquad \bar{p}_\varepsilon = p_{-\varepsilon} \,, \tag{7.0.6}$$

subjected to DPBs

$$\{w_\varepsilon, p_\varepsilon\} = \{\bar{w}_\varepsilon, \bar{p}_\varepsilon\} = 1 \,, \tag{7.0.7}$$

Applying transformation (7.0.3) to the quantum analogs of this set of coordinates we produce

$$(\hat{w}_\varepsilon,\, \hat{p}_\varepsilon) \rightarrow \Big(\sqrt{\tfrac{2\hbar}{|B|}}\,\hat{b}_{-\varepsilon}^+,\, -i\sqrt{\tfrac{\hbar|B|}{2}}\,\hat{b}_{-\varepsilon}^-\Big) \,, \tag{7.0.8}$$

$$(\hat{\bar{w}}_\varepsilon,\, \hat{\bar{p}}_\varepsilon) \rightarrow \Big(\sqrt{\tfrac{2\hbar}{|B|}}\,\hat{b}_{\varepsilon}^+,\, -i\sqrt{\tfrac{\hbar|B|}{2}}\,\hat{b}_{\varepsilon}^-\Big) \,, \tag{7.0.9}$$

which are “circularly polarized" creation-annihilation operators that are linear combinations of $\hat{a}_j^{\pm}$ analogous to (6.1.17).

Transformation over the Wick rotated $i\hat{D}$ and $\hat{M}$ generators produces

$$i\hat{D} = \tfrac{i}{2}(\hat{q}_j\hat{p}_j + \hat{p}_j\hat{q}_j) \;\to\; \hat{\check{M}} = \hbar(\hat{N}_\varepsilon + \hat{N}_{-\varepsilon} + 1)\,, \tag{7.0.10}$$

$$\hat{M} = \epsilon_{ij}\hat{q}_i\hat{p}_j \;\to\; \hat{M} = \varepsilon\hbar(\hat{N}_{-\varepsilon} - \hat{N}_\varepsilon)\,, \tag{7.0.11}$$

where $\hat{N}_\varepsilon = \hat{b}^+_\varepsilon\hat{b}^-_\varepsilon$, $\hat{N}_{-\varepsilon} = \hat{b}^+_{-\varepsilon}\hat{b}^-_{-\varepsilon}$,

In coordinate representation, these generators can be written as

$$i\hat{D} = \hbar(w_\varepsilon\partial_{w_\varepsilon} + \bar{w}_\varepsilon\partial_{\bar{w}_\varepsilon} + 1)\,, \qquad \hat{M} = \varepsilon\hbar(w_\epsilon\partial_{w_\varepsilon} - \bar{w}_\varepsilon\partial_{\bar{w}_\varepsilon})\,, \tag{7.0.12}$$

where the set of functions

$$\Phi_{n_+,n_-} = w_\varepsilon^{n_+}\bar{w}_\varepsilon^{n_-}\,, \qquad n_+,\, n_- = 0, 1, \ldots, \tag{7.0.13}$$

constitute a set of formal eigenstates of the Wick rotated dilatation and momentum operators with eigenvalues

$$i\hat{D}\Phi_{n_+,n_-} = \hbar(n_+ + n_- + 1)\Phi_{n_+,n_-}\,, \qquad \hat{M}\Phi_{n_+,n_-} = \hbar(n_+ - n_-)\Phi_{n_+,n_-}\,. \tag{7.0.14}$$

Hamiltonian operator becomes

$$\hat{H}_0 = \frac{1}{2m}\hat{p}_i{}^2 = -\frac{1}{m}\partial_{w_\varepsilon}\partial_{\bar{w}_\varepsilon}\,. \tag{7.0.15}$$

By virtue of $[\hat{M},\hat{H}_0] = 0$, $[i\hat{D},\hat{H}_0] = -2\hbar\hat{H}_0$, states (7.0.13) also constitute eigenstates of free particle Hamiltonian corresponding to zero energy [32, 64],

$$(\hat{H}_0)^{n_+ + n_- + 1}\Phi_{n_+,n_-} = 0\,. \tag{7.0.16}$$

These are called Jordan states.

The state $\Phi_{0,0}$ is annihilated by the operators $\hat{p}_\varepsilon$ and $\hat{\bar{p}}_\varepsilon$, and in accordance with (2.2.10), is transformed, up to a normalization into

$$\Phi'_{0,0} = \hat{\mathfrak{S}}\Phi_{0,0} \propto \exp\left(-\frac{m|\Omega|}{\hbar}w_\varepsilon\bar{w}_\varepsilon\right). \tag{7.0.17}$$

Also, in the Fock representation the ket states $|n_+, n_-\rangle$ are the common eigenstates of the number operators $\hat{N}_\varepsilon$ and $\hat{N}_{-\varepsilon}$, which, up to a normalization, correspond to the transformed states (7.0.13), $\Phi'_{n_+,n_-} = \hat{\mathfrak{S}}\Phi_{n_+,n_-}$ in holomorphic representation.

7.1. Non-chiral phase CBT

In order to obtain the non-chiral phase, we identify the canonical variables (6.1.1) with the canonical coordinates and momenta (q_i, p_i). Thus, we generate the non-chiral phase of NCLP from the free particle in a plane by applying CBT (7.0.3).

With the construction made in Sec.6.1, we find the pre-images of $\hat{\mathcal{P}}_i$ and $\hat{\mathbb{P}}_i$ operators

$$(\hat{w}_\varepsilon,\, \hat{p}_\varepsilon) \to \big(-\tfrac{i}{|B|}\tfrac{1}{\sqrt{2}}(\hat{\mathcal{P}}_1 + i\varepsilon\hat{\mathcal{P}}_2),\, \tfrac{1}{\sqrt{2}}(\hat{\mathcal{P}}_1 - i\varepsilon\hat{\mathcal{P}}_2)\big)\,, \tag{7.1.1}$$

$$(\hat{\bar{w}}_\varepsilon,\, \hat{\bar{p}}_\varepsilon) \to \big(-\tfrac{i}{|B|}\tfrac{\lambda}{\sqrt{2}}(\hat{\mathbb{P}}_1 - i\varepsilon\hat{\mathbb{P}}_2),\, \tfrac{\lambda}{\sqrt{2}}(\hat{\mathbb{P}}_1 + i\varepsilon\hat{\mathbb{P}}_2)\big)\,. \tag{7.1.2}$$

where the operators corresponding to the free particle are shown on the left side.

The quadratic integrals pre-images are also established as

$$\hat{M} \to \hat{\mathbb{M}}\,, \qquad i\hat{D} \to \hat{\mathring{\mathbb{M}}}\,, \qquad \tfrac{|\Omega|}{1-\beta}(i\hat{D} - \varepsilon\hat{M}) \to \hat{\mathbb{H}}\,. \tag{7.1.3}$$

The pre-images of the non-commutative coordiantes $\hat{\mathbb{X}}_i$ and $\hat{\mathbb{Y}}_i$ can be found from (7.1.1), (7.1.2) by using Eq. (3.2.16).

7.2. Chiral phase CBT

For chiral phase, we identify the canonical variables with $\widetilde{\mathfrak{q}}_i$ and $\widetilde{\mathfrak{p}}_i$ given by Eq. (6.2.1) as the canonical variables of a free particle. Then, CBT transformation (7.0.3) maps

$$(\hat{w}_\varepsilon,\, \hat{p}_\varepsilon) \to \big(-\tfrac{i}{|B|}\tfrac{1}{\sqrt{2}}(\hat{\mathcal{P}}_1 + i\varepsilon\hat{\mathcal{P}}_2),\, \tfrac{1}{\sqrt{2}}(\hat{\mathcal{P}}_1 - i\varepsilon\hat{\mathcal{P}}_2)\big)\,, \tag{7.2.1}$$

$$(\hat{\bar{w}}_\varepsilon,\, \hat{\bar{p}}_\varepsilon) \to \big(-\tfrac{i}{|B|}\tfrac{\widetilde{\lambda}}{\sqrt{2}}(\hat{\mathring{\mathbb{P}}}_1 - i\varepsilon\hat{\mathring{\mathbb{P}}}_2),\, \tfrac{\widetilde{\lambda}}{\sqrt{2}}(\hat{\mathbb{P}}_1 + i\varepsilon\hat{\mathbb{P}}_2)\big)\,. \tag{7.2.2}$$

Pre-images of the operators $\hat{\mathbb{X}}_i$ and $\hat{\mathbb{Y}}_i$ can be found from (7.2.1), (7.2.2) by using Eq. (3.2.16).

The quadratic integrals pre-images are

$$\hat{M} \to \hat{\mathring{\mathbb{M}}}\,, \qquad i\varepsilon\hat{D} \to \hat{\mathbb{M}}\,, \qquad \tfrac{|\Omega|}{\beta-1}(i\hat{D} - \varepsilon\hat{M}) \to \hat{\mathbb{H}}\,. \tag{7.2.3}$$

For non-chiral (sub-critical) phase of NCLP, the angular momentum of the free particle $\hat{M}$ transforms into the angular momentum while the Wick rotated generator of dilatations multiplied by ε converts into the integral $\hat{\mathring{\mathbb{M}}}$. Instead, for the chirial (super-critical) phase, CBT transmutes the free particle integrals $\hat{M}$ and $i\varepsilon\hat{D}$ into the integrals $\hat{\mathring{\mathbb{M}}}$ and $\hat{\mathbb{M}}$, respectively.

Because for free particle in non-commutative plane we have ${\mathcal{P}^i}_{|B=0} = {\mathbb{P}^i}_{|B=0} = \mathfrak{p}^i$, and $\mathfrak{q}^i$ reduces to the coordinate $\mathcal{X}^i$ introduced in Eq. (3.3.3), the conformal bridge transformation can be reinterpreted as a non-unitary mapping from a free particle system in non-commutative plane ($B = 0$, $\theta \neq 0$) into the NCLP ($B \neq 0$, $\theta \neq 0$) in non-chiral and chiral phases by changing the coordinate variable q_i of the free particle for $\mathcal{X}_i$ in the above relations.

7.3. Two-vortex system CBT

The prescription to obtain the two-vortex system by CBT transformation, can be realized in accordance with correspondences (5.0.1) and (5.0.2) where the integral (6.0.1) and spatially reflected vector (6.0.3) are introduced in analogous form

$$\breve{\mathbb{M}} \sim \breve{M}_\Gamma = M_\Gamma + \varrho\, r_i^2\,, \qquad \breve{\mathbb{P}}_i \sim \gamma_2 \breve{r}_i\,, \qquad \breve{r}_i = (r_2, r_1)\,. \tag{7.3.1}$$

Also, we define a direct analog of NCLP Hamiltonian, (see Eq. (4.2.12)) as

$$\mathbb{H} \sim \mathcal{H}_\Gamma := \tfrac{1}{2}\gamma_2^2 r_i^2 = \tfrac{1}{2}\gamma_2^2 \exp\!\big(-2\kappa^{-1} H_\Gamma\big)\,, \tag{7.3.2}$$

Now, we prescribe the adequate canonical pair of coordinates for two-vortex system in non-chiral phase $\kappa = \gamma_1\gamma_2 < 0$. Coordinates (q_i, p_i) can be defined, in correspondence with NCLP and Eq. (6.1.1), as

$$q_i = \tfrac{1}{\Gamma}\left((\gamma_1 + \lambda\gamma_2)x_1^i + \gamma_2(1-\lambda)x_2^i\right)\,, \qquad p_i = \tfrac{1}{2}\epsilon_{ik}\left((\gamma_1 - \lambda\gamma_2)x_1^k + \gamma_2(1+\lambda)x_2^k\right)\,, \tag{7.3.3}$$

where $\lambda = \sqrt{-\gamma_1/\gamma_2}$.

We can verify that the set of coordinates has appropriate limits when we tend one of the parameters to zero. The limit $\Gamma \to 0$ provides

$$q^i_{|\Gamma\to 0} = \frac{1}{2}(x_1^i + x_2^i) = \chi^i\,, \qquad p^i_{|\Gamma\to 0} = \Pi^i\,, \tag{7.3.4}$$

where χ^i and Π^i are the canonical variables of the vortex-antivortex system (see Eqs. (4.2.2) and (4.2.1)).

The inverse relations of (7.3.3) are

$$x_1^i = \tfrac{1}{2}(1+\lambda^{-1})q_i - \tfrac{1}{\Gamma}(1-\lambda^{-1})\epsilon_{ij}p_j\,, \qquad x_2^i = \tfrac{1}{2}(1+\lambda)q_i - \tfrac{1}{\Gamma}(1-\lambda)\epsilon_{ij}p_j\,, \tag{7.3.5}$$

beside, we obtain

$$P_i = p_i + \tfrac{1}{2}\Gamma\epsilon_{ij}q_j\,, \qquad \lambda\gamma_2\, r_i = \epsilon_{ij}p_j + \tfrac{1}{2}\Gamma q_i\,. \tag{7.3.6}$$

The quadratic integrals M_Γ, $\breve{M}_\Gamma$ and $\mathcal{H}_\Gamma$ are presented in a analogous way to (6.1.11), (6.2.5), in canonical variables terms (7.3.3)

$$M_\Gamma = \epsilon_{ij}q^ip^j := \mathcal{M}\,, \qquad \breve{M}_\Gamma = \Omega^{-1}\mathcal{H}_{\rm osc}\,, \qquad \mathcal{H}_{\rm osc} = \tfrac{1}{2}p_i^2 + \tfrac{1}{2}\Omega^2 q_i^2\,, \tag{7.3.7}$$

$$\mathcal{H}_\Gamma = \lambda^{-2}(\mathcal{H}_{\rm osc} - \Omega\mathcal{M})\,, \qquad \Omega = -\tfrac{1}{2}\Gamma\,. \tag{7.3.8}$$

And the free particle dilatation generator takes the form (see Eq. (3.3.12))

$$D = q_ip_i = \operatorname{sgn}\gamma_2\,\sqrt{-\kappa}\,\epsilon_{ij}x_1^ix_2^j\,, \tag{7.3.9}$$

On the other hand, if we consider the two-vortex system in the chirial phase, $\kappa > 0$. The adequate canonical variables $(\widetilde{q}_i, \widetilde{p}_i)$, definded by analogy to (6.2.1), are

$$\widetilde{q}_i = -\tfrac{1}{\Gamma}\epsilon_{ij}(P_j - \gamma_2\widetilde{\lambda}\,\check{r}_j)\,, \qquad \widetilde{p}^{\,j} = P_i + \gamma_2\widetilde{\lambda}\,\check{r}_i\,, \tag{7.3.10}$$

where $\widetilde{\lambda} = \sqrt{\gamma_1/\gamma_2}$.

The inverse relations of (7.3.10) are

$$\begin{aligned} x_1^j &= \tfrac{1}{2}\left(1-(-1)^j\widetilde{\lambda}^{-1}\right)\widetilde{q}^{\,j} - \tfrac{1}{\Gamma}(1+\widetilde{\lambda}^{-1})\epsilon^{jk}\widetilde{p}_k\,, \\ x_2^j &= \tfrac{1}{2}(1+(-1)^j\widetilde{\lambda})\widetilde{q}^{\,j} - \tfrac{1}{\Gamma}(1-(-1)^j\widetilde{\lambda})\epsilon^{jk}\widetilde{p}_k\,, \end{aligned} \tag{7.3.11}$$

and also obtaining

$$P_i = \widetilde{p}_i + \tfrac{1}{2}\Gamma\epsilon_{ij}\widetilde{q}_j\,, \qquad \gamma_2\,\check{r}_i = \widetilde{p}_i - \tfrac{1}{2}\Gamma\epsilon_{ij}\widetilde{q}_j\,. \tag{7.3.12}$$

This quadratic integrals $\breve{M}_\Gamma$, M_Γ and $\mathcal{H}_\Gamma$, analogous to (6.2.4), are presented in terms of this set of canonical coordinates

$$\breve{M}_\Gamma = \epsilon_{ij}\widetilde{q}^i\widetilde{p}^j := \breve{\mathcal{M}}\,, \qquad M_\Gamma = \Omega^{-1}\widetilde{\mathcal{H}}_{\text{osc}}\,, \qquad \widetilde{\mathcal{H}}_{\text{osc}} = \tfrac{1}{2}\widetilde{p}_i{}^2 + \tfrac{1}{2}\Omega^2\widetilde{q}_i{}^2\,, \tag{7.3.13}$$

$$\mathcal{H}_\Gamma = \tilde{\lambda}^{-2}(\widetilde{\mathcal{H}}_{\text{osc}} - \Omega\breve{\mathcal{M}})\,, \tag{7.3.14}$$

where Ω is defined in (7.3.8).

The dilation generator obtained is

$$D = \widetilde{q}_i\widetilde{p}_i = \operatorname{sgn}\gamma_2\,\tfrac{\sqrt{\kappa}}{\Gamma}\,\epsilon_{ij}P_i\check{r}_j\,. \tag{7.3.15}$$

With these correspondences in dynamical variables and integrals of motion, we are in a position to carry out the procedure presented above and proceed in a similar way to NCLP in its respective two phases. As well, we can apply CBT to a free particle obtaining the quantum integral $\hat{\mathcal{H}}_\Gamma$ with nonzero total vorticity $\Gamma \neq 0$. Finally by means of Eq. (7.3.2) we obtain the correct integral operator as

$$\hat{H}_\Gamma = -\tfrac{1}{2}\kappa\log\Big(2\gamma_2^{-2}\hat{\mathcal{H}}_\Gamma\Big)\,. \tag{7.3.16}$$

We can also generate the two-vortex systems with $\kappa = \gamma_1\gamma_2 < 0$, $\Gamma \neq 0$, and $\kappa > 0$ from the vortex-antivortex system with $\Gamma = 0$, identifying the (q_i, p_i) canonical coordinates with the vortex-antivortex canonical coordinates (χ_i, Π_i). Also, by means of the composition of the corresponding inverse and direct conformal bridge transformations, the two-vortex system with $\kappa < 0$, $\Gamma \neq 0$, and the system with $\kappa > 0$ can be related via the "virtual" free particle, or via the vortex-antivortex system with $\Gamma = 0$.

8. Conclusions and Outlook

The correspondence between non-commutative Landau problem and two-vortex system is given. Introducing an "imaginary mirror particle" coordinate in NCLP, we established a relation between a set of its coordinates and parameters with those of the two-vortex system, based on their dynamics. In coherence with the central charge for their respective $\mathfrak{e}_B(2)$ and $\mathfrak{e}_\Gamma(2)$ algebras, we found that parameter $\Gamma = \gamma_1 + \gamma_2$ in the two-vortex system corresponds to the minus magnetic field, $-B$, in the NCLP. Consequently, one of the vortex strength γ is mapped to $-\theta^{-1}$, where θ corresponds to the non-commutativity parameter of the NCLP.

The i) sub- ($\beta = B\theta < 1$), ii) super- ($\beta > 1$), and iii) critical ($\beta = 1$) phases in NCLP correspond to the i) non-chiral ($\kappa = \gamma_1\gamma_2 < 0$), ii) chiral ($\kappa > 0$), and iii) stationary (when one of the vortex strengths γ_a is set to zero) cases of two-vortex system. For both systems, the angular momentum takes values of i) both signs, ii) one sign, and iii) discrete values of one sign, respectively.

The NCFP ($B = 0$,$\theta \neq 0$) with its hidden $(1+1)D$ "isospace" Lorentz symmetry corresponds to the vortex-antivortex system ($\Gamma = 0$). The usual Landau problem ($\theta = 0$, $B \neq 0$) also is covered by the correspondence with the two-vortex system via a limit procedure applied to the case $\kappa < 0$. In this limit, the vortices coordinates coincide, and their coordinates components commute.

The obtained correspondences between the systems showed us that permutation symmetry of the vortex pair generates weak-strong coupling duality in NCLP, $\mathbb{H} \to (1-\beta)^{-2}\mathbb{H}$, where $\mathbb{H}$ is the NCLP Hamiltonian. In this scenario, two points $\beta = 0$ and $\beta = 2$ constitute stable points under this weak-strong duality. The chiral phase at $\beta = 2$ corresponds to vortices of the same strength $\gamma_1 = \gamma_2$. Conversely, the non-chiral phase at $\beta = 0$ corresponds to the cases of vortex-antivortex system $\Gamma = 0$ or to the indicated limit case to be similar to the usual Landau problem with $\theta = 0\,, B \neq 0$.

As a consequence of the chirality, we introduced a linear combination $\breve{\mathbb{M}}$ of the angular momentum $\mathbb{M}$ and Hamiltonian $\mathbb{H}$ integrals in NCLP. This particular integral generates rotations of the guiding center vector integral of motion and of the spatially reflected non-commutative momentum. This allowed us to identify the appropriate sets of canonical coordinates and momenta for the 2D vector coordinates that transforms as vectors either with respect to the angular momentum $\mathbb{M}$ or the defined integral $\breve{\mathbb{M}}$ in the sub-critical and super-critical phases, respectively. With the found correspondences, we were able to find the two-vortex system analogs.

The nature of the chiral and non-chiral phases in both quantum systems are related to the outer Z_2 automorphism of the conformal $\mathfrak{sl}(2,\mathbb{R})$ algebra.

The appropriate set of canonical coordinates enabled us to construct two similarly but different forms of the conformal bridge transformation, which constitute a non-unitary similarity transformation that maps a 2D quantum free particle system into sub-critical and super-critical phases of NCLP and the non-chiral and chiral phases of two-vortex systems. The CBT, based on the conformal $\mathfrak{sl}(2,\mathbb{R})$ symmetry, is a kind of the Dyson map appearing in the $\mathcal{PT}$ symmetry [12, 13]. It acts as the eighth-order root of an identity transformation in the phase space, which changes the topological nature of the $\mathfrak{sl}(2,\mathbb{R})$ generators.

The action of the CBT, over a linear combination of the Wick rotated dilatation and angular momentum operator generators corresponding to the free particle, is transformed into Hamiltonian operators of the two quantum systems in corresponding sub-critical (non-chiral) or super-critical (chiral) phases. In particular, when we performed the mapping into the non-chiral phase, the angular momentum operator of the free particle was transformed into the angular momentum integral of NCLP, or its analog in two-vortex system, while, on the other hand, for chiral phase, the Wick rotated dilatation generator of the free particle was mapped into the integral $\hat{\mathbb{M}}$ of NCLP, or its analog in the two-vortex system. In this sense, the corresponding transformation into the chiral phase of the two systems, mapped the Wick rotated dilatation generator into the angular momentum operator, while its angular momentum mapped into the integral $\hat{\mathbb{M}}$ of NCLP or its analog in the two-vortex system. The described picture have not appeared in earlier CBT applications.

Since the CBT only involves the conformal algebra realization, it can be generated in analogous way starting from the vortex-antivortex system ($\Gamma = 0$) and free particle in non-commutative plane ($B = 0$, $\theta \neq 0$) mapping any one of these systems into the sub-critical (non-chiral) and super-critical (chiral) phases of NCLP or its analog in two-vortex system, respectively.

Because the CBT is an invertible non-unitary transformation, we can also relate the sub-critical (non-chiral) and super-critical (chiral) phases of NCLP, and its analog in two-vortex system, by taking the inverse CBT from the sub-critical phase into a "virtual" free particle (in commuting or non-commuting space) or, equivalently into the vortex-antivortex system, and then, by CBT composition, relate it to the super-critical phase. In this composition of CBT, it is convenient to take the same value for the magnetic field B in both phases, and choose the non-commutative parameter θ, such that $\beta = B\theta < 1$ at the beginning, and then, choose θ' so that $\beta' = B\theta' > 1$ for the second CBT composition. The inverse relation from super-critical to sub-critical is analogous.

We have investigated the relationships between the classical and quantum dynamics of NCLP and the planar two-vortex system. An interesting perspective for study is to generalize the obtained results for non planar spaces with spherical and hyperbolic geometries [39, 65]. The case of the hyperbolic geometry has a particular interest in the light of the non-relativistic conformal-invariant Schwartzian mechanics associated with the low energy limit of the Sachdev-Ye-Kitaev model [47, 48], problem related to the particle dynamics on AdS_2 and Landau problem on hyperbolic space [49].

We expect that the established relationships and mappings can be useful for the theory of anyons and fractional Hall effect, where the point vortices and non-commutative quantum mechanics play an important role.

Appendices

A. Conformal Group

Assume a metric structure $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ in a space-time manifold. Symmetries, in general, are associated with isometries of this metric, i.e. transformations of coordinates that preserve the interval invariant $ds^2 \to ds'^2 = ds^2$. This is often understood as a space-time diffeomorphism.

For instance, consider Minkowski $g_{\mu\nu} = \eta_{\mu\nu}$ space-time with Cartesian coordinates. There exist some transformations that preserve the metric $\eta_{\mu\nu}$ invariant. These transformations include translations, rotations and Lorentz boosts and they generators are

$$M_{\mu\nu} = -i\left(\eta_{\mu\rho}x^\rho\partial_\nu - \eta_{\nu\rho}x^\rho\partial_\mu\right)\,, \qquad P_\mu = -i\partial_\mu\,, \tag{.0.1}$$

where $M_{\mu\nu}$ contains the space rotations and Lorentz boosts, and P_μ corresponds to translation generators.

Preserve the metric invariant means $\eta_{\mu\nu}$ is a fixed specified matrix and the Poincare transformations keep the interval $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ invariant. Nevertheless, a coordinate transformation can be postulated for which $g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = h(x)g_{\mu\nu}(x)$. This transformation does not contradict the second postulate of relativity that requires light in vacuum fulfills $ds^2 = 0$. However, this kind of transformation produces the rate of clocks will depend on their history.

In this flat Minkowski space-time considering two points with interval $ds^2 = 0$ and performing a transformation $x^\mu \to x'^\mu = f^\mu(x)$ demanding $ds'^2 = 0$, we imply

$$\eta_{\mu\nu}dx^\mu dx^\nu = 0 \to \eta_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma dx^\rho dx^\sigma = 0\,, \tag{.0.2}$$

where $\Lambda^\mu_\nu = \partial_\nu f^\mu$. Taking account an infinitesimal transformation $\Lambda^\mu_\nu = \delta^\mu_\nu + \partial_\nu\epsilon^\mu$ in Eq(.0.2), implicates

$$\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right)dx^\mu dx^\nu = 0\,. \tag{.0.3}$$

In order to satisfy this relation, is required

$$\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right) = K\eta_{\mu\nu}\,, \tag{.0.4}$$

where K can be determined by multiplying the equality above with $\eta^{\mu\nu}$. The result yields $K = \frac{2}{d}(\partial_\mu\epsilon^\mu)$, where d is the dimension of the metric tensor. We conclude

$$\left(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu\right) = \frac{2}{d}(\partial_\rho\epsilon^\rho)\eta_{\mu\nu}\,, \tag{.0.5}$$

which refers to the constraint equation for conformal transformations. This equation can be manipulated to obtain

$$(1-d)\,\partial_\mu\partial^\mu(\partial_\rho\epsilon^\rho) = 0\,, \tag{.0.6}$$

which becomes a trivial transformation at $d = 1$, and fails at $d = 2$, case that will be considered in detail later. This transformation is precisely a change in the scale of metric $g_{\mu\nu}(x) \to g'_{\mu\nu}(x) = h(x)g_{\mu\nu}(x)$. From Eq. (.0.6) is obtained that ϵ^μ is constrained to be at most quadratic in the coordinates arguments. Any cubic or higher dependence would violate the third derivative being zero everywhere.

Taking into account the zero order term in coordinates, we obtain the aforementioned translations. A new generator $D = ix^\mu\partial_\mu$ is obtained considering a linear term in coordinates. This transformation correspond to *dilatations* that act over the coordinates as

$$x^\mu \to x'^\mu = \alpha x^\mu\,. \tag{.0.7}$$

Before considering the quadratic transformation in coordinates, is important to mention another transformation which preserves the null directions. The *inversion* $I : x^\mu \to x'^\mu = x^\mu/x^2$, for which the Jacobian transformation matrix is $\Lambda^\mu_\nu = \delta^\mu_\nu/x^2 - 2x^\mu x_\nu/(x^2)^2$ leads to

$$\eta_{\mu\nu}\Lambda^\mu_\rho\Lambda^\nu_\sigma dx^\rho dx^\sigma = \frac{1}{(x^2)^2}\eta_{\rho\sigma}dx^\rho dx^\sigma\,. \tag{.0.8}$$

Thereby, relation (.0.2) holds. The nature of I is singular on the light cone and do not constitute an infinitesimal transformation, however, we can define an infinitesimal generator as $K_\mu := IP_\mu I$. Then, transformation $exp\,(-ib^\mu K_\mu)$ maps

$$x^\mu \to x'^\mu = \frac{\left(x^\mu + b^\mu x^2\right)}{\left(1 + 2b_\mu x^\mu + x^2 b^2\right)}\,. \tag{.0.9}$$

Its generator is $K_\mu = i\left(x^2\delta^\nu_\mu - 2x^\nu x_\mu\right)\partial_\nu$, which correspond to the quadratic coordinate argument infinitesimal transformation and is called *special conformal transformation.*

In this way, adding the *dilatations* and *special conformal transformations* to the Poincare algebra, we obtain a 15 dimensional Lie algebra

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i\eta_{\mu\rho}M_{\nu\sigma} + i\eta_{\nu\sigma}M_{\mu\rho} - i\eta_{\mu\sigma}M_{\nu\rho} - i\eta_{\nu\rho}M_{\mu\sigma}\,,\\
[M_{\mu\nu}, P_\rho] &= i\eta_{\mu\rho}P_\nu - i\eta_{\nu\rho}P_\mu\,,\\
[P_\mu, P_\nu] &= 0\,,\\
[P_\mu, D] &= iP_\mu\,,\\
[M_{\mu\nu}, D] &= 0\,,\\
[K_\mu, K_\nu] &= 0\,,\\
[M_{\mu\nu}, K_\rho] &= i\eta_{\mu\rho}K_\nu - i\eta_{\nu\rho}K_\mu\,,\\
[K_\mu, D] &= -iK_\mu\,,\\
[K_\mu, P_\nu] &= 2i\eta_{\mu\nu}D - 2iM_{\mu\nu}\,,
\end{aligned} \tag{.0.10}$$

called *conformal symmetry group.*

There are d generators for both translations and special conformal transformations, rotations (with their constraint of antisymmetry), add $\frac{d(d-1)}{2}$ generators and one other generator for dilatations. We calculate a total of $\frac{(d+2)(d+1)}{2}$, hence the conformal group in d dimensions is isomorphic to the group $\mathcal{SO}(d+1,1)$ with $\frac{(d+2)(d+1)}{2}$ parameters.

Applying the aforementioned to the two-dimensional case we get

$$\mathcal{SO}(3,1) \cong \mathcal{SL}(2,\mathbb{C})\,. \tag{.0.11}$$

This is indeed true for the global conformal group for $d = 2$ but we expect the infinitesimal structure to be quite different due to equation (.0.6). From equation (.0.5) for $\mu = \nu = 0, 1$ we obtain in both cases

$$\partial_0\epsilon_0 = \partial_1\epsilon_1\,. \tag{.0.12}$$

For $\mu \neq \nu$ we are on the off-diagonal of the metric tensor (which we just take to be the identity for a moment - no Minkowski space involved) and get

$$\partial_0\epsilon_1 = -\partial_1\epsilon_0\,. \tag{.0.13}$$

If we identify $\epsilon_{0,1}$ with the real and imaginary part of a complex function respectively, obtaining the Cauchy-Riemann equations This means that the infinitesimal conformal maps are exactly the holomorphic ones on the complex plane. A general locally holomorphic map can be written down as

$$f(z) = z + \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1})\,, \tag{.0.14}$$

which can be decomposed into functions $f_n = \epsilon_n(-z^{n+1})$ corresponding to the summands in the expansion. Each of those infinitesimal transformations has a corresponding generator $l_n = -z^{n+1}\partial_z$. The commutation relation of these generators are

$$[l_n, l_m] = (n-m)l_{n+m}\,. \tag{.0.15}$$

These generators constituting Laurent series together with this commutation relation are called *Witt Algebra*. It is infinite-dimensional. But, on the other hand the group of conformal transformations in d dimensions is isomorphic to $\mathcal{SO}(d+1,1)$, for the case $d=2$ this would only allow for six dimensions, not infinitely many. This paradox comes from the fact that equation (.0.6) was used to derive the correlation between those groups but it does not hold for the infinitesimal transformations considered here. Let us instead review global conformal transformations, i.e. those which map the Riemann sphere in a $1-1$ and holomorphic way onto itself. From complex analysis we know that those functions have to fulfill a couple of constraints. It may not have essential singularities. Injectivity requires that it may only have one pole of order one and one zero of multiplicity one. It must satisfy the group property, i.e. the combination must again be such a transformation. The only holomorphic function left are thus

$$f(z) = \frac{az+b}{cz+d} \qquad ad-bc = \text{const}\,. \tag{.0.16}$$

Those transformations are called *Mobius Transformations*, where the composition of two Mobius Transformations results in another Mobius Transformation. We can identify every Mobius transform with an element of $\mathcal{SL}(2,\mathbb{C})$ which is in turn isomorphic to $\mathcal{SO}(3,1)$. We have thus shown that for *global conformal transformations*, we recover the behavior of the transformation group (.0.10).

Let us reconsider the generators l_n. We note that the principal part of the Laurent series diverges at $z=0$ for $n<-1$. Now we study the behavior for $z\to\infty$. Again from complex analysis we know that $f(z)$ is defined to have a singularity at $z\to\infty$ if $f(w := \frac{1}{z})$ has a singularity at $w=0$. Thus we first have to recast the generators into that shape. Since

$$z = \frac{1}{w} \to \frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}\,. \tag{.0.17}$$

Thus, we obtain

$$l_n = -z^{n+1}\partial_z = -\left(-\frac{1}{w}\right)^{n-1}\partial_w\,. \tag{.0.18}$$

A singularity at $w=0$ in the latter expression occurs for $n>1$. Thus, the only generators that are allowed in a holomorphic $1-1$ Laurent series are l_{-1}, l_0 and l_1. Consequently the local and global perspective of the transformation are inequivalent. This is a particularity of the two-dimensional case.

B. Hamiltonian Formalism

From the classical mechanics perspective, the possible physical system configurations are described by a generalized phase space, given by a symplectic manifold, while the dynamics are obtained from a Hamiltonian function

A *symplectic manifold* is given by a pair $(\mathcal{M}, \sigma)$, where $\mathcal{M}$ is a differentiable manifold and $\sigma : T\mathcal{M} \times T\mathcal{M} \to \mathbb{R}$ is an antisymmetric two-form. This symplectic form is non-degenerate $det(\sigma) \neq 0$ and closed, $d\sigma = 0$. Local coordinates on $\mathcal{M}$ are given by $\{x^a\}$, with $a = 1, \ldots, 2n$. The non-degeneracy, which implies $\mathcal{M}$ has an even dimension allows to write the symplectic two-form as $\sigma = \frac{1}{2}\sigma_{ab} dx^a \wedge dx^b$. By Darboux's theorem, it can be find local coordinates $x^a = (q^i, p_j)$, where $i = 1, \ldots, n$, such that $\sigma = dp_i \wedge dq^i$.

A Hamiltonian function $H : \mathcal{M} \to \mathbb{R}$ induces dynamics on $\mathcal{M}$ according to the evolution equation

$$\frac{dx^a}{dt} = \sigma^{ab}\partial_b H \,, \qquad \sigma^{ac}\sigma_{cb} = \delta^a_b \,, \tag{.0.19}$$

rewritten in terms of local coordinates (q^i, p_i) as

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i} \,, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i} \,. \tag{.0.20}$$

As a matter of fact, for any phase space function $f : \mathcal{M} \to \mathbb{R}$ is possible to define a symplectic gradient as a vector with components $X^a_f = \sigma^{ab}\partial_b f$. So, Eq. (.0.19) can be expressed as

$$\frac{dx^a}{dt} = X^a_H \,, \tag{.0.21}$$

where X^a_H is the Hamiltonian flow, and is tangent to the trajectory in $\mathcal{M}$ associated with the dynamical evolution of the system. In general, for any phase space function f, the derivative along the Hamiltonian flow is

$$\frac{df}{dt} = X_H(f) \,. \tag{.0.22}$$

In particular, H itself is conserved along the flow, and can therefore be set as a constant E, associated with an energy for a particular trajectory. It is defined a bilinear operation called Poisson bracket that acts as

$$(f, g) \to X_g(f) = -X_f(g) = \{f, g\} \,, \tag{.0.23}$$

providing a Lie algebra structure, the Poisson Algebra in which the Poisson brackets satisfy the Jacobi's identity. In local coordinates

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i} \,. \tag{.0.24}$$

Canonical transformation is a diffeomorphism $\psi : \mathcal{M} \to \mathcal{M}, x \to x' = \psi(x)$ such that σ is preserved under the pullback transformation $\psi : \tilde{\psi}(\sigma) = \sigma$. An infinitesimal canonical transformation generated by a vector field X is a canonical transformation that maps points of $\mathcal{M}$ to points along the integral lines of X. Then, σ is preserved under an infinitesimal transformation if $\mathcal{L}_X(\sigma) = 0$, where $\mathcal{L}$ is the Lie derivative.

Now, suppose there is a phase space function $C : \mathcal{M} \to \mathbb{R}$ which commutes with H in the sense of Poisson brackets $\{H, C\} = 0$. Then, according to Eq. (.0.22), C is conserved along the Hamiltonian flow and constitute a constant of motion. On the other hand, C is associated with an infinitesimal canonical transformation whose tangent vector is X_C, that remains invariant under the transformation. Such kind of canonical transformation is referred as *symmetry*, since transforms the original Hamiltonian system into itself, and consequently, transforms a trajectory into another trajectory associated with the same energy value E of the aforesaid system. If exist two conserved quantities, C_1 and C_2, then using Jacobi identity, it is obtained a third quantity $C_3 = \{C_1, C_2\}$ which is conserved, too.

A common case of symplectic manifold is given by the *cotangent bundle* over a base manifold that may be considered as a configuration space with local coordinates q^i. If exists a metric g defined on this cotangent configuration, then among all symmetries, one special class is given by isometries. They are generated by constants of motion C such that the metric is invariant along C flows, $\mathcal{L}_{X_C}(g) = 0$. Symmetries that are not isometries are typically called *hidden symmetries*.

Hidden symmetries play an important role in physics. They are commonly quadratic or higher order in the momenta coordinates [66]. At the quantum level, hidden symmetries explain some interesting particular properties of the spectra such as the unusual energy levels degeneration. An important example corresponds to the Kepler-Coulomb [67] problem with its conical section trajectories that lie in the orthogonal plane to the angular momentum vector which is integral of the system along with the energy. However, in order to obtain all the geometric properties of the system there is one more conserved quantity that must be consider, which specifies the orientation of the trajectories with respect to a particular point. This hidden symmetry is described by the Laplace-Runge-Lentz integral which is second-order in the canonical momenta. Its quantum analog integral explains the accidental degeneration in the spectrum of the hydrogen atom model [68], and also, allows to find the spectrum by purely algebraic methods.These methods are attributed in first instance to Pauli. Unfortunately not all hidden symmetries have a simple geometric interpretation like this example, and in general the study of their geometric interpretation are related to Killing, conformal Killing and Killing-Yano tensors [66].

C. Point vortices

Let us consider the Euler equation

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right) \vec{u} = -\vec{\nabla} p + \vec{f}, \tag{.0.25}$$

where p is the pressure and f is a conservative force. We restricted our study to two dimensions. By taking the curl in the equation above we obtain the evolution equation of the vorticity

$$\frac{\partial \omega}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right) \omega = 0 \qquad \frac{D\omega}{Dt} = 0\,, \tag{.0.26}$$

where operator $\frac{D}{Dt}$ is the material derivative and describe the evolution along the flow lines. From the last equation it corresponds that vorticity is conserved and it is transported along the flow lines. Supposing the vorticity ω is known it is possible to determine the vector field $\vec{u}$ that generates it. In two dimensions we can recast the fluid equations (.0.25) and (.0.26) into a Hamiltonian formalism. Consider $\vec{u} = (\dot{x}, \dot{y})$, we can represent the vorticity functions as

$$\dot{x} = \frac{d\psi}{dy}\,, \qquad \dot{y} = -\frac{d\psi}{dx}\,, \tag{.0.27}$$

by means of ψ called stream function. Formally it plays the roll of Hamiltonian for the pair of conjugate variables (x, y) and it is used to describe the dynamics of a test particle advected by the flow. Substituting equation (.0.27) into (.0.25) we obtain

$$\nabla^2 \psi(\vec{r}) = \omega(\vec{r})\,, \tag{.0.28}$$

that is a Poisson equation with ω as source term. Then, inverting the above relation we obtain the stream function to be

$$\psi(\vec{r}) = \int d\vec{r'} G(\vec{r},\vec{r'})\omega(\vec{r'})\,, \tag{.0.29}$$

where $G(\vec{r},\vec{r'})$ is the Green function solution to the equation $\nabla^2 G(\vec{r},\vec{r'}) = \delta(\vec{r},\vec{r'})$. For the plane and the sphere the Green function is

$$G(\vec{r},\vec{r'}) = -\frac{1}{4} log|\vec{r}-\vec{r'}|^2\,, \tag{.0.30}$$

where $|\vec{r}-\vec{r'}|^2 = (x-x')^2+(y-y')^2$. Once specify the vorticity we can compute ψ from equation (.0.29) and then the vorticity field becomes

$$\vec{u} = \int d\vec{r'} K(\vec{r},\vec{r'})\omega(\vec{r'})\,, \tag{.0.31}$$

where $K(\vec{r},\vec{r'}) = \frac{-(\vec{r}-\vec{r'})}{2\pi|\vec{r}-\vec{r'}|^2}$ represents the velocity field generated by point vortex. Then, considering the vorticity field associated with N point vortex as $\sum_{n=0}^{N} \gamma_n \delta(\vec{r}-\vec{r}_n)$ we obtain

$$\psi = \frac{1}{4}\sum_{n=0}^{N} log|\vec{r}-\vec{r}_n|^2\,. \tag{.0.32}$$

This equation describes the dynamics of a test particle at a point $\vec{r} = (x, y)$. In this way, for N-vortex system we obtain the evolution equations

$$\dot{x} = -\frac{1}{2}\sum_{n=0}^{N}\gamma_m(y_n - y_m)r_{nm}^{-2}\,,\,, \qquad \dot{y} = \frac{1}{2}\sum_{n=0}^{N}\gamma_m(x_n - x_m)r_{nm}^{-2}\,, \qquad m = 1,2,\ldots,N \tag{.0.33}$$

where $r_{nm}^2 = (x_n - x_m)^2 + (y_n - y_m)^2$. Kirchhoff showed that the above system can be expressed in a Hamiltonian canonical form. [69]. The symplectic structure for the system can be obtained by setting $m = 0$ in Landau problem, see [70].

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