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STUDY OF VISCOSITY IN COSMOLOGY

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Abstract

In this thesis we explore the different types of singularities that arises in the ΛCDM model when dissipative process are considered, in the framework of the Eckart's theory. In particular, we study the late-time behavior of ΛCDM model with viscous cold dark matter (CDM) and a early-time viscous radiation domination era with cosmological constant (CC). The fluids are described by the barotropic equation of state (EoS) $p = (\gamma - 1)\rho$, where p is the equilibrium pressure of the fluid, ρ their energy density, and γ is the barotropic index. We explore two particular cases for the bulk viscosity ξ, a constant bulk viscosity $\xi = \xi_0$, and a bulk viscosity proportional to the energy density of the fluid $\xi = \xi_0\rho$. Due to some previous investigations that have explored to describe the behavior of the universe with a negative CC, we extend our analysis to this case. We found that future singularities like Big-Rip are allowed but without having a phantom EoS associated to the DE fluid. Big-Crunch singularities also appears when a negative CC is present, but also de Sitter and even Big-Rip types are allowed due to the negative pressure of the viscosity, which opens the possibility of an accelerated expansion in AdS cosmologies. We also discuss a very particular solution without Big Bang singularity that arises in the early-time radiation dominant era of our model known as Soft-Big Bang.

In addition to this, we study an exact solution, which describe the evolution of a viscous warm ΛDM model (ΛWDM) at late times, where the DM component obeys a polytropic EoS and experiment a dissipation with a bulk viscosity proportional to its the energy density, leading a behavior very similar to the ΛCDM model for a small values of dissipation, evolving also to a de Sitter type expansion at the very far future. In the present thesis, the study of this solution lies in the fulfillment of the two following conditions: the near equilibrium condition, that it is assumed in Eckart' theory of non-perfect fluids; and the positiveness of the entropy production. We explore the conditions on the range of parameters of the model that allow

fulfilling the two conditions at the same time, founding that a viscous warm DM component, its compatible with these two conditions, being in this sense, a viable model from the thermodynamic point of view, which also behaves very close to the standard model, with the same asymptotic de Sitter expansion. We also show that our viscous ΛWDM model can describe the combined SNe Ia + OHD data in the same way as the ΛCDM, being in this sense a more general model with two more free parameters, contrary to ΛCDM which assumes beforehand a CDM describing like a perfect fluid. Also with the cosmological data constraint, measured at 3σ confidence level, we found an upper limit on the bulk viscosity, to be of the order of $\xi_0 \sim 10^6 Pa \times s$ in agreement with some previous investigations.

Keywords: Cosmological constant, Dark Energy, Dark Matter, Eckart theory, near equilibrium condition, Warm dark matter, ΛCDM, cosmological data.

Resumen

En esta tesis exploramos los diferentes tipos de singularidades que surgen en el modelo ΛCDM cuando se considera el proceso disipativo, en el marco de la teoría de Eckart. En particular, estudiamos el comportamiento tardío del modelo ΛCDM con materia oscura fría viscosa (CDM) y una era de dominación de radiación viscosa temprana con constante cosmológica (CC). Los fluidos se describen mediante la ecuación barotrópica de estado (EoS) $p = (\gamma - 1)\rho$, donde p es la presión de equilibrio del fluido, ρ su densidad de energía, y γ es el índice barotrópico. Exploramos dos casos particulares para la viscosidad de bulto ξ, una viscosidad de bulto constante $\xi = \xi_0$, y una viscosidad de bulto proporcional a la densidad de energía del fluido $\xi = \xi_0\rho$. Debido a algunas investigaciones previas que se han explorado para describir el comportamiento del universo con una CC negativa, extendemos nuestro análisis a este caso. Descubrimos que las singularidades futuras como Big-Rip están permitidas pero sin tener una EoS fantasma asociada al fluido DE. Las singularidades big-crunch también aparecen cuando una CC negativa está presente, pero también se permiten los tipos de Sitter e incluso Big-Rip debido a la presión negativa de la viscosidad, lo que abre la posibilidad de una expansión acelerada en cosmología AdS. También discutimos una solución muy particular sin singularidad Big Bang, que surge en la era temprana dominada por radiación de nuestro modelo, conocida como Soft-Big Bang.

Ademas de esto, estudiamos una solución exacta, que describe la evolución de un modelo viscoso tibio ΛDM (ΛWDM) en épocas tardías, donde la componente DM obedece a una EoS politrópica y experimentamos una disipación con una viscosidad de bulto proporcional a su densidad de energía, llevando un comportamiento muy similar al modelo ΛCDM para pequeños valores de disipación, evolucionando también a una expansión tipo de Sitter en un futuro muy lejano. En el presente trabajo, el estudio de esta solución radica en el cumplimiento de

las dos condiciones siguientes: la condición de equilibrio cercano, que se asume en la teoría de fluidos no perfectos de Eckart; y la positividad de la producción de entropía. Exploramos las condiciones en el rango de parámetros del modelo que permiten cumplir las dos condiciones al mismo tiempo, mostrando que una componente de DM tibia viscosa, es compatible con estas dos condiciones, siendo en este sentido, un modelo viable desde el punto de vista termodinámico, que además se comporta muy cercano al modelo estándar, con la misma expansión asintótica de Sitter. También mostramos que nuestro modelo viscoso ΛWDM puede describir los datos combinados SNe Ia + OHD de la misma manera que el ΛCDM, siendo en este sentido un modelo más general con dos parámetros libres más, al contrario de ΛCDM que asume de antemano una CDM descrito como un fluido perfecto. También con la restricción de datos cosmológicos, medidos a un nivel de confianza de 3σ, encontramos un límite superior en la viscosidad de bulto, del orden de $\xi_0 \sim 10^6 Pa \times s$ de acuerdo con algunas investigaciones anteriores.

Palabras clave: constante cosmologíca, condición de equilibrio cercano, energia oscura, teoría de Eckart ,materia oscura, materia oscura tibia, ΛCDM, data cosmologica.

Dedication

Throughout this journey of years of study and learning, it is impossible to show my gratitude to the people in my personal life who were with me side by side throughout this process, that's why the most important work of my academic life is dedicated to Andrea Manriquez,Glenn Manriquez, Rosa Ortiz, and my son Santiago Jovel. Thank you all, for supporting me me during this difficult period, plainly, this work would not be ready without the support of each one of you, and especially my son, that you know that every time the moments were difficult, I always remembered that I am the world for someone. Also, I want to thank professors Ph.D Norman Cruz and Ph.D Esteban Gonzalez, who always had the patience and affection to teach me throughout this time, and without them this thesis would not be possible, even in the most difficult moments derived from the pandemic process, they always showed their humanity and their vocation to train my learning and encourage me to continue forward. I just want to say thank you also on behalf of my family for everything.

“The known is finite, the unknown infinite; from the intellectual point of view we are on a small island in the middle of an illimitable ocean of inexplicability. Our task in each generation is to reclaim a little more land."

T. H. Huxley 1887.

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Introduction

It is well known in current cosmology that the accelerated expansion of the universe is one of the most fascinating puzzles in physics. This behavior is supported by the cosmological data coming from measurements of Supernovae type Ia (SNe Ia) [1–3], the observational Hubble parameter data (OHD) [4], the baryonic acoustic oscillations (BAO) [5], the cosmic microwave background (CMB) [6], and information from large-scale structures (LSS) formation coming from WMAP [7]; showing also that the universe is spatially flat.

There are different approaches in order to describe this accelerated expansion of the universe. One of them is to add in the energy-momentum tensor $T_{\mu\nu}$, in the right hand side of the Einstein gravity equation, an exotic fluid with negative pressure, dubbed dark energy (DE), which can cause an overall repulsive behavior of the gravity at large cosmological scales (see [8–10] for some excellent reviews). The other approach is by modifying the left hand side of the Einstein equation, i.e., the geometry of the space time, that leads to different ideas of modify gravity (for some theories of modified gravity which involve this idea see [11–14]). For the first approach, the most simple model is the ΛCDM model which it is also the best cosmological model in order to describe the cosmological data [3, 7]. In this model, the current universe is dominated by dark matter (DM) and DE, representing approximately the 30% and 70% of the total energy density of the universe, respectively. The DM is described as a pressureless fluid known as cold DM (CDM), and the DE is given by the cosmological constant (CC) Λ, which can be characterized by a barotropic EoS with barotropic index $\gamma = 0$, causing the acceleration in the universe expansion [1]. However, this model is not absents of problems, of which we can highlight:

- The CC problem: the value of the CC predicted from field theoretical estimations is about 60-120 order of magnitude larger than the observed value

[15–17].

- The coincidence problem: current energy densities of DM and DE have the same order of magnitude, but in the ΛCDM model, these energy densities evolves differently, so, it is necessary a fine-tuning between them in the early universe in order to that both densities match in magnitude at the current time [17–19].

- The H_0 tension: measurements of the Hubble parameter at the current time, H_0, shows a discrepancy of 4.4σ between the value inferred from Planck CMB and the locally meassurements obtained by A. G. Riess *et al.* [20].

- EDGES: most recently, results of the experiment to detect the global EoR signature (EDGES) detect an excess of radiation that is not predicted by the ΛCDM model in the reionization epoch, specifically at z≈ 17 [21].

One approach to overcome some of these problems, without going further than ΛCDM or modify the gravity, is to consider disipative fluids as a more realistic way of treating cosmic fluids [22–24]. In this sense, several authors have shown that a bulk viscous DM in different models without DE can cause the accelerated expansion of the universe [25–35], due to the negativeness of the viscous pressure, which allows to alleviate in principle the CC and the coincidence problems. The excess of radiation predicted by EDGES are explained in [36], where the authors consider a viscous nature in DM. In [37, 38] the authors address the H_0 tension problem as a good chance to construct new cosmological models with viscous/inhomogeneous fluids in the context of a bulk viscosity. Furthermore, tensions in the measurements of $\sigma_8 - \Omega_m$ (where σ_8 is the r.m.s. fluctuations of perturbations at $8h^{-1}Mpc$ scale) and $H_0 - \Omega_m$ coming from LSS observations and the extrapolated from Planck CMB parameters using the ΛCDM model, can be alleviate if one assumes a small amount of viscosity in the DM [24]. Some authors have also used bulk viscosity in inflationary phases of the universe [39, 40].

It is important to mention that from Landau and Lifshitz [41] we know that the bulk viscosity in the cosmic evolution seems to be significant and we can interprete, from the macroscopic point of view, that is equivalent to the existence of slow processes restoring equilibrium state. Some authors propos that bulk viscosity of the cosmic fluid may be the result of non-conserving particle interactions [42] and others have shown that different cooling rates of the components of the cosmic medium can produce bulk viscosity [43–45]. Also, for neutralino CDM bulk viscosity pressure appears in the CDM fluid due to the energy transferred from the CDM fluid to the radiation fluid [46].

Many observational properties of disk galaxies can be reproduced by a dissipative DM component, which appears as a residing component in a hidden sector [47, 48]. On the other hand, at perturbative level, viscous fluid dynamic provides a simple and accurate framework for extending the description into the nonlinear regime [49]. In addition, the viscous effect could be the result of the interaction between DM and DE fluids [50]. In this sense, diffusion in cosmology plays an important role, since a diffusive exchange of energy between dark energy and dark matter could occur [51, 52]. Since, up to date it is unknown the nature of the DM and the dissipative effect in cosmology can not be discarded, it is of physical interest to explore the behavior of this type of DM in the ΛCDM model.

In order to study dissipative processes in cosmology it is necessary to develop a relativistic thermodynamic theory out of equilibrium, being Eckart the first who developed it [53]. Later, it was discovered that Eckart's theory was not really relativistic, since it is a non-causal theory [54, 55]. A causal theory was proposed by Israel and Stewart [56, 57], but it presents a much greater mathematical difficulty than the Eckart's theory, even in scenarios where the bulk viscosity does not present very exotic forms. Therefore, many authors work in the Eckart's formalism in order to have a first approximation of the cosmological behavior with dissipative fluids [22, 58–62], since the Israel-Stewart's theory is reduced to the

Eckart's theory if the relaxation time for transient viscous effects is equal to zero. [63]

As we mentioned before, in both Eckart's and Israel-Stewart's theory it is possible to describe the accelerated expansion of the universe without the inclusion of a CC. Nevertheless, as it was previously discussed by Maartens [63], in the context of dissipative inflation, the condition to have an accelerated expansion due only to the negativeness of the viscous pressure enters into direct contradiction with the near equilibrium condition that is assumed in the Eckart's and Israel-Stewart's theory

$$\left|\frac{\Pi}{p}\right| \ll 1, \tag{1}$$

which means that the viscous stress Π must be smaller than the equilibrium pressure p of the dissipative fluid. So, following this line, it as been proved in [64, 65] that the inclusion of a positive CC could preserve the near equilibrium condition (1) in some regime. The price to pay is to abandon the idea of unified DM models with dissipation as models that can consistently describe the late time evolution of the universe. It is important to mention that a negative CC can not be ruled out from study in cosmology [66–74]. For example, a negative CC appears naturally in superstring theory in the dual space $AdS_5 \times S^5$ [75–77]. Some authors even mentioned the possibility of a transition between a negative CC to a positive one [67, 78]. Even more, a negative CC has been explored by many authors with the aim of alleviating the H_0 tension [68–74].

Works with dissipation where the CC is included have been studied in recent times, for example the authors in [60] already work in Eckart formalism with CC and a bulk viscosity proportional to the Hubble parameter, or more interesting scenarios can be seen in [79] where the authors also include a CC that is variable in time. On the other hand, some authors have shown that the presence of bulk viscosity in the DE could cause that their effective barotropic index can be less than 0 [61, 62]. Fluids with a barotropic index $\gamma < 0$ are dubbed “phantom" [80] and can not be ruled out of the current cosmological data. For example, some works indicated that the barotropic index of the DE is inconsistent with the value of 0 at 2.3σ level [7, 81]. The possibility of phantom EoS for the DE open an interesting scenario known as Big-Rip, in which the scale factor presents a singularity in a finite future time [82]. Following this line, the authors in [83, 84] have made a classification of the different singularities obtained in models with phantom DE.

Other interesting issue related to viscous fluid is the possibility of avoid singularities. In the framework of general relativity, many studies found cosmological scenarios where there are no singularities, corresponding to emergent and bouncing universes [85–89]. A regular universe without Big Bang was found in [58], where the viscosity drives the early universe to a phase with a finite space-time curvature. This regular scenario called “soft Big-Bang" are also discussed in other contexts [88, 89], describing universes with eternal physical past time, that come from a static universe with a radius greater than the Planck radius to be far out of the regime where quantum gravity has to be employed.

Another important possible extension of the standard cosmological model has also been investigated in recent decades. It is well known that a CDM is capable to explain the observed structure very well above $\sim$ 1 Mpc, while it has issues explaining small-scale structure observations [90, 91], such as the missing satellite problem [92, 93]— this refers to the discrepancy of approximately 10 times more dwarf galaxies between the values obtained by the numerical simulations based on the ΛCDM model and the observed ones in the cluster of galaxies. In this sense, a warm DM (WDM) can potentially be a good candidate to explain small-scale structure observations that currently represent a challenge for a CDM model. Many studies of the number of satellites in the Milky Way, or small halos with dwarf galaxies, appear to be in better agreement with the observations for a WDM than they are for a CDM [91, 94, 95]. One of the well-motivated WDM hypotheses implies an extension of the standard model of particle physics by three sterile (right-handed, gauge singlet) neutrinos [96–98], produced via mixing with active neutrinos in the early universe [96, 99–103].

On the other hand, from the perturbation point of view, the no-linear effects make the power spectrum of the WDM look very much similar to a CDM, and LSS such as filaments, sheets, and a large void suggest that a WDM reproduces the observed ones well[104]; the WDM is an interesting alternative from the point of view of cosmology and particle physics.

All the discrepancies mentioned above implies extensions of the ΛCDM model like the considerations of dissipative effects in a WDM component, which leads to taken into account a relativistic thermodynamic theory of non-perfect fluids out of equilibrium.

The aim of this thesis is to explore exact solutions of a viscous ΛCDM like model, looking for the conditions that leads to early and late time singularities, considering a bulk viscosity term constant and proportional to the energy density of the dissipative fluid. These two simple cases open up a great variety of behaviors which will allow us to study different singularity scenarios in the framework of the Eckart's theory. We will discuss our results according to the classification given in [83]. Also, we will investigate solutions which represent regular universes, as it was found by Murphy in [58], but with the inclusion of a CC. It is important to note that many authors have studied these type of singularities within the framework of cosmological models filled with a phantom DE [10, 62, 83, 84, 105]. In our case, the model can be characterized by an effective EoS that represents the behavior of the two fluids of the model, the dissipative fluid and the CC, as a whole. Even more, we will study solutions where the CC can take negative values. Therefore, in this thesis we will try to give a more complete understanding of the early and late-time singularities when dissipative process are considered in a ΛCDM like cosmological model. We also study an analytical solution obtained in chapter 3.2 (and it was published in [106]). This solution was obtained using the expression $\xi = \xi_0 \rho$ for the bulk viscosity, where $\xi_0 > 0$ is a bulk viscous constant, and have the important characteristics that, for a positive CC, behaves very similarly to the ΛCDM model for all the cosmic time when $\xi_0 \to 0$, without singularity towards the past in an asymptotic behavior known as "soft-Big Bang", and with an asymptotic de Sitter expansion towards the future. This last behavior is of interest because the solution tends to the de Sitter expansion regardless of the value of ξ_0 and γ, as long as $\xi_0 < \gamma/3H_0$. Therefore, we focus our study to the late-time behavior of this solution assuming that $\gamma \neq 1$ but close to 1, which represents a dissipative ΛWDM model with the same asymptotic late-time behavior that the ΛCDM model. In particular, we study the near equilibrium condition and the positiveness of entropy production, additionally, we study the mathematical stability of the Hubble parameter of this solution to find the constraints that these criteria impose on the model's free parameters. We focus on the possibility to have a range of them satisfying all of these conditions, and compare it with the best fit values obtained from the cosmological constraint with the SNe Ia + OHD data. This study leads to obtaining some important clues about the physical behavior of the analytical solution, which represents a particular extension of the standard cosmological model. In this model, we have two more free parameters, namely, γ and ξ_0; contrary to the standard cosmological model ΛCDM, which try to give a more complete description of the nature of the DM component, suggested by previous investigation made in the context to alleviate tensions in the standard cosmological model. Despite the fact that we are not facing in this work none of the mentioned tensions, our primary intention is to explore if the two extension made to the standard model present also a consistent relativistic fluid description and not only a well suitable fit with the cosmological data.

The outline of this thesis is as follow: In chapter 1 the basic aspects of Modern Cosmology will be discussed, as well as the study of the perfect and non-perfect fluids, additionally, we describe the non-causal Eckart's theory for the study of non-perfect fluids. In Chapter 2 we present the field equations of Eckart's theory, and we find the general differential equation to solve. Also, we present the possibility of de Sitter like solutions that arises from the general differential equation previously found. In chapter 3.2 we describing the different types of singularity that arises for a Friedmann–Lemaître–Robertson–Walker (FLRW) metric, in the context of geodesic incompleteness. We study the late-time singularities that arises in our model for a constant dissipation, an a dissipation proportional to the energy density of the dissipative fluid, for a positive CC. We will do the same for the case of a negative CC. Also, we discuss early-time singularities for the case of positive CC and negative CC. Finally, we will finish the chapter discussing an early-time solution without Big Bang singularity called "soft Big-Bang". In chapter 4 we summarize a particular solution that was found in 3, and it was published in [106]. We present general results about the near equilibrium condition, the mathematical stability in the Hubble parameter, and the entropy production for this particular solution. We study the solution at late-times, described by a ΛWDM model, and we constrain the free parameters of our model with the SNe Ia and OHD data; we discuss this results, comparing them with ΛCDM model, and we study the completeness of both, the near equilibrium condition, and entropy production for the actual data. In addition, we find a upper limit for the present value of the bulk viscous constant. In chapter 5 we present some conclusions and final discussions.

Chapter 1

Basic foundations of cosmology

In this chapter, the fundamental aspects of modern cosmology will be discussed, starting from the Friedmann equations that will allow us to understand and deepen both the Standard Model of Cosmology, as well as the recent accelerated expansion of the Universe and its evolution. Then relativistic hydrodynamics for perfect fluids will be discussed, followed by the case of hydrodynamics for non-perfect fluids from Eckart's point of view [53].

1.1 The Friedmann equations

The Friedmann[1] equations govern the expansion of space in homogeneous and isotropic models of the universe, within the context of general relativity. Here we will find these equations, to do this we need to describe briefly the General Theory of Relativity and the metric for a homogeneous and isotropic universe.

The General Theory of Relativity describes the fundamental interaction of

[1] Using the Library of Congress transliteration system for Cyrillic, his name would be "Aleksandr Fridman." However, in the German scientific journals where he published his main results, he alternated between the spellings "Friedman" and "Friedmann" for his last name. The two-n spelling is more popular among historians of science.

gravitation as a result of space-time being curved by matter and energy. They were first published by Einstein in 1915 [107], the dynamics are described by Einstein's field equations, which are a set of ten differential equations,second-order, nonlinear, not all of them independent, nonlinear, given by the expression

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1.1}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R the Ricci scalar or curvature scalar, $g_{\mu\nu}$ the metric tensor, Λ is the CC, $T_{\mu\nu}$ the energy-momentum tensor, G the universal gravitational constant and c the speed of light. We will consider the natural units $8\pi G = c = 1$.

In the case of cosmology, the so-called Cosmological Principle is imposed, which establishes that, at sufficiently large scales, the Universe is homogeneous and isotropic, an affirmation today supported by observations that indicate that this principle is valid for scales greater than $100Mpc$ [108]. In the 1930s, the physicists Howard Robertson and Arthur Walker asked, "What form can the metric of spacetime assume if the universe is spatially homogeneous and isotropic at all time and if distances are allowed to expand or contract as a function of time?" The metric they derived (independently of each other) is called the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, given by

$$ds^2 = -g^{\mu\nu}dx^\mu dx^\nu = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2\right), \tag{1.2}$$

where t is the cosmic time, r is the radial comoving coordinate, θ and ϕ the angular coordinates, and k describes the type of curvature of the universe, where $k = 1$ is a spherical space, $k = 0$ is a flat space, and $k = -1$ is a hyperbolic space, here the function $a(t)$ is the scale factor, equal to one at the present moment $t = t_0$ and totally independent of location or direction. The scale factor $a(t)$ tells us how the expansion (or possibly contraction) of the universe depends on time. With Eq. (1.2), the left side of the Einstein's field equations can be solved.

If we consider also, the energy-momentum tensor of an ideal fluid

$$T^{\mu\nu} = (\rho + p)\, u^{\mu} u^{\nu} + p g^{\mu\nu}, \tag{1.3}$$

where u^{μ}, ρ and p are the four-velocity, energy density an pressure of the cosmic fluid respectively. We can solved the right-hand side of the Einstein field equations, and we will get the Friedmann and acceleration equations, and for a perfect fluid we have the conservation equations (see appendix A, Eqs. (A.10), (A.29), (A.33) for technical details) given respectively, by

$$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho + \Lambda}{3} - \frac{k}{a^2}, \tag{1.4}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6}\,(\rho + 3p) + \frac{\Lambda}{3}, \tag{1.5}$$

$$\dot{\rho} + 3H\,(\rho + p) = 0, \tag{1.6}$$

where the dot represents the derivative with respect to cosmic time and H is known as the Hubble parameter.It is important to mention that, from Eq. (1.5), if the pressure p is positive, then it provides a negative acceleration that is, it decreases the value of $\dot{a}$ and reduces the relative velocity of any two points in the universe, contrary to the sign provided by the CC, whose sign increase the relative velocity.

This system of equations describe the evolution of the universe, being only two of them linearly independent, for three unknowns ρ, p, and a. To close the system, the energy density of the fluid is related to its pressure by an EoS.

In general, EoS can be dauntingly complicated. But cosmology usually deals with dilute gases, for which the EoS is simple. For substances of cosmological importance, the equation of state can be written in a simple linear form

$$p = (\gamma - 1)\,\rho, \tag{1.7}$$

which corresponds to a barotropic fluid, and γ is know as a barotropic index.

Many known fluids can be described by this type of EoS, for example, radiation can be represented by a barotropic EoS whose barotropic index is $\gamma = 4/3$, while in the case of dust, $\gamma = 1$ (fluid without pressure or non-relativistic). Also, it is important to note that, the inclusion of the CC in the Friedmann equation (2.2), tells that this term is equivalent to add a new component to the universe with energy density $\rho_\Lambda = \Lambda$. Therefore, if Λ remains constant in time, then so does its associated energy density ρ_Λ. The conservation Eq. (2.6) tells us that to have ρ_Λ constant with time, the Λ term must have a pressure

$$p_\Lambda = -\rho_\Lambda = -\Lambda. \tag{1.8}$$

Thus, we can think of the CC, according to Eq. (1.7), as a component of the universe that has a negative energy pressure with a barotropic index $\gamma = 0$.

Another important point to mention is, if we consider a spatially flat universe ($\kappa = 0$) contributed only by a CC (Λ), this it is, with zero energy density [2]. For a flat, lambda-dominated universe, the Friedmann equation Eq. (2.2) takes the form

$$\dot{a}^2 = \frac{\Lambda}{3} a^2. \tag{1.9}$$

The solution of this equation is

$$a(t) = \exp^{H_0(t-t_0)}, \tag{1.10}$$

this scale factor shows an expanding universe. Therefore, a spatially flat universe with nothing but a CC is exponentially expanding.

Since the energy density and pressure for the different components of the universe are additive, if we assume that ρ_i and γ_i represent the energy density and barotropic index of the ith component of the universe, from Eq. (2.6) one gets

$$\rho_i = \rho_{i,0} a^{-3\gamma_i}, \tag{1.11}$$

[2]Such a universe is sometimes called de Sitter universe, after Willem de Sitter, who pioneered its study in 1917.[109]

which relates the energy density of each component of the universe with the scale factor.

1.2 Cosmological standard model

The cosmological standard model, or ΛCDM model, has a good fit to the currently available observational data. It is spatially flat, and contains radiation, matter, and a CC (some of its properties are listed in Table 1.1[3, 5, 7, 110, 111]).

Considering (1.11) we can see that the energy density for matter has the dependence $\rho_m = \rho_{m,0}/a^3$; for radiation it will be $\rho_r = \rho_{r,0}/a^4$. The evidence indicates the presence of a CC with energy density $\rho_\Lambda = \rho_{\Lambda,0} =$constant. So, we consider a universe with contributions from matter ($\gamma = 1$), radiation ($\gamma = 4/3$) and a CC ($\gamma = 0$). In general the Friedmann equation, (2.2) using Eq. (1.11), takes the form

$$H = H_0\sqrt{\frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3} + \Omega_{\Lambda,0} + \frac{\Omega_{k,0}}{a^2}}, \tag{1.12}$$

where $\Omega_{r,0} = \rho_{r,0}/3H_0^2$, $\Omega_{r,0} = \rho_{m,0}/3H_0^2$, $\Omega_{\Lambda,0} = \rho_{\Lambda,0}/3H_0^2$, $\Omega_{k,0} = -k/3H_0^2$ and H_0 is the Hubble constant and in the ΛCDM is assumed the value obtained from Planck [3] to be $H_0 = 68kms^{-1}Mpc^{-1}$; additionally, we have the follow Friedmann's constraint $1 = \Omega_{r,0} + \Omega_{m,0} + \Omega_{\Lambda,0} + \Omega_{k,0}$.

TheΛCDM model has $\Omega_{k,0} = 0$, and hence is spatially flat. However, although a perfectly flat universe is consistent with the data, it is not *demanded* by the data. Thus many research of investigation [112–114] considering the possibility that the curvature term in Eq. (1.12), might be nonzero. However, although this line of research may be interesting, in this thesis we consider a spatially flat Universe.

From (1.1) we can see that the Universe can be divided in three distinct eras: the era of radiation dominance (before than $t_{rm} = 0.050Myr$), the era of matter dominance (before $t_{m\Lambda} = 10.2Gyr$), and the era of DE domination (after $t_{m\Lambda} = 10.2Gyr$).

At present times we are in the final transition stage from the era of domination of matter to the era of domination of DE. Note that, at late times in the evolution of the Universe, when $a \to \infty$, the Hubble constant given by Eq. (1.12) tends asymptotically to $H(t \to \infty) \to H_0\sqrt{\Omega_{\Lambda,0}}$, and the integral solution for the scale factor for this last expression it would be given by Eq. (1.10), that is, under this model our Universe tends at late times to an exponential expansion. Backward in time ($a \to 0$), Eq. (1.12) shows that the radiation density predominated over matter density and DE density.

Our model would be based on the composition of only two fluids: (i) dissipative matter, and (ii) dark energy modeled as a CC. Therefore, the density components of each element in our universe are restricted by Friedmann's constraint

$$1 = \Omega_m + \Omega_\Lambda. \tag{1.13}$$

Note that, in the study of early times of the Universe, radiation is imposed as the dominant fluid in relation to the value of Ω_Λ, therefore for an arbitrary very early radiation time we can consider the value of $\Omega_\Lambda = 10^{-6}$, in order to use the exact solution found, and explore its behavior to the past.

Table 1.1: Properties of the ΛCDM model

List of ingredients		
Photons:	$\Omega_{\gamma,0} = 5.35 \times 10^{-5}$	
Neutrinos:	$\Omega_{\nu,0} = 3.65 \times 10^{-5}$	
Total radiation:	$\Omega_{r,0} = 9.0 \times 10^{-5}$	
Baryonic matter	$\Omega_{bary,0} = 0.048 \times 10^{-5}$	
Nonbaryonic dark matter	$\Omega_{dm,0} = 0.262 \times 10^{-5}$	
Total Matter:	$\Omega_{m,0} = 0.31 \times 10^{-5}$	
Cosmological constant:	$\Omega_{\Lambda,0} = 0.69 \times 10^{-5}$	
Important epochs		
Radiation–matter equality:	$a_{rm} = 2.9 \times 10^{-4}$	$t_{rm} = 0.050 Myr$
Matter–lambda equality:	$a_{m\Lambda} = 0.77$	$t_{m\Lambda} = 10.2 Gyr$
Now:	$a_{m\Lambda} = 1$	$t_0 = 13.7 Gyr$

1.3 Perfect and non-perfect fluids

The study of fluids in the context of general relativity is of vital importance to understand the dynamics of the Universe. We must first understand that, a perfect fluid is defined as a medium for which at every point there is a locally inertial Cartesian frame of reference, moving with the fluid, in which the fluid appears the same in all directions. In such a locally inertial co-moving frame[3] the components of the energy-momentum tensor take the following form

$$T^{ij} = \delta_{ij}, \;\; T^{i0} = T_{0i} = 0, \;\; T^{00} = \rho, \tag{1.14}$$

[3]A co-moving frame being characterized by the condition that at a given point, the velocity four-vector is $u^0 = 1$ and $u^i = 0$.

where the index i and j run over the three Cartesian coordinate directions $1, 2, 3.$ (The reason for this index label, is because the non-zero value of T^{i0} and any term in T^{ij} other than one proportional to δ_{ij} would select out special directions in space, such as the direction of T_{i0}). In a locally inertial Cartesian frame with an arbitrary velocity, the energy-momentum tensor takes the form (see appendix A.2 Eq. (A.47) for all technical details)

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)\, u^{\alpha}u^{\beta}, \tag{1.15}$$

where ρ and p are defined to be the same as in the co-moving inertial frame, and u^{α} is a four-vector known as *velocity vector*, with components in locally Cartesian co-moving inertial frame $u_0 = 1$ and $u_i = 0$, and is normalized so that, in any inertial frame, $\eta_{\alpha\beta}u^{\alpha}u^{\beta} = -1$.

It follows that in a general gravitational field the energy-momentum tensor of a perfect fluid is

$$T^{\alpha\beta} = pg^{\alpha\beta} + (p+\rho)\, u^{\alpha}u^{\beta}, \quad g_{\alpha\beta}u^{\alpha}u^{\beta} = -1. \tag{1.16}$$

This formula for $T^{\alpha\beta}$ is generally covariant [4] and it is true in locally inertial Cartesian coordinate systems. In addition, if the pressure depends on the density n of some conserved quantity such as baryon number, then we need the equation of conservation, which in locally inertial Cartesian frames reads (followed by (A.54))

$$\frac{\partial}{\partial x^{\alpha}}\left(nu^{\alpha}\right) = 0, \tag{1.17}$$

thus in a general coordinate system in an arbitrary gravitational field, we have

$$\left(nu^{\alpha}\right)_{;\mu} = 0. \tag{1.18}$$

Here $;$ accounts for covariant derivative. Let's go now to the study of non-perfect

[4] The physical quantities must transform covariantly, that is, under coordinate transformations in the frame of Lorentz.

fluid. For this we add a small correction $\Delta T^{\alpha\beta}$ to $T_{\mu\nu}$ in Eq. (1.15) in locally inertial Cartesian coordinate system. Them

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)\, u^{\alpha} u^{\beta} + \Delta T^{\alpha\beta}, \tag{1.19}$$

and a small correction ΔN^{α} to the particle current

$$\frac{\partial}{\partial x^{\alpha}} \left(n u^{\alpha} + \Delta N^{\alpha} \right) = 0. \tag{1.20}$$

The scalar ρ is defined as the energy density observed in a co-moving frame in which $u^i = 0$, so that in this frame $\Delta T^{00} = 0$. Therefore, in all locally inertial Cartesian frames $u^{\alpha} u^{\beta} T_{\alpha\beta} = 0$, since this quantity is a scalar and vanishes in a co-moving frame. It is important to mention that the scalar n can be defined as the value of the conserved density observed in such a co-moving frame, and therefore $\Delta N^0 = 0$, so by the same reasoning in all locally inertial Cartesian frames we have $u_{\alpha} \Delta N^{\alpha} = 0$.

The pressure can be defined as a function of ρ in a static homogeneous fluid. But the definition of the velocity four-vector u^{α} remains somewhat ambiguous. We could define u^i to be the velocity of particle transport (this is the option adopted by C. Eckart [53]). With this definition of velocity, the second law of thermodynamics, together with the condition $u_{\alpha} \frac{\partial T^{\alpha\beta}}{\partial x^{\beta}} = 0$ (see appendix A.3 for all technical details) gives for a bulk viscous fluid, the following expression (Eq. (A.85))

$$\Delta T^{\alpha\beta} = -\xi \left(\eta^{\alpha\beta} + u^{\alpha} u^{\beta} \right) \frac{\partial u^{\gamma}}{\partial x^{\gamma}}, \tag{1.21}$$

where ξ is the coefficient of bulk viscosity and it has to be defined.It is then, an immediate consequence of the Equivalence Principle, that in general coordinate systems in arbitrary gravitational field we have

$$\Delta T^{\alpha\beta} = -\xi \left(g^{\alpha\beta} + u^{\alpha} u^{\beta} \right) u^{\gamma}_{;\gamma}. \tag{1.22}$$

Therefore, in a general coordinate system $u^{\gamma}_{;\gamma} = \partial u^{\gamma}/\partial x^{\gamma} + u^{\alpha}\Gamma^{\gamma}_{\alpha\gamma}$ and ussing Eq. (A.4), we get $u^{\gamma}_{;\gamma} = 3H$. Then, Eq. (1.21) tourn out to be

$$\Delta T^{\alpha\beta} = -3H\xi \left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right), \tag{1.23}$$

and the Eq. (1.19) in a general coordinate systems is

$$T^{\alpha\beta} = (p + \Pi)g^{\alpha\beta} + (p + \Pi + \rho)\, u^{\alpha}u^{\beta}, \tag{1.24}$$

where $\Pi = -3H\xi$. We can interpret this last result as and effective cosmological pressure given by $P_{eff} = p + \Pi$, and then, the effect of the bulk viscosity is to produce a negative pressure Π that leads to an acceleration in the universe expansion according to Eq. (1.5).

Chapter 2

Dissipative Cosmology

In this chapter we study dissipative cosmology in the framework of the Eckart's theory. We establish the bases that will be useful in chapter 3.2. The fluids are described by the barotropic equation of state (EoS) $p = (\gamma - 1)\rho$, where p is the equilibrium pressure of the fluid, ρ their energy density, and γ is the barotropic index. We explore two particular cases for the bulk viscosity ξ: *a)* A constant bulk viscosity $\xi = \xi_0$,*b)* A bulk viscosity proportional to the energy density of the fluid $\xi = \xi_0\rho$. Due to some previous investigations that have explored the behavior of the Universe with a negative CC, we extend our analysis to this case.

2.1 Theory of Eckart with CC

In what follows, we will consider a flat FLRW cosmological spacetime, dominated by only two matter components: a DE given by Λ, and a barotropic fluid with EoS $p = (\gamma - 1)\rho$, where p is the equilibrium pressure of the fluid, ρ their energy density and γ is the barotropic index that takes the values of $\gamma = 1$ for CDM and $\gamma = 4/3$ for radiation.

Only the barotropic fluid experience dissipative processes during their cosmic evolution, with a bulk viscosity coefficient ξ that depends on their energy density through the power-law

$$\xi = \xi_0 \rho^m, \quad \xi_0 > 0, \tag{2.1}$$

where ξ_0 and m are constant parameters, with $\xi_0 > 0$ in order to be consistent with the second law of thermodynamics [111].

The behavior described by Eq. (2.1) for the viscosity has been widely investigated in the literature as one of the simplest and most natural choices since the bulk viscosity of fluids depends, particularly, on its temperature and pressure, and therefore it is physically suitable to take this dependence. Other choices include, for example, the function $\xi = \xi_0 + \xi_1 H$ [60], but in this case, and since we are including a CC, this choice implies that the viscosity of the fluid is a function not only of its properties but also of the CC.

In the Eckart's theory, the field equations in presence of bulk viscous are

$$H^2 = \frac{\rho}{3} + \frac{\Lambda}{3}, \tag{2.2}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6}\left(\rho + 3P_{eff}\right) + \frac{\Lambda}{3}, \tag{2.3}$$

P_{eff} is an effective pressure given by

$$P_{eff} = p + \Pi, \tag{2.4}$$

being Π the bulk viscous pressure defined in the Eckart's theory by

$$\Pi = -3H\xi. \tag{2.5}$$

The conservation equation takes the form

$$\dot{\rho} + 3H(\rho + p + \Pi) = 0. \tag{2.6}$$

Therefore, we can obtain from Eqs. (2.1)-(2.6) a single evolution equation for H, given by (see appendix Eq. (C.9))

$$2\dot{H} + 3\gamma H^2 - 3\xi_0 H(3H^2 - \Lambda)^m - \Lambda\gamma = 0. \tag{2.7}$$

Since we are interested in comparing some solutions of Eq. (2.7) for different values of m with the standardΛCDM model, we display below the solution for $H(t)$ y $a(t)$ with the initial conditions $H(t=0)=H_0$ and $a(t=0)=1$, for the case without dissipation ($\xi=0$) (see appendix C.2, Eqs. (C.24) y (C.32))

$$H(t)=\frac{H_0\sqrt{\Omega_\Lambda}\left(\left(\sqrt{\Omega_\Lambda}+1\right)e^{3\gamma H_0 t\sqrt{\Omega_\Lambda}}-\sqrt{\Omega_\Lambda}+1\right)}{\left(\sqrt{\Omega_\Lambda}+1\right)e^{3\gamma H_0 t\sqrt{\Omega_\Lambda}}+\sqrt{\Omega_\Lambda}-1}, \tag{2.8}$$

$$a(t)=\left(\cosh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}H_0 t}{2}\right)+\frac{\sinh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}H_0 t}{2}\right)}{\sqrt{\Omega_\Lambda}}\right)^{\frac{2}{3\gamma}}, \tag{2.9}$$

where $\Omega_\Lambda=\Lambda/(3H_0^2)$. From Eq. (2.8) we can see that $H=\sqrt{\Lambda/3}$ for very late times, corresponding to the de Sitter behavior.

2.2 The de Sitter like solutions

Before to make a complete integration of Eq. (2.7), we will explore the possibility of de Sitter like solutions. Knowing this behavior will help us to compare with the asymptotic behaviors in cases when $\dot{H}\neq 0$. Taking $H=H_{dS}$ with $\dot{H_{dS}}=0$, Eq. (2.7) reduces to the following algebraic equation

$$3\gamma H_{dS}^2-3\xi_0 H_{dS}(3H_{dS}^2-\Lambda)^m-\Lambda\gamma=0. \tag{2.10}$$

A general solution can be quickly found if the above equation is written as

$$(3H_{dS}^2-\Lambda)\left[\gamma-3\xi_0 H_{dS}(3H_{dS}^2-\Lambda)^{m-1}\right]=0, \tag{2.11}$$

which indicates that the values of H_{dS} given by

$$H_{dS}=\pm\sqrt{\frac{\Lambda}{3}}, \tag{2.12}$$

are two real solutions of Eq. (2.11), for $m\geq 1$ and $\Lambda>0$. Note that the positive solution corresponds to the usual de Sitter one, and the contracting solution

$H_{dS} < 0$ it is not of physical interest. The other possible de Sitter solutions are obtained taking the square bracket of the left-hand side of Eq. (2.11) equals to zero, for different values of m.

2.2.1 Case $m = 0$

In this case, the dissipation of the fluid is constant and Eq. (2.11) becomes a quadratic equation of the form

$$H_{dS}^2 - \frac{\xi_0}{\gamma} H_{dS} - \frac{\Lambda}{3} = 0, \tag{2.13}$$

with a discriminant given by

$$\Delta_0 = \left(\frac{\xi_0}{\gamma}\right)^2 + \frac{4\Lambda}{3}. \tag{2.14}$$

Then, two solutions are allowed for the Hubble constant

$$H_{dS\pm} = \frac{(\xi_0/\gamma) \pm \sqrt{\Delta_0}}{2}. \tag{2.15}$$

The above equation depends on the values of ξ_0, γ and Λ, and three types of solutions are obtained, depending if Δ_0 is positive, zero or negative. This last one, where $\Lambda < -3\xi_0^2/4\gamma^2$, is discarded because represents a complex Hubble constant without physical interest. If $\Delta_0 = 0$, the solution reduces to

$$H_{dS} = \frac{\xi_0}{2\gamma} \text{ for } \Lambda = -\frac{3\xi_0^2}{4\gamma^2}, \tag{2.16}$$

being the only de Sitter like solution of the model for this case, which is driven by the dissipative processes. Since ξ_0 can be expressed in terms of $|\Lambda|$ it is straightforward to find that in this case H_{dS} in Eq. (2.16) can also be expressed as $H_{dS} = \sqrt{\frac{|\Lambda|}{3}}$. If $\Delta_0 > 0$, then $\Lambda > -3\xi_0^2/4\gamma^2$, and the model for this case has only two de Sitter like solutions, H_{dS+} and H_{dS-}, again directly driven by the dissipative processes. But, H_{dS-} only represent an expanding solution when $\Lambda < 0$. But, H_{dS-} only represent an expanding solution when $\Lambda \geq 0$.

On the other hand, using the Eq. (2.2) it is possible to obtain the energy density associated to the de Sitter solutions (2.15) and (2.16), given respectively by

$$\rho_{\pm} = \frac{3\xi_0}{2\gamma}\left(\frac{\xi_0}{\gamma} \pm \sqrt{\Delta_0}\right), \tag{2.17}$$

$$\rho = \frac{3\xi_0^2}{2\gamma^2}. \tag{2.18}$$

From the above expressions it is possible to see that $\rho_+ > 0$ and $\rho > 0$, i. e., the de Sitter like solution given by Eqs. (2.15) (positive one) and (2.16) do not have null fluid energy density, contrary to the usual de Sitter solution (2.12) (positive one), where $\rho_{dS} = 0$ (DE dominant solution). It is important to note that $\rho_- > 0$ leads to the constraint $\Lambda < 0$, expression that it's according with the constraint obtained in order to H_{dS-} represent an expanding solution. Therefore, the de Sitter like solutions with physical interest for $m = 0$ are $H_{dS\pm}$ and H_{dS}.

2.2.2 case m=1

In this case the dissipation is proportional to the energy density of the dissipative fluid, and the other real solution of Eq. (2.11), besides the positive de Sitter solution (2.12) when $\Lambda > 0$, is given by

$$H_{dS} = \frac{\gamma}{3\xi_0}, \tag{2.19}$$

which depends only of the values of γ and ξ_0, i. e., being a de Sitter like solution that is a function of the parameters related to the dissipative processes and, in principle, independent of the values of Λ. But, using Eq. (2.2), we obtain that the fluid energy density for this solution is given by

$$\rho = \frac{\gamma^2}{3\xi_0^2} - \Lambda, \tag{2.20}$$

expression that, when we impose $\rho > 0$, leads to $\Lambda < \gamma^2/3\xi_0^2$. Again, this de Sitter like solution do not have null energy density, contrary to the usual de Sitter solution, except when $\Lambda = \gamma^2/3\xi_0^2$. The same result given by Eq. (2.19) was found in [58] for the case of a null CC.

A surprising results in both, $m = 0$ and $m = 1$ cases, is the possibility of de Sitter like solutions of physical interest despite the presence of a negative CC. It will find that the corresponding exact solutions behaves asymptotically like the de Sitter like evolution found in this section.

Chapter 3

Study of singularities

A singularity is the region where the curvature and tidal forces are infinite. In this region the geodesics[1] cannot be extended beyond that limit [115–117]. Also, we can have information from the presence of a singularity through the curvature, which is measured by the Riemann tensor, but it is hard to say when a tensor becomes infinite, since its components are coordinate-dependent. But from the curvature we can construct various scalar quantities, and since scalars are coordinate-independent it is significant to say that they become infinite in the context of singularities.

The more simple scalar is the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$, but we can also construct higher-order scalars such as $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R_{\mu\nu\rho\sigma}R^{\rho\sigma\lambda\tau}R^{\mu\nu}_{\lambda\tau}$, and so on. If any of these scalars (but not necessarily all of them) goes to infinity as we approach some point, we regard that point as a singularity of the curvature [118]. We should also check that the point is not infinitely far away; that is, that it can be reached by traveling a finite distance along a curve.

In this seance we will study singularities from the point of view of the incompleteness of the geodesic and the divergence of the Ricci scalar in the remainder

[1] In General Relativity geodesics describe the trajectories and the fate followed by unaccelerated test particles

of this chapter.

3.1 Incompleteness of geodesic

One way to explore singularities is to study the incompleteness of the space-time path that the universe follows during its cosmic evolution. To develop this idea we must first understand some preliminary concepts.

A metric $g_{\mu\nu}(x)$ is said to be *form-invariant* under a coordinate transformation of the type $x \to x'$, when the transformed metric $g'_{\mu\nu}(x')$ does not change its shape relative to the metric $g_{\mu\nu}(x)$, so it is possible to set the next expression

$$g'_{\mu\nu}(y) = g_{\mu\nu}(y) \forall y. \tag{3.1}$$

For a given point with coordinate x, the transformed metric is given by the relation

$$g'_{\mu\nu}(x') = \frac{\partial x^{\rho}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} g_{\rho\sigma}(x), \tag{3.2}$$

or, equivalently,

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g'_{\rho\sigma}(x'). \tag{3.3}$$

When the relation (3.1) is valid, we can replace $g'(x')$ by $g_{\rho\sigma}(x')$ and from this, we obtain the condition needed for the form invariance of the metric

$$g_{\mu\nu}(x) = \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} g_{\rho\sigma}(x'). \tag{3.4}$$

Any transformation $x \to x'$ that satisfies Eq. (3.4) is known as isometry.

In general, the Eq. (3.4) can become a very complex constraint for the function $x'^{\mu}(x)$. But a simplified form can be worked out for the special case of an infinitesimal coordinate transformation

$$x'^{\mu} = x^{\mu} + \epsilon\, \xi^{\mu}(x) \text{ with } |\epsilon| \ll 1, \tag{3.5}$$

to first order in ϵ the Eq. (3.4) reduces to

$$0 = \frac{\partial \xi^{\mu}(x)}{\partial x^{\rho}} g_{\mu\sigma}(x) + \frac{\partial \xi^{\nu}(x)}{\partial x^{\sigma}} g_{\rho\nu}(x) + \xi^{\mu}(x) \frac{\partial g_{\rho\sigma}}{\partial x^{\mu}}, \tag{3.6}$$

The latter can be further reduced if we consider that the first 2 terms are related to the total derivative of the term $\partial \xi_{\sigma} \equiv \partial(g_{\mu\sigma}\xi^{\mu})$ as follows

$$\begin{aligned} 0 &= \frac{\partial \xi^{\mu}(x)}{\partial x^{\rho}} g_{\mu\sigma}(x) + \frac{\partial \xi^{\nu}(x)}{\partial x^{\sigma}} g_{\rho\nu}(x) + \xi^{\mu}(x) \left[\frac{\partial g_{\rho\sigma}}{\partial x^{\mu}} - \frac{g_{\mu\sigma}}{x^{\rho}} - \frac{\partial g_{\rho\mu}}{x^{\sigma}} \right], \\ 0 &= \frac{\partial \xi^{\mu}(x)}{\partial x^{\rho}} g_{\mu\sigma}(x) + \frac{\partial \xi^{\nu}(x)}{\partial x^{\sigma}} g_{\rho\nu}(x) - 2\xi_{\mu}(x)\Gamma^{\mu}_{\rho\sigma}, \end{aligned} \tag{3.7}$$

or written more compactly

$$0 = \xi_{\sigma;\rho} + \xi_{\rho;\sigma}. \tag{3.8}$$

Where the “;" refers to the covariant derivative

$$\nabla_{\beta}\theta_{\alpha} = \frac{\partial \theta_{\alpha}}{\partial x^{\beta}} - \Gamma^{\mu}_{\alpha\beta}\theta_{\mu}. \tag{3.9}$$

Any four-vector that satisfies Eq. (3.9) is known as a Killing vector [119] of the metric $g_{\mu\nu}(x)$. The problem of determining the infinitesimal isometries for a given metric is now reduced to the problem of determining all the Killing vectors of the metric. For a FLRW cosmologies, which is homogeneous and isotropic, the metric is given by (1.2), and we have a six-dimensional group of isometries generated by the Killing fields [120]

$$\xi_1 = \sin\theta\cos\phi\partial_r + \frac{\cos\theta\cos\phi}{r}\partial_{\theta} - \frac{\sin\theta}{r\sin\theta}\partial_{\phi}, \tag{3.10}$$

$$\xi_2 = \sin\theta\sin\phi\partial_r + \frac{\cos\theta\sin\phi}{r}\partial_{\theta} + \frac{\cos\phi}{r\sin\theta}\partial_{\phi}, \tag{3.11}$$

$$\xi_3 = \cos\phi\,\partial_r - \frac{\sin\theta}{r}\partial_{\theta}, \tag{3.12}$$

$$\zeta_1 = \cos\phi\,\partial_{\theta} - \cos\theta\sin\phi\partial_{\phi}, \tag{3.13}$$

$$\zeta_2 = \sin\phi\,\partial_{\theta} + \cot\theta\cos\phi\partial_{\phi}. \tag{3.14}$$

$$\zeta_3 = \partial_{\phi}, \tag{3.15}$$

This is the maximum number of Killing vectors for the metric given by (1.2), and they are all independent [111]. These Killing vectors yield to six different constants of geodesic motion,corresponding to the components of linear momenta and and components of angular momenta

$$
\begin{aligned}
P_1 &= a(t)\left\{r\left(\cos\theta\cos\phi\dot{\theta}-\sin\theta\sin\phi\dot{\phi}\right)+\sin\theta\cos\phi\dot{r}\right\}, && (3.16)\\
P_2 &= a(t)\left\{r\left(\cos\theta\sin\phi\dot{\theta}+\sin\theta\cos\phi\dot{\phi}\right)+\sin\theta\sin\phi\dot{r}\right\}, && (3.17)\\
P_3 &= a(t)\left(\cos\theta\dot{r}-r\sin\theta\dot{\theta}\right), && (3.18)\\
L_1 &= a(t)r^2\left(\cos\phi\dot{\theta}-\sin\theta\cos\theta\sin\phi\dot{\phi}\right), && (3.19)\\
L_2 &= a(t)r^2\left(\sin\phi\dot{\theta}+\sin\theta\cos\theta\cos\phi\dot{\phi}\right), && (3.20)\\
L_3 &= a(t)r^2\sin^2\theta\dot{\phi}, && (3.21)
\end{aligned}
$$

the dots accounts for derivative with respect to proper time [2]. We now define

$$\delta = -\dot{t}^2 + a(t)\left\{\dot{r}^2 + r^2\left(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2\right)\right\}, \tag{3.22}$$

where δ is zero for null geodesics and -1 for time-like geodesics. With this conserved quantities, geodesic equations reduce to first order differential equations

$$
\begin{aligned}
\dot{t}^2 &= \frac{P^2}{a(t)}-\delta, && (3.23)\\
\dot{r} &= \frac{P_1\sin\theta\cos\phi+P_2\sin\theta\sin\phi+P_3\cos\theta}{a(t)}, && (3.24)\\
\dot{\theta} &= \frac{L_1\cos\phi+L_2\sin\phi}{a(t)r^2}, && (3.25)\\
\dot{\phi} &= \frac{L_3}{a(t)r^2\sin^2\theta}, && (3.26)
\end{aligned}
$$

we can rewrite the previous expression in terms of total linear momentum and angular momentum

$$P^2 = P_1^2+P_2^2+P_3^2 \qquad L^2 = L_1^2+L_2^2+L_3^2. \tag{3.27}$$

[2]Proper time is defined as the time as measured by a clock following a time-like geodesic (a time-like connects two events that are causally connected, that is the second event is in the light cone of the first even) world line.

Due to spherical symmetry, every geodesic may be fit in the hypersurface[3] $\theta = \frac{\pi}{2}$, whit $L_1 = L_2 = 0 = P_3$, by a suitable choice of the coordinates. Then, we can simplify our previous differential equations as follows

$$\dot{t}^2 = \frac{P^2}{a(t)} - \delta, \tag{3.28}$$

$$\dot{r} = \frac{P_1 \cos\phi + P_2 \sin\phi}{a(t)}, \tag{3.29}$$

$$\dot{\phi} = \frac{L_3}{a(t) r^2}. \tag{3.30}$$

It can be easily noticed that these equations are singular if and only if $a(t)$ has a zero, which corresponds to either a Big-Bang or Big-Crunch singularity. The authors in [82, 83, 83, 121–129] explore different forms of scale factors by viscous effects, and produce an incompleteness of geodesic equation, finding the following set of classifications for different singularities (see appendix B for all technical details) given by [83, 84]:

- **Type 0A** ("Big Bang"): for $t \to 0$, $a \to 0$, $\rho \to \infty$ and $|p| \to \infty$.

- **Type 0B** ("Big Crunch"): for $t \to t_s$, $a \to 0$, $\rho \to \infty$ and $|p| \to \infty$.

- **Type I** ("Big Rip"): for $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$.

- **Type** I_l ("Little-Rip"): for $t \to \infty$, $a \to \infty$, $\rho \to \infty$ and $|p| \to \infty$.

- **Type II** ("Sudden"): for $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|p| \to \infty$.

[3] This is a generalization for surface of a Euclidean space, this definition is used in order to describe the "surface" of a space than contains the maximun Killing vector.

- **Type III** ("Big freeze") : for $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|p| \to \infty$.

- **Type IV** ("Generalized Sudden"): for $t \to t_s$, $a \to a_s$, $\rho \to 0$ and $|p| \to 0$, and higher derivatives of H diverge.

In the next chapter, we will study the presence of some of these singularities in our model.

3.2 Singularities in viscous ΛCDM models

In what follows we will study the solutions that arise from Eq. (2.7), for the particular cases when $m = 0$ and $m = 1$, and we discuss their behavior in terms of the free parameters ξ_0, γ and Λ. The solutions for each case will be compared with the ΛCDM model.

We will focus our study in the existence of different types of early and late time singularities, which can occur for some values of the free parameters of each model, following the classifications given in section 3.1. These singularities are typical in the following cosmological scenarios: (i) type I emerges at late times in phantom DE dominated universes [82, 121–125]; (ii) type II corresponds to a sudden future singularity [83, 126], also know as a big brake or a big démarrage, which appear under the conditions $\rho > 0$ and $\rho + 3p > 0$ (SEC) in an expanding universe [127]; type III occurs for models with $p = -\rho - A\rho^{\alpha}$ and the difference with the Big-rip type I is that here the scale factor has a finite value in a finite time [83, 128]; and (iv) type IV which also appears in the context of phantom DE of the form $p = -\rho - f(\rho)$, explored in [83] with a particular form of $f(\rho)$ called "32", and in the context of quantum cosmology [129].

Since the singularities are characterized by the divergences in the curvature scalar, we will use in our study the Ricci scalar, given by the following expression

$$R = 6\left(\frac{\ddot{a}(t)}{a(t)} + \frac{\dot{a}^2(t)}{a^2(t)}\right) = 6\left(\dot{H} + 2H^2\right), \tag{3.31}$$

The following results were published in [106].

3.2.1 Late-times singularities with $\Lambda > 0$

In this subsection we will study the singularities that arise from the solutions of Eq. (2.7) for a positive CC when $m = 0$ and $m = 1$. In order to compare with ΛCDM model, we will set in the general solutions $\gamma = 1$ (CDM) and $\Omega_\Lambda = 0.69$, which is the current value given by the cosmological data [3]. From now, all solutions will be expressed in terms of the dimensionless density parameters Ω_Λ and $\Omega_\xi = 3^m \xi_0 H_0^{2m-1}$, using the initial conditions $H(t=0) = H_0$ and $a(t=0) = 1$, where $t = 0$ is the present time.

Cases for $m = 0$

The integration of Eq. (2.7) is straightforward and leads to an integral of the form $\int \frac{dH}{R} = -\frac{3\gamma}{2}t + C$, where $R = H^2 - (\xi_0/\gamma)H - (\Lambda/3)$ is a polynomial in H.

In principle, three different types of solutions emerge depending if the discriminant Δ_0, given by Eq. (2.14), is positive, negative or zero. For $\Lambda > 0$ the only solution is with $\Delta_0 > 0$. The condition (2.14), in terms of dimensionless densities, takes the form $\Delta_0 = H_0^2 \bar{\Delta}_0$, where

$$\bar{\Delta}_0 = \left(\frac{\Omega_\xi}{\gamma}\right)^2 + 4\Omega_\Lambda > 0, \tag{3.32}$$

and $\Omega_\xi = \xi_0/H_0$. The exact solution for this case is (see appendix C.3.1 Eqs. (C.47), (C.55).)

$$E(T) = \frac{\sqrt{\bar{\Delta}_0}}{2}\tanh\left[\frac{3\gamma\sqrt{\bar{\Delta}_0}T}{4} + \operatorname{arctanh}\left(\frac{2 - \frac{\Omega_\xi}{\gamma}}{\sqrt{\bar{\Delta}_0}}\right)\right] + \frac{\Omega_\xi}{2\gamma}, \tag{3.33}$$

$$a(T)=\exp\left(\frac{\Omega_\xi}{2\gamma}T\right)\left\{\frac{\cosh\left[\frac{3\gamma\sqrt{\bar{\Delta}_0}}{4}T+\text{arctanh}\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{\bar{\Delta}_0}}\right)\right]}{\cosh\left[\text{arctanh}\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{\bar{\Delta}_0}}\right)\right]}\right\}^{\frac{2}{3\gamma}}, \tag{3.34}$$

where $E(T)=H(T)/H_0$ and $T=H_0t$ is a dimensionless time, therefore their positive values represents future evolution. It is important to note that the Hubble constant (3.33) does not exhibit a singularity for any time T and the scale factor (3.34) represents a bouncing universe. Even more, the asymptotic behavior of the Hubble constant for $T\rightarrow\infty$ give us H_{ds+} and for $T\rightarrow-\infty$ gives us H_{ds-}, both solutions given by Eq. (2.15), being H_{dS+} the de Sitter-like solution of this model.

Case for $m=1$

In this case the polynomial in H is $R=(1-3\xi_0H/\gamma)(H^2-\Lambda/3)$ and the solution takes the dimensionless form (see appendix C.4 Eq. (C.76))

$$T(E)=\frac{\Omega_\xi\sqrt{\Omega_\Lambda}\log\left(\frac{(1-\Omega_\Lambda)(\gamma-E\Omega_\xi)^2}{(E^2-\Omega_\Lambda)(\gamma-\Omega_\xi)^2}\right)}{3\sqrt{\Omega_\Lambda}\left(\gamma^2-\Omega_\xi^2\Omega_\Lambda\right)}+\frac{\gamma\log\left(\frac{\left(\sqrt{\Omega_\Lambda}-1\right)\left(\sqrt{\Omega_\Lambda}+E\right)}{\left(\sqrt{\Omega_\Lambda}+1\right)\left(\sqrt{\Omega_\Lambda}-E\right)}\right)}{3\sqrt{\Omega_\Lambda}\left(\gamma^2-\Omega_\xi^2\Omega_\Lambda\right)}, \tag{3.35}$$

where $\Omega_\xi=3\xi_0H_0$. In Fig. 3.1 we have numerically found the behavior of $E(T)$ given by the above equation. Note that $T\rightarrow\infty$, $\forall E$ when $\Omega_\xi=\gamma$, in other words, this case represents the de Sitter case given by Eq. (2.19), that is $H(t)=H_0$, $\forall t$, as it can be seen from Fig. 3.1.

From Eq. (3.35) a singularity time, T_s, appears if we take $E\rightarrow\infty$, which gives

$$T_s=\frac{2\Omega_\xi\log\left[\left(\frac{1-\sqrt{\Omega_\Lambda}}{1+\sqrt{\Omega_\Lambda}}\right)^{\frac{\gamma}{2\Omega_\xi\sqrt{\Omega_\Lambda}}}(1-\Omega_\Lambda)^{\frac{1}{2}}\left(\frac{-\Omega_\xi}{\gamma-\Omega_\xi}\right)\right]}{3\left(\gamma^2-\Omega_\xi^2\Omega_\Lambda\right)}. \tag{3.36}$$

At this future singularity, from Eqs. (2.2), (2.5) and the EoS, we can see that ρ, p and Π are divergent. If

$$\Omega_\xi>\gamma, \tag{3.37}$$

then the argument of the logarithm in Eq. (3.36) is always positive. Even more, if $\Omega_\xi = \gamma/\sqrt{\Omega_\Lambda} > \gamma$, the numerator and denominator of the Eq. (3.36) are zero, however

$$\lim_{\Omega_\xi \to \frac{\gamma}{\sqrt{\Omega_\Lambda}}} T_s = \frac{\left(\sqrt{\Omega_\Lambda} - 1\right) \log\left(\frac{1-\sqrt{\Omega_\Lambda}}{\sqrt{\Omega_\Lambda}+1}\right) - 2\sqrt{\Omega_\Lambda}}{6\gamma\left(\Omega_\Lambda - \sqrt{\Omega_\Lambda}\right)}, \tag{3.38}$$

therefore T_s is continue for $\Omega_\xi > \gamma$ and there are a change of sign in $\Omega_\xi = \gamma/\sqrt{\Omega_\Lambda}$ for both, the numerator and denominator in Eq. (3.36), yielding that T_s is always positive because when $\Omega_\xi > \gamma/\sqrt{\Omega_\Lambda}$ the argument of the logarithm is lower than 1 (negative numerator) and the denominator is negative, as can be seen in Fig. 3.2, where T_s given by Eq. (3.36) is plotted as a function of Ω_ξ. In Fig. 3.1 the red dashed lines represent two times of singularities according to Eq. (3.36), for $\Omega_\xi = 1.5$ and $\Omega_\xi = 1.1$, where the time of singularities are $T = 0.812997$, which is roughly equivalent to 11.6969 Gyrs (0.86 times the lifetime ofΛCDM universe); and $T = 2.26375$, corresponding to 32.5695 Gyrs (2.4 times the lifetime ofΛCDM universe), respectively.

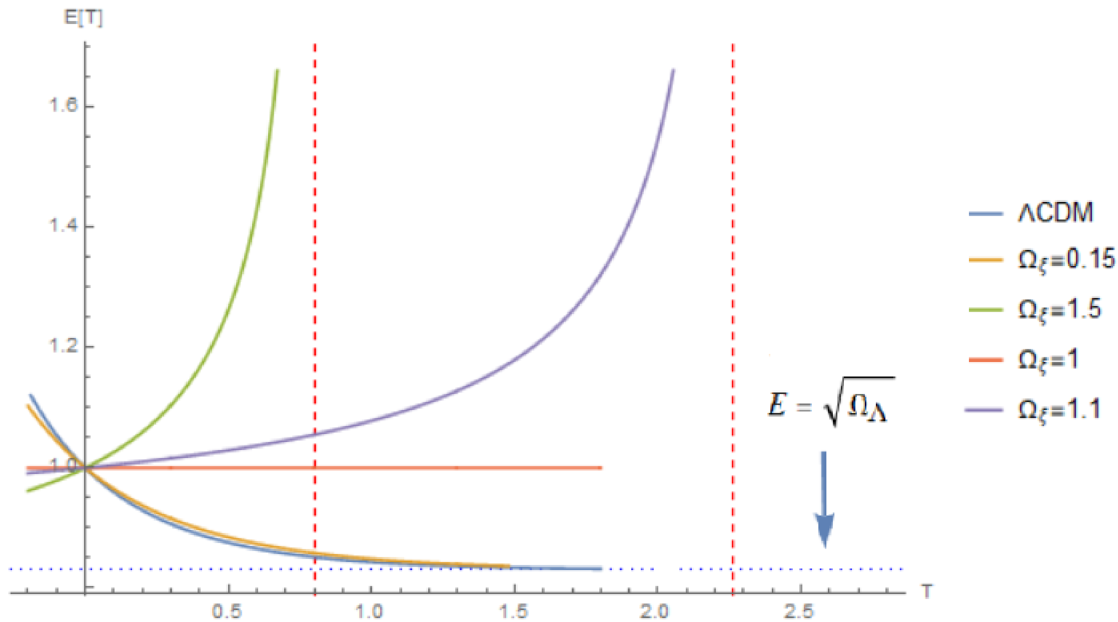

Figure 3.1: Numerical behavior of $E(T)$, given by Eq. (3.35), for different values of Ω_ξ and for the particular values of $\gamma = 1$ and $\Omega_\Lambda = 0.69$. We also plotted the ΛCDM model. The red dashed lines represent the times of singularities given by Eq. (3.36) for $\Omega_\xi = 1.5$ and $\Omega_\xi = 1.1$, respectively.

For $\Omega_\xi < \gamma$, there are no future singularities (no finite time is obtained from Eq. (3.36)). From Eq. (3.35) we can see that $E(T)$ follows very close the behavior of standard model, ending with a de Sitter behavior at $T \to +\infty$, which can be seen taking $E = \sqrt{\Omega_\Lambda}$ (equivalent to the solution given by Eq. (2.12)) in Eq. (3.35).

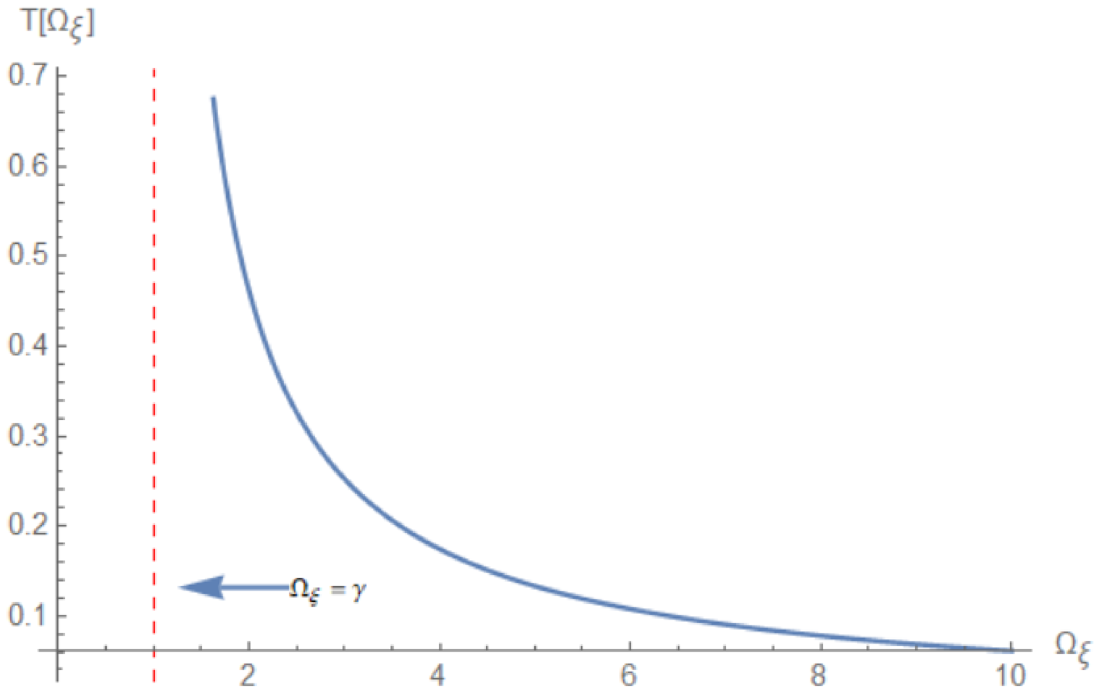

Figure 3.2: Behavior of the time for singularities given by Eq. (3.36) as a function of Ω_ξ, for the particular values of $\gamma = 1$ and $\Omega_\Lambda = 0.69$.

In order to classify these singularities we need to explore the effective EoS of the models found. From Eqs. (2.4), (2.6) and the EoS one obtains in agreement with [25] that

$$\gamma_{eff} = \gamma + \frac{\Pi}{3H^2}, \tag{3.39}$$

and from Eq. (2.3) it is possible to find an expression for the viscous pressure given by

$$\Pi = -2\dot{H} - 3\gamma H^2, \tag{3.40}$$

using the above expression, we have (for (3.39))

$$\gamma_{eff} = -\frac{2\dot{H}}{3H^2}, \tag{3.41}$$

and using Eq. (2.7) in our dimensionless notation we have (see appendix D.3, Eq.

(D.20))

$$\gamma_{eff} = \gamma - \Omega_\xi E + \frac{\Omega_\xi \Omega_\Lambda}{E} - \frac{\Omega_\Lambda \gamma}{E^2}. \tag{3.42}$$

This γ_{eff} represents the effective EoS of a universe with a DE component modeled by a CC and a dissipative component.

The phantom behavior of our solutions can be associated to the global composition of the universe. Fig. 3.3 shows the behavior of γ_{eff} as a function of T for the solutions found, for different values of Ω_ξ.

Let see now the type of singularities that we found in the dissipative CDM case ($\gamma = 1$). For the solutions without singularities, i.e., $\Omega_\xi < 1$, γ_{eff} evolves to 0, representing the dominance of the CC at very far future times. In the solution with $\Omega_\xi > 1$, ρ and p diverges and therefore, from Eq. (2.2) H and a diverges, that is, these solutions present Big-Rip singularities because γ_{eff} from Eq. (3.42) is always phantom, as can be seen from Fig. 3.3. It is important to note that since H and $\dot{H}$ go to infinity for this singularity, then the Ricci scalar given by Eq. (3.31) also diverge.

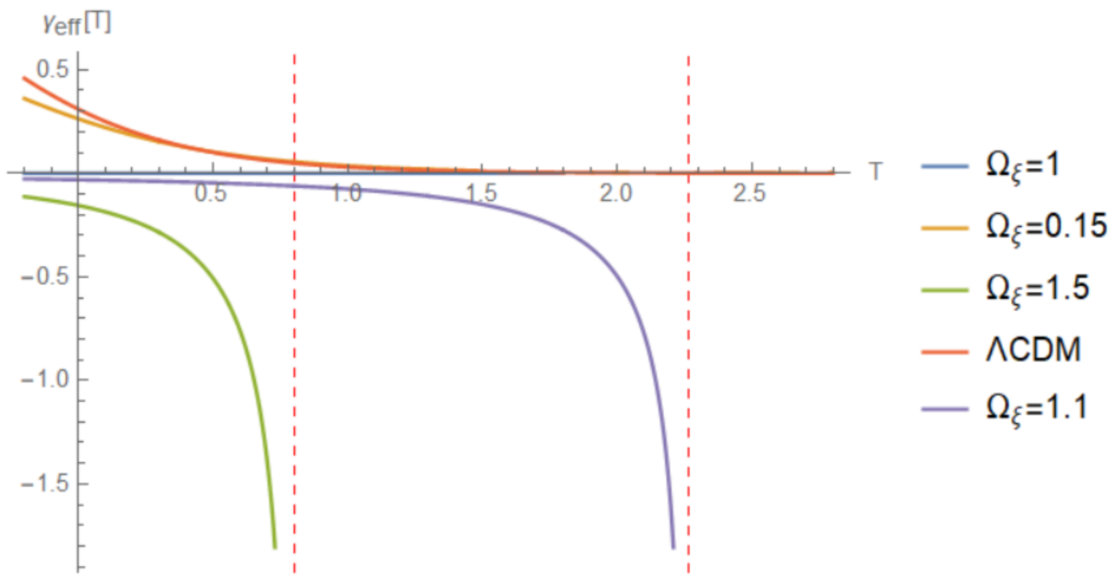

Figure 3.3: Behavior of γ_{eff} given by Eq. (3.42) as a function of T for the solutions with $m = 1$, $\gamma = 1$ and $\Omega_\Lambda = 0.69$, for different values of Ω_ξ. We also plotted γ_{eff} for the ΛCDM model. The red dashed lines represent the times of singularities given by Eq. (3.36) for $\Omega_\xi = 1.5$ and $\Omega_\xi = 1.1$, respectively.

3.2.2 Late-times singularities with $\Lambda < 0$

In this subsection we will study the singularities that arise from the solutions of Eq. (2.7) for a negative CC when $m = 0$ and $m = 1$. In this case, in order to compare with ΛCDM model, we will set in the general solutions $\gamma = 1$ (CDM) and $\Omega_\Lambda = -0.69$. It is important to note from Eq. (2.3) that the model with a negative CC can still gives an accelerated solution because of the negative pressure due to the bulk viscosity. Therefore, the election of $\Omega_\Lambda = -0.69$ is the first natural election in order to a further comparison, because, from Eqs. (2.2) and (2.6) the usual Friedmann's constraint $\Omega_m + \Omega_\Lambda = 1$ is not already valid and the values of Ω_Λ can, in principle, take any negative value.

Cases for $m = 0$

In this case we have three different types of solutions depending if the discriminant, Δ_0, given by Eq. (2.14) is greater, equal, or lower than zero.

(i) Case $\Delta_0 > 0$. In this case the constraint for the values of a negative CC is

$$-\left(\frac{\Omega_\xi}{2\gamma}\right)^2 < \Omega_\Lambda < 0. \tag{3.43}$$

We already have explained that this solution does not present any kind of singularity due to its bouncing behavior and the solution was already found in Eq. (3.33) (for the Hubble constant) and in Eq. (3.34) (for the scale factor).

It is interesting to mention that, despite having a negative CC, this solution does not present Big-Crunch singularity, and at late times displays a de Sitter like expansion.

(ii) Case $\Delta_0 = 0$. In this case the CC takes the particular value

$$\Omega_\Lambda = -\left(\frac{\Omega_\xi}{2\gamma}\right)^2, \tag{3.44}$$

and the solution for $E(T)$ takes the form (see appendix C.3.2, Eq.(C.61))

$$E(T) = \frac{4 + 3\Omega_\xi(1 - \frac{\Omega_\xi}{2\gamma})T}{4 + 6\gamma(1 - \frac{\Omega_\xi}{2\gamma})T}. \tag{3.45}$$

The corresponding scale factor is given by (see appendix C.3.2, Eq.(C.68))

$$a(t) = \exp\left[\frac{\Omega_\xi}{2\gamma}T\right]\left[\frac{3\gamma}{2}T(1 - \frac{\Omega_\xi}{2\gamma}) + 1\right]^{\frac{2}{3\gamma}}. \tag{3.46}$$

It is straightforward to see from Eq. (3.45) that for $\Omega_\xi = 2\gamma$, $E = 1$ for all time, corresponding to our de Sitter like solution given by (2.16) . If $\Omega_\xi > 2\gamma$, E goes to zero in a future time T_c, given by

$$T_c = -\frac{4}{3\Omega_\xi\left(1 - \frac{\Omega_\xi}{2\gamma}\right)} > 0, \tag{3.47}$$

which indicates that the scale factor takes a maximum value at this time and, from Eq. (3.46), goes to zero at a time given by

$$T_s = -\frac{2}{3\gamma\left(1-\frac{\Omega_\xi}{2\gamma}\right)} > 0. \tag{3.48}$$

From Eq. (3.45) we can see that at the above time $E \to -\infty$, which means, from Eq. (2.2), that ρ diverge and, from the EoS, p diverge, indicating that in this case the future singularity corresponds to a Big-Crunch (Type OB singularity). It is important to note that since H and $\dot{H}$ go to minus infinity for this singularity, then the Ricci scalar given by Eq. (3.31) also diverge.

On the other hand, if $\Omega_\xi < 2\gamma$, then $E > 0$ for all time and goes to the value $\Omega_\xi/2\gamma$ when $T \to +\infty$. Therefore, this solution asymptotically takes a de Sitter like behavior given by Eq. (2.16). Note that for $\Omega_\xi \leq 2\gamma$ effectively we can drive the acceleration expansion of the universe when a negative CC is considered in our model, due only to the negativeness of the viscous pressure.

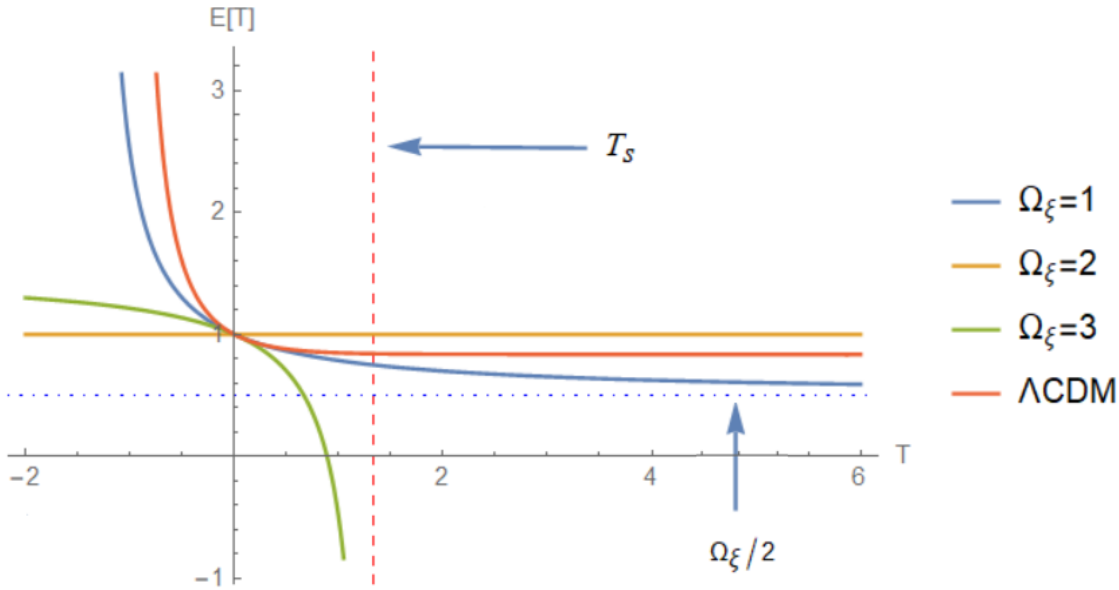

Figure 3.4: Behavior of $E(T)$, given by Eq. (3.45) for different values of Ω_ξ and for the particular value of $\gamma = 1$. Ω_Λ is given by Eq. (3.44). We also plotted the ΛCDM model. The red dashed line represent the singularity time given by Eq. (3.48) for $\Omega_\xi = 3$.

In Fig. 3.4 we display the behavior of the Hubble constant (3.45) for $\gamma = 1$. The Big-Crunch singularity appears for the particular values of $\gamma = 1$ and $\Omega_\xi = 3$ evaluated in Eq. (3.48), and leads to $T_s = 4/3$, which is roughly equivalent to 19.18 Gyrs (1.45 times the lifetime of theΛCDM universe).

From Eq. (3.41) the effective barotropic index for this solution is

$$\gamma_{eff} = \gamma - \frac{\Omega_\xi}{E} + \frac{|\Omega_\Lambda|\,\gamma}{E^2}, \tag{3.49}$$

and from the solution given by Eq. (3.45), we have (see appendix D.1, Eq. (D.15))

$$\gamma_{eff} = \frac{16\gamma\left(\frac{\Omega_\xi}{2\gamma} - 1\right)^2}{\left(4 - 3\Omega_\xi T\left(\frac{\Omega_\xi}{2\gamma} - 1\right)\right)^2}. \tag{3.50}$$

Note that, if we substitute Eq. (3.47) (where E=0 and a takes his maximum value) in Eq. (3.50), we will get $\gamma_{eff} \to +\infty$, and if we substitute Eq. (3.48) (Big Crunch time) we will get $\gamma_{eff} = \gamma$ (according to Eq. 3.49). The behavior of this γ_{eff} is presented in the Fig. (3.5).

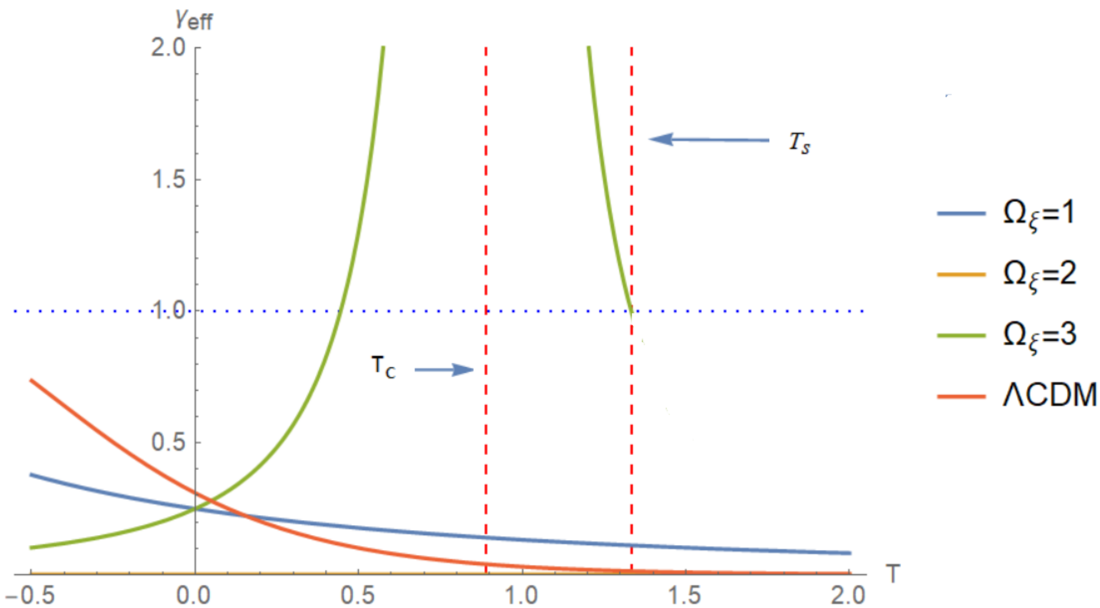

Figure 3.5: Behavior of γ_{eff} given by Eq. (3.50) as a function of T for the solution with $m = 0$ and $\Delta_0 = 0$, for the particular value of $\gamma = 1$ and for Ω_Λ given by Eq. (3.44), for different values of Ω_ξ. T_c and T_s are given by (3.47) and Eq. (3.48), respectively. We also plotted the γ_{eff} for the ΛCDM model.

It is important to mention that as the viscosity increases, the value of Ω_Λ also increases, which can be seen from Eq. (3.44); also the time T_c, where the scale factor takes its maximum value, occurs after the current time.

(iii) Case $\Delta_0 < 0$. In this case the dimensionless density parameter associated to the negative CC satisfied the following inequality

$$\Omega_\Lambda < -\left(\frac{\Omega_\xi}{2\gamma}\right)^2, \tag{3.51}$$

and the exact solution takes the following form (see appendix C.3.3, Eq. (C.69))

$$
\begin{aligned}
E(T) &= -\frac{\sqrt{|\bar{\Delta}_0|}}{2}\tan\left(\frac{3\gamma\sqrt{|\bar{\Delta}_0|}T}{4}-\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right) \\
&+ \frac{\Omega_\xi}{2\gamma},
\end{aligned}
\tag{3.52}
$$

with a scale factor given by (see appendix C.3.3, Eq. (C.70))

$$
a(T)=\exp\left(\frac{\Omega_\xi}{2\gamma}T\right)\left\{\frac{\cos\left[\frac{3\gamma\sqrt{|\bar{\Delta}_0|}}{4}T-\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right]}{\cos\left[\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right]}\right\}^{\frac{2}{3\gamma}}, \tag{3.53}
$$

In order to explore the possibility of future singularities, we found from Eq. (3.52) T as a function of E, obtaining

$$
T(E)=\frac{4}{3\gamma\sqrt{|\bar{\Delta}_0|}}\times\left(\arctan\left(\frac{\frac{\Omega_\xi}{\gamma}-E}{\sqrt{|\bar{\Delta}_0|}}\right)+\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right). \tag{3.54}
$$

From this equation we can notice that E is zero in a time given by

$$
T_c=\frac{4}{3\gamma\sqrt{|\bar{\Delta}_0|}}\times\left(\arctan\left(\frac{\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)+\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right), \tag{3.55}
$$

which indicates that the scale factor takes a maximum value at this time and goes to zero when $E\rightarrow-\infty$, as can be seen from Eq. (3.54), in a time given by

$$
T_s=\frac{4}{3\gamma\sqrt{|\bar{\Delta}_0|}}\times\left(\frac{\pi}{2}+\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right), \tag{3.56}
$$

i. e., $a\rightarrow 0$, as can be shown if we substitute the time given in Eq. (3.56) in (3.53). So, from Eq. (2.2) ρ diverge and from the EoS equation p also diverge. Therefore, at this time occurs a Big-Crunch singularity (Type 0B). It is important to note that since H and $\dot{H}$ go to minus infinity for this singularity, then the Ricci scalar given by Eq. (3.31) also diverge. This is the only scenario that we have for this solution and a de Sitter asymptotic expansion is not possible, as it can be check from Eq. (2.15). In Fig. 3.6 we present the behavior for E as a function of T, given by Eq.(3.52). The value of time of singularity shown in this figure is $T = 0.609495$, which is roughly equivalent to 8.76906Gyrs (0.64 times the life of theΛCDM universe).

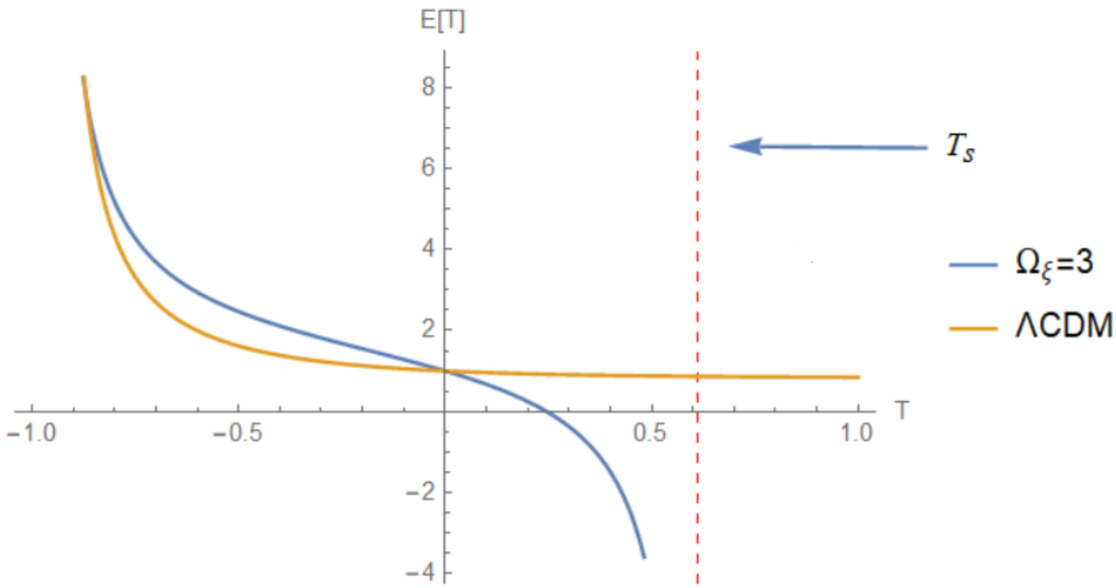

Figure 3.6: Behavior of $E(T)$ given by Eq. (3.52), for different values of Ω_ξ and for the particular values $\gamma = 1$, $\Omega_\Lambda = -4$ and $\Omega_\xi = 3$, according to restriction (3.51). We also plotted the ΛCDM model. The red dashed line represent the singularity time given by Eq. (3.56).

For the solution given by Eq. (3.52) we have, from Eq. (3.49), that (see appendix D.1 Eq. (D.18))

$$\gamma_{eff} = \frac{\gamma\left|\bar{\Delta}_0\right| \sec\left(\frac{3}{4}\gamma\sqrt{\left|\bar{\Delta}_0\right|}T - \arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{\left|\bar{\Delta}_0\right|}}\right)\right)^2}{4\left(\frac{\Omega_\xi}{2\gamma} - \frac{\sqrt{\left|\bar{\Delta}_0\right|}}{2}\tan\left(\frac{3}{4}\gamma\sqrt{\left|\bar{\Delta}_0\right|}T - \arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{\Delta_0}}\right)\right)\right)^2}, \tag{3.57}$$

Note that, if we substitute Eq. (3.55) (where E=0) in Eq. (3.57) we will get $\gamma_{eff} \to +\infty$, and if we substitute Eq. (3.56) we will get $\gamma_{eff} = \gamma$ (according to Eq.

3.49). The behavior of the above expression is presented in Fig. 3.7.

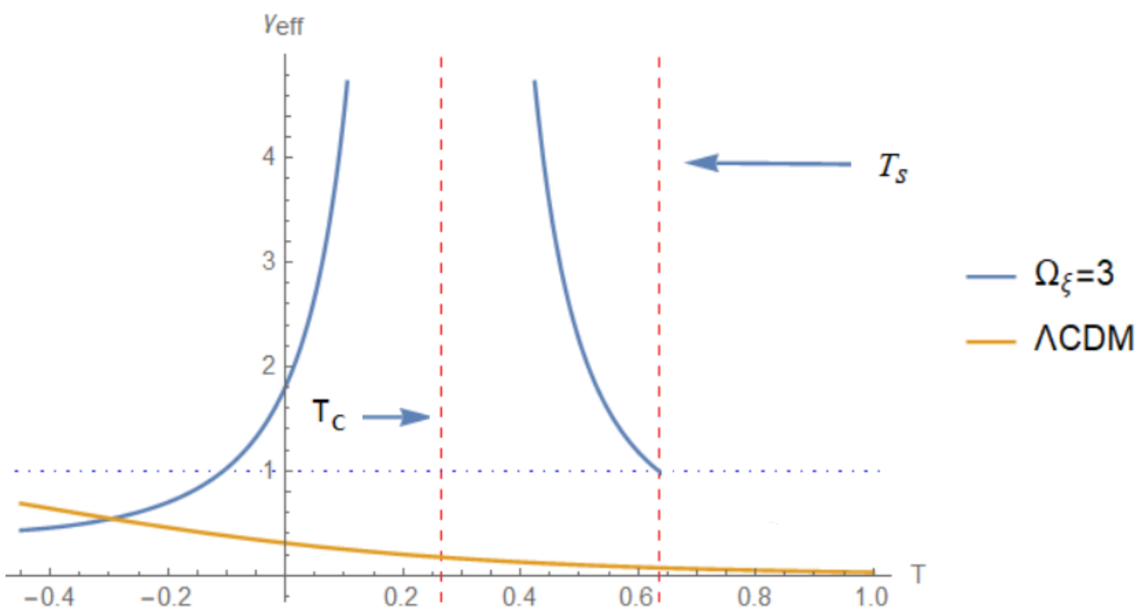

Figure 3.7: Behavior of $\gamma_{eff}(T)$, given by Eq. (3.52), for the solution with $m = 0$ and $\Delta_0 < 0$, for the particular values of $\gamma = 1$ and $\Omega_\Lambda = -4$. We use $\Omega_\xi = 3$ according to restriction (3.51). T_c and T_s are given by (3.55) and Eq. (3.56), respectively. We also plotted γ_{eff} for the ΛCDM model.

Cases for $m = 1$

In this case the solution is given by (see appendix C, Eq. (C.5))

$$T(E) = \frac{2\Omega_\xi\sqrt{|\Omega_\Lambda|}\log\left(\frac{(1+|\Omega_\Lambda|)^{\frac{1}{2}}(\gamma-E\Omega_\xi)}{(E^2+|\Omega_\Lambda|)^{\frac{1}{2}}(\gamma-\Omega_\xi)}\right)}{3\sqrt{|\Omega_\Lambda|}\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)} + \frac{2\gamma\left(\arctan\left(\frac{1}{\sqrt{|\Omega_\Lambda|}}\right) - \arctan\left(\frac{E}{\sqrt{|\Omega_\Lambda|}}\right)\right)}{3\sqrt{|\Omega_\Lambda|}\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)} \tag{3.58}$$

Note that $T \to \infty,\ \forall E$ when $\Omega_\xi = \gamma$, in other words, this case represents the de Sitter case given by Eq. (2.19), that is $H(t) = H_0$, $\forall t$, as it can be seen from Fig.

3.8. If $\Omega_\xi < \gamma$, $E = 1$ at $T = 0$ and goes to zero in a future time T_c given by

$$T_c = \frac{2\Omega_\xi\sqrt{|\Omega_\Lambda|}\log\left(\frac{(1+|\Omega_\Lambda|)^{\frac{1}{2}}(\gamma)}{\left(\sqrt{|\Omega_\Lambda|}\right)(\gamma-\Omega_\xi)}\right)}{3\sqrt{|\Omega_\Lambda|}\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)} + \frac{2\gamma\left(\arctan\left(\frac{1}{\sqrt{|\Omega_\Lambda|}}\right)\right)}{3\sqrt{|\Omega_\Lambda|}\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)},$$

indicating that the scale factor takes a maximum value at this time, and goes to zero when $E \to -\infty$ at a time given by

$$T_{s1} = \frac{2\Omega_\xi\log\left((1+|\Omega_\Lambda|)^{\frac{1}{2}}\left(\frac{\Omega_\xi}{\gamma-\Omega_\xi}\right)\right)}{3\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)} + \frac{2\gamma\left(\arctan\left(\frac{1}{\sqrt{|\Omega_\Lambda|}}\right)+\frac{\pi}{2}\right)}{3\sqrt{|\Omega_\Lambda|}\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)}.$$

Therefore, if $a \to 0$ and since $E \to -\infty$, from Eq. (2.2) ρ diverge and from the Eos p also diverge, so this solution represents a universe with a Big-Crunch type future singularity (Type 0B) and γ_{eff} from (3.42) goes to infinity. It is important to note that since H and $\dot{H}$ go to minus infinity for this singularity, then the Ricci scalar given by Eq. (3.31) also diverge. In Fig. 3.8 we have numerically found the behavior of E as a function of T, given by Eq. (3.58). The value of time of singularity show in this figure is $T_{s1} = 1.79003$, which is roughly equivalent to 25.7539Gyrs (1.87 times the life of theΛCDM universe).

If $\Omega_\xi > \gamma$, $E > 0$ for all time and goes to infinite in a finite time given by

$$T_{s2} = \frac{2\Omega_\xi\log\left((1+|\Omega_\Lambda|)^{\frac{1}{2}}\left(\frac{-\Omega_\xi}{\gamma-\Omega_\xi}\right)\right)}{3\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)} + \frac{2\gamma\left(\arctan\left(\frac{1}{\sqrt{|\Omega_\Lambda|}}\right)-\frac{\pi}{2}\right)}{3\sqrt{|\Omega_\Lambda|}\left(\gamma^2+\Omega_\xi^2|\Omega_\Lambda|\right)}. \tag{3.59}$$

As in the case of a positive CC, Eq. (3.59) is always positive for any value of $\Omega_\xi > \gamma$. Now, if we substitute $\gamma = 1$ (dust case) and if we use $\Omega_\Lambda = -0.69$ (to compare with the case of positive CC), then for $\Omega_\xi > 1$, ρ and p diverges and therefore, from Eq. (2.2) H and a diverges, i. e., these solutions present Big-Rip singularities because γ_{eff} from (3.42) is always phantom, as can be seen from Fig. 3.9. In this figure for $\Omega_\xi = 1.5$, we have a Big Rip singularity time of $T = 0.315246$, which is roughly equivalent to 4.53558Gyrs (0.33 times the life of theΛCDM universe). It is important to note that since H and $\dot{H}$ go to infinity for this singularity, then the Ricci scalar given by Eq. (3.31) also diverge.

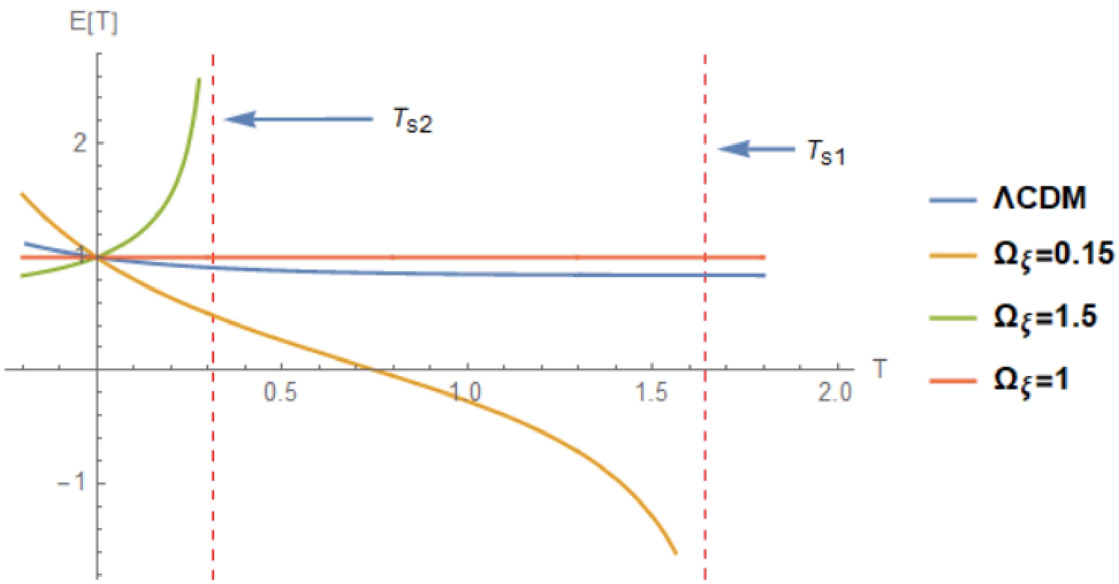

Figure 3.8: Numerical behavior of $E(T)$, given by Eq. (3.58), for different values of Ω_ξ and for the particular values of $\gamma = 1$ and $\Omega_\Lambda = -0.69$. We also plotted the ΛCDM model. The red dashed line represent T_{s1} and T_{s2} given by Eq. (3.59) and Eq. (3.59) respectively.

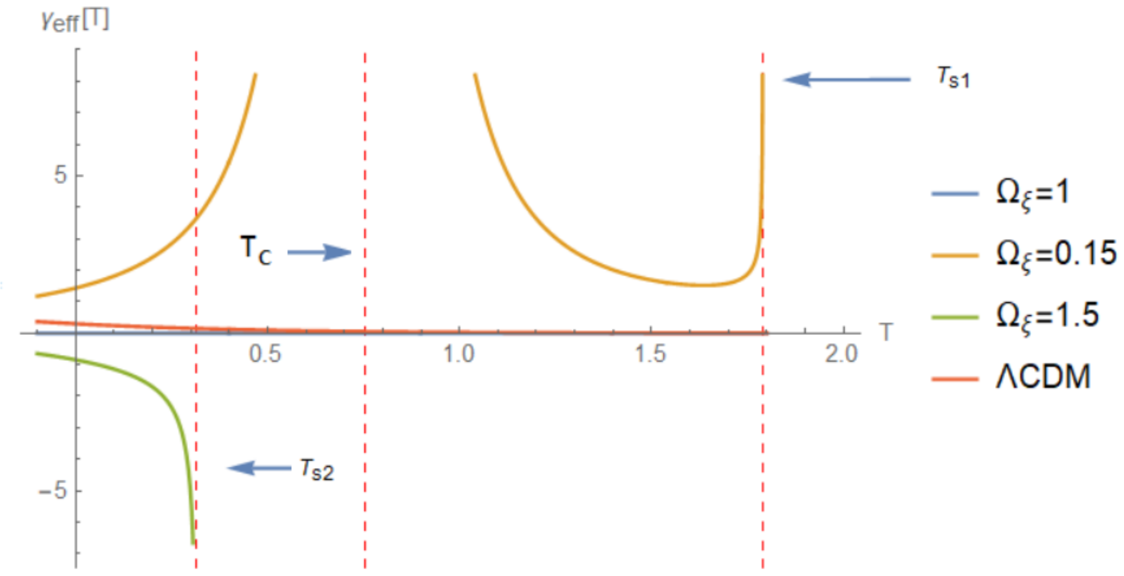

Figure 3.9: Behavior of γ_{eff} given by Eq. (3.42) as a function of T for the solutions with $m = 1$, for the particular values of $\gamma = 1$ and $\Omega_\Lambda = -0.69$, and different values of Ω_ξ. We also plotted γ_{eff} for the ΛCDM model. T_c, T_{s1} and T_{s2} are given by Eq. (3.59), Eq. (3.59) and Eq. (3.59) respectively.

3.2.3 Early-Time singularities for the case of $\Lambda > 0$

In the case of early-singularities we explore the behavior of our exact solutions backward in time, taking $\gamma = 4/3$ (radiation) or even $\gamma \leq 2$ (cuasi stiff fluid), assuming that some kind of dissipation is possible at these very early stages. As an initial condition for our solutions, we will assume that $\Omega_{radiation}$ takes values very close to one, which is reasonable to assume during the radiation dominant era. In order to make comparisons we will consider an early evolution stage of theΛCDM model. Our model is based on the composition of only two fluids, (i) dissipative matter (ii) dark energy modeled as a CC. According to our previous discussion, radiation is imposed as the dominant fluid in relation to the value of Ω_Λ, therefore from Eq. (4) for an arbitrary very early radiation time we can consider the value of $\Omega_\Lambda = 10^{-6}$, in order to use the exact solution found and explore its behavior to the past. On the contrary, during the current DE era the actual value of radiation density, according to observation, is $\Omega_{radiation} = 9.72 \times 10^{-5}$ [7, 130]

The below discussion correspond to the case of a dissipative radiation fluid. The initial condition chosen, $T = H_0 t = 0$, represents the arbitrary moment during the radiation dominance when $\Omega_\Lambda = 10^{-6}$ and $1 - \Omega_\Lambda = \Omega_{radiation}$ is very close to one. Here H_0 and $a_0 = 1$ are the Hubble constant and the scale factor at this arbitrary moment and we keep the definition for $E(T)$. Clarifying these new particular initial conditions, we can use the solutions previously found looking their behavior backward in time. The value of Ω_ξ represents then the dimensionless density of dissipation at this initial time chosen above.

Case for $m = 0$

We have found that the only solution for a positive CC is given, in this case, when the discriminant of Eq. (2.14) is positive, but this solution presents a bouncing behavior as it was discussed in section 3.2.1, so this solution describe a regular universe without an early singularity.

Case for $m = 1$

The general solution for an arbitrary γ for this case corresponds to the expression (3.35). The expression (3.36) corresponds a time when the energy density and E tends to infinity, which are the same conditions required to have an early Type 0A (Big-Bang) singularity, with the difference that, in this case, the scale factor tends to zero. We already have discussed analytically and graphically (see Fig. 3.2) Eq. (3.36), showing that is strictly positive, so this universe does not have early singularities. We will later discuss in detail in section 3.3 the behavior of this solution at early times.

3.2.4 Early time singularity for the case of $\Lambda < 0$

Cases for $m = 0$

For this case the only solutions that present a singularities are those with a discriminant equal or less than zero; recall when $\Delta_0 > 0$ the solution has a Bouncing-type behavior given by (3.34).

(i) Case $\Delta_0 = 0$**.** We consider the behavior backwards in time of expression $E(T)$ and $a(T)$ given by (3.45) and (3.46), respectively. Even more, for this solution the time for singularity is given by (3.48), resulting in a scale factor of null value, and since H and $\dot{H}$ go to infinity for this singularity, then the Ricci scalar given by Eq. (3.31) also diverge. To get a early singularity we have a restriction for Ω_ξ from (3.48) given by $\Omega_\xi < 8/3$ for the case of radiation. In this sense, ρ and p diverges, so we will get a Type $0A$ singularity (Big-Bang). From the value of the CC, Eq. (3.44) leads to $\Omega_\xi = 7 \times 10^{-3}$. In Fig. 3.10 we present the behavior for E as a function of T, given by Eq.(3.52).

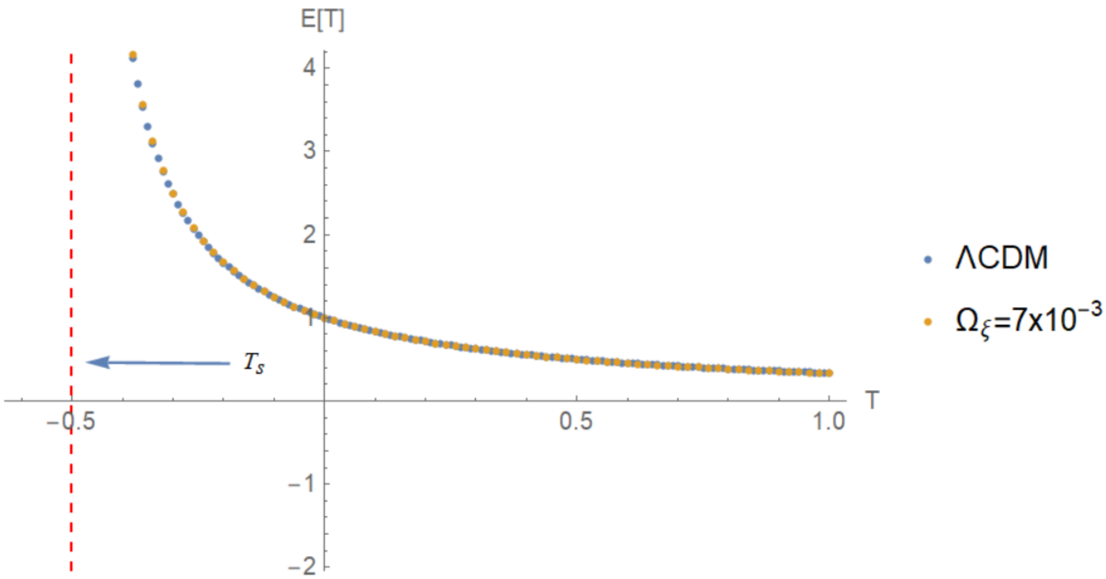

Figure 3.10: Behavior of $E(T)$, given by Eq. (3.45) for $\gamma = 4/3$, $\Omega_\Lambda = -10^6$. Ω_ξ is restricted by Eq. (3.44). We also plotted the ΛCDM model. The red dashed line represent the singularity time given by Eq. (3.48)

(ii) Case $\Delta_0 < 0$. We consider the behavior backwards in time of expression $E(T)$ and $a(T)$ given by Eqs. (3.52) and (3.53), respectively. Even more, for this solution the time for singularity is given by (3.56), resulting in a scale factor of null value, and since H and $\dot{H}$ go to infinity for this singularity, then the Ricci scalar given by Eq. (3.31) also diverge. For early time singularity we need to considered, from (3.54), $E \to +\infty$ to get

$$T_s = \frac{4}{3\gamma\sqrt{|\bar{\Delta}_0|}} \times \left(-\frac{\pi}{2} + \arctan\left(\frac{2 - \frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}} \right) \right). \tag{3.60}$$

From the previous expression ρ and p diverges, so this is a type 0A singularity (Big-Bang). The value of $\Omega_\Lambda = -10^{-6}$ leads to $\Omega_\xi < 3 \times 10^{-3}$ from Eq. (3.51). In Fig. 3.11 we present the behavior for E as a function of T, given by Eq.(3.52).

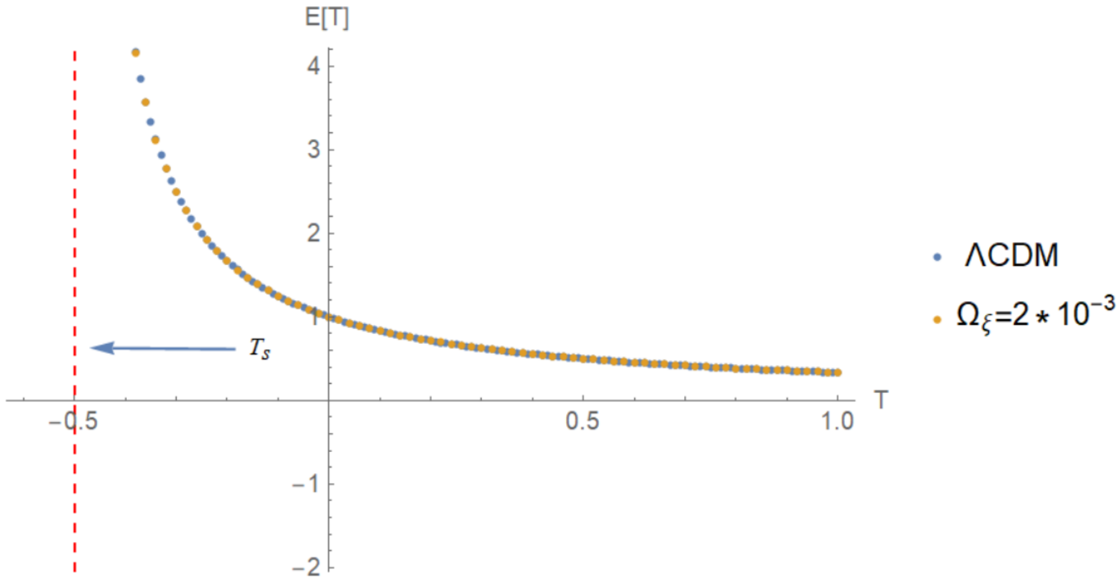

Figure 3.11: Behavior of $E(T)$ given by Eq. (3.52), for the particular values $\gamma = 4/3$, $\Omega_\Lambda = -10^{-6}$ and $\Omega_\xi = 2 \times 10^{-3}$, according to restriction (3.51). We also plotted the ΛCDM model. The red dashed line represent the singularity time given by Eq. (3.60).

case $m = 1$

We discuss in seccion 3.2.2 that the time for singularity is given by (3.59) and is strictly positive, so this solution as in the case of positive CC does not have a singularity in early stages either. A detailed discussion about this behavior will be done in section 3.3

3.3 Soft-Big Bang

As we have discussed in section 3.2.3, the solution given by Eq. (3.35) (case with $m = 1$ and $\Omega_\Lambda > 0$), when $\Omega_\xi < \gamma$, describe a universe without initial singularity. In this particular solution, when $T \to -\infty$, we obtain that the Hubble constant is given by Eq. (2.19), and the same behaviour is obtained if we considered a negative CC as can be seen in Eq.(3.58) . In Fig. 3.12 we show this behavior.

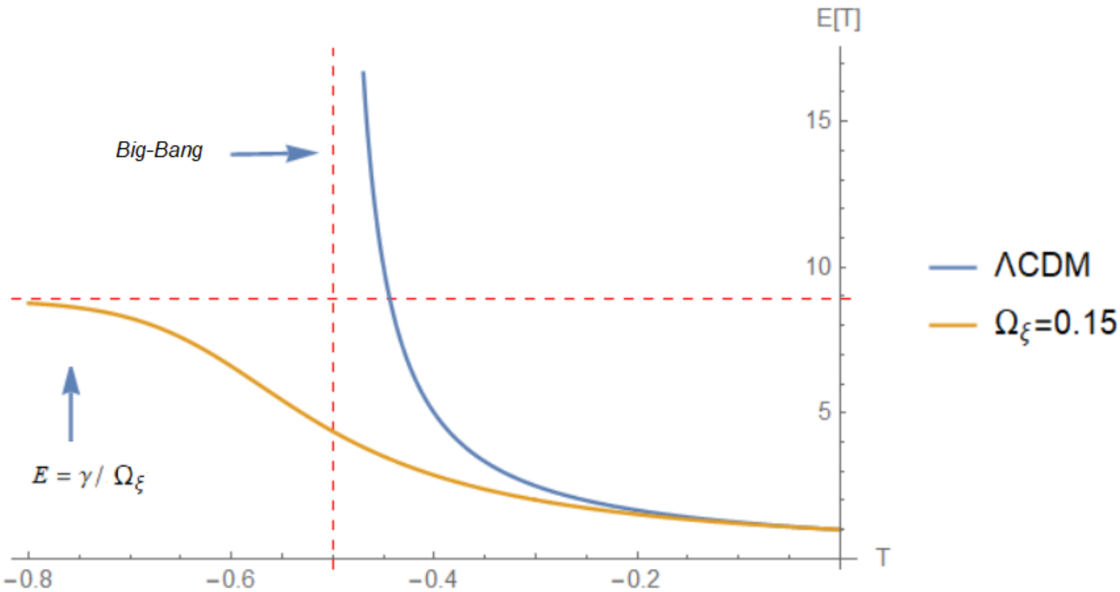

Figure 3.12: Numerical behavior for $E(T)$ given by Eq. (3.35) for $m = 1$, $\Omega_\Lambda = 10^{-6}$ and $\gamma = 4/3$. We also plotted the behaviour of the ΛCDM model.

Note that taking the limit $\Omega_\xi \to 0$ in Eq. (2.19) we obtain that $E \to \infty$, which is the behavior corresponding to a Big Bang singularity in the past. Hence, this solution turns into aΛCDM model with a Big-Bang singularity when dissipation is neglected. The behavior of the scale factor is shown in Fig. 3.13.

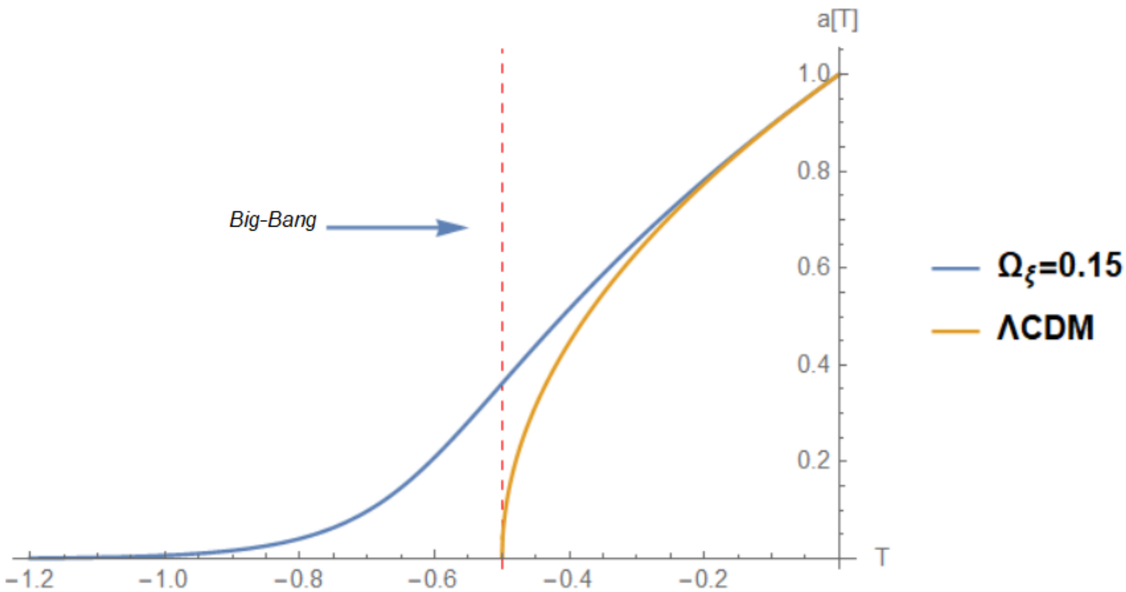

Figure 3.13: Behavior of numerical integration from Eq. (3.35) to get $a(T)$ for $m = 1$, $\Omega_\Lambda = 10^{-6}$ and $\gamma = 4/3$. We also plotted the behaviour of the ΛCDM model.

At $T \to -\infty$, $H \to \gamma/(3\xi_0)$ and $\dot{H} \to 0$, so the Ricci scalar given by Eq. (3.31) takes the value

$$R = \frac{4\gamma^2}{3\xi_0^2}, \tag{3.61}$$

indicating that there is no curvature singularity in this solution. In the infinity past $a = 0$ and H takes a constant value. If $\xi_0 \to 0$ the behavior of the standard model is recovered with $R \to \infty$ when $a = 0$ in some finite time in the past. The inclusion of dissipation without a CC led to these soft-Big Bang scenarios [58].

This solution is different from the soft-Big Bang studied in [88, 89] or from other singularity-free models these suggested by Israel $\&$ Rosen [131], or by Blome $\&$ Priester [132] where the universe begin from either by a tiny bubble in a homogeneous and isotropic quantum state with the diameter of a Planck length as an initial condition, or start from an Einstein static universe, with a radius determined by the value of Λ, before entering a never-ending period of de Sitter expansion. The solution discussed in [88] has the particularity of having a finite scale factor in the infinite past.

Chapter 4

Study of a viscous ΛWDM model: Near equilibrium condition, mathematical stability, entropy production, an cosmological constraints

In previous chapters, we explore the presence of singularity in viscous cosmology, and since the nature of the DM is unknown up to date, and dissipative effect can not be discarded[133], it is of physical interest to explore how a viscous DM behaves in the ΛCDM model.

The following results were recently published in [134]. In order to motivate the beginning of this second investigation, it is important to mention that in Eckart's and Israel Stewart theories, the bulk viscous pressure Π has to be lower than the equilibrium pressure p of the dissipative fluid, according to the Eq.(1), we define the following quantity

$$l = \left|\frac{\Pi}{p}\right| \ll 1, \tag{4.1}$$

which is known as *the near-equilibrium condition*, and represents the assumption that the fluid is close to thermodynamic equilibrium.

According to Maartens, in the context of dissipative inflation [63], the condition to have an accelerated expansion due only to the negativeness of the viscous pressure Π in Eckart's and IS theories enters into direct contradiction with the near-equilibrium condition given by Equation (4.1). In this sense, as it has been proposed in [64, 65], if a positive CC is considered in these theories, then the near-equilibrium condition could be preserved in some regimes. In addition, it was shown by J. Hua and H. Hu [60] that a dissipative DM in Eckart's theory with CC has a significantly better fit with the cosmological data than the ΛCDM model, which indicates that this model is competitive to fit the combined SNe Ia + CMB + BAO + OHD data. Nevertheless, the inclusion of the CC implies abandoning the idea of unified DM models with dissipation, whose advantage is to avoid the CC problem, but that leads to reinfor the proposal of extending the standard model, keeping a DE component modeled by a CC.

Another important point of the near-equilibrium condition given by Equation (4.1) is that we need a non-zero equilibrium pressure for the dissipative fluid discarding the possibility of a CDM component. In this sense, a relativistic approach to dissipative fluids is consistent with a WDM component [90, 95, 96, 104, 135–137] in order to satisfy the near-equilibrium condition.

It is important to mention that, for cosmologies with perfect fluids, there is no entropy production because these fluids are in equilibrium and their thermodynamic are reversible.

However, for cosmologies with non-perfect fluids, where irreversible process exists, there is a positive entropy production during the cosmic evolution [34, 79, 138–140]. The near-equilibrium condition and entropy production has been previously discussed in the literature. The near-equilibrium condition was studied, for example, in [79] for the IS theory with gravitational constant G and Λ that vary over time; while in [141], it was studied in Eckart's and IS theories for the case of a dissipative Boltzmann gas and without the inclusion of a CC. The entropy production was studied in [138] in Eckart's and IS theories for a dissipative DE; while in [34], the authors studied the entropy production in the full IS theory with a matter content represented by one dissipative fluid component, and the kinematics and thermodynamics properties of the solutions are discussed (the entropy production in cosmological viscous fluids has been more widely studied, and more references can be found in [142–147]).

4.1 An Exact analytical solution in Eckart's theory with CC

In this section we briefly resume a de Sitter-like solution and an analytical solution found in chapter 2 and published in [106].

For a flat FLRW universe composed with a dissipative DM which obeys the barotropic EoS $p = (\gamma - 1)\rho$, with a bulk viscosity of the form $\xi = \xi_0 \rho^m$, and DE given by the CC, it is possible to obtain, in the framework of the Eckart's theory, a single evolution equation for the Hubble constant $H = \dot{a}/a$, where a is the scale factor and "dot" accounts for the derivative with respect to the cosmic time t, which is given by Eq. (2.7)

$$2\dot{H} + 3\gamma H^2 - 3\xi_0 H (3H^2 - \Lambda)^m - \Lambda\gamma = 0. \tag{4.2}$$

The results were compared with theΛCDM model and, for this purpose, the differential equation (2.7) is solved for $\xi_0 = 0$ with the initial conditions $H(t = 0) =$

H_0 and $a(t=0)=1$, which leads to Eqs. (2.8) and (2.9)

$$H(t) = \frac{H_0\sqrt{\Omega_{\Lambda_0}}\left(\left(\sqrt{\Omega_{\Lambda_0}}+1\right)e^{3\gamma H_0 t\sqrt{\Omega_{\Lambda_0}}} - \sqrt{\Omega_{\Lambda_0}}+1\right)}{\left(\sqrt{\Omega_{\Lambda_0}}+1\right)e^{3\gamma H_0 t\sqrt{\Omega_{\Lambda_0}}} + \sqrt{\Omega_{\Lambda_0}}-1}, \tag{4.3}$$

$$a(t) = \left(\cosh\left(\frac{3\gamma\sqrt{\Omega_{\Lambda_0}}H_0 t}{2}\right) + \frac{\sinh\left(\frac{3\gamma\sqrt{\Omega_{\Lambda_0}}H_0 t}{2}\right)}{\sqrt{\Omega_{\Lambda_0}}}\right)^{\frac{2}{3\gamma}}, \tag{4.4}$$

where $\Omega_{\Lambda_0} = \Lambda/(3H_0^2)$. Note that Eq.(4.3) tends asymptotically at very late times ($t\to\infty$) to the de Sitter solution $H_{dS} = H_0\sqrt{\Omega_{\Lambda_0}}$.

We are particularly interested in the case of $m=1$ of Eq. (4.2) with a positive CC, where the following de Sitter-like solution ($\dot{H}=0$) is (Eq. (2.12))

$$E_{dS} \;=\; \sqrt{\Omega_{\Lambda_0}}, \tag{4.5}$$

with $E = H(t)/H_0$. It is important to note that Eq. (4.5) is the usual de Sitter solution, written in its dimensionless form, and naturally appears in this dissipative scenario. On the other hand, the exact analytical solution ($\dot{H}\neq 0$) found, takes the following form in terms of the dimensionless parameters (Eq. (3.35) [1])

$$\tau = \frac{\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}\log\left(\frac{(1-\Omega_{\Lambda_0})(\gamma-E\Omega_{\xi_0})^2}{\left(E^2-\Omega_{\Lambda_0}\right)(\gamma-\Omega_{\xi_0})^2}\right)}{3\sqrt{\Omega_{\Lambda_0}}\left(\gamma^2-\Omega_{\xi_0}^2\Omega_{\Lambda_0}\right)} + \frac{\gamma\log\left(\frac{\left(\sqrt{\Omega_{\Lambda_0}}-1\right)\left(\sqrt{\Omega_{\Lambda_0}}+E\right)}{\left(\sqrt{\Omega_{\Lambda_0}}+1\right)\left(\sqrt{\Omega_{\Lambda_0}}-E\right)}\right)}{3\sqrt{\Omega_{\Lambda_0}}\left(\gamma^2-\Omega_{\xi_0}^2\Omega_{\Lambda_0}\right)}, \tag{4.6}$$

obtained with the boundary condition $H(t=0)=H_0$. Here $\Omega_{\xi_0} = 3\xi_0 H_0$, and $\tau = tH_0$ are the dimensionless bulk viscous constant and cosmic time, respectively. The above solution is an implicit relation of $E(\tau)$. Eq. (4.6) presents a future singularity in a finite time known as Big-Rip [62, 82–84] when $\Omega_{\xi_0} > \gamma$. At this singularity we have an infinite a, ρ and p, and, therefore, the Ricci scalar diverges. Also, is discussed that one interesting behavior of this solution can be obtained if

[1] We change the notation for dimensionless time $T\to\tau$ in order to not confuse with temperature symbol T using in this chapter

we considered the opposite condition, i.e.,

$$\Omega_{\xi_0} < \gamma, \tag{4.7}$$

which leads to a universe with a behavior very similar to theΛCDM model, which coincide as $\Omega_{\xi_0} \to 0$, as can be seen in Fig. 4.1, where we have numerically found the behavior of E as a function of τ from Eq. (4.6), taking into account the condition (4.7) with $\gamma = 1.002$, $\Omega_{\xi_0} = 0.001$ and $\Omega_{\Lambda_0} = 0.69$. For a comparison, we also plotted theΛCDM model.

Note that the solution (4.6) tends asymptotically, for $\tau \to \infty$, to the usual de Sitter solution (4.5), which can be seen in the Fig. 4.1. Therefore, for the condition given by Eq. (4.7) and for $\gamma \neq 1$, but close to 1, solution (4.6) represents a viscous ΛWDM model with a late-time behaviour very similar to theΛCDM model and with the same asymptotic de Sitter expansion.

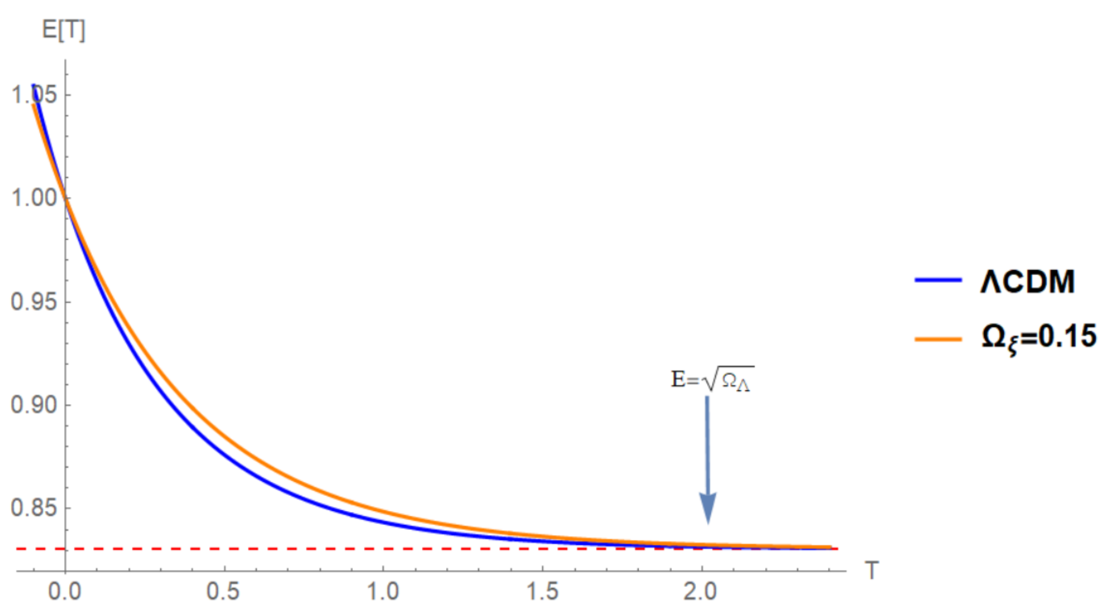

Figure 4.1: Numerical behavior of $E(\tau)$ obtained from Eq. (4.6) at late times, for $\Omega_{\Lambda_0} = 0.69$, $\Omega_{\xi_0} = 0.001$ and $\gamma = 1.002$. For a comparison, we also plotted the ΛCDM model obtained from Eq. (4.3).

4.2 Near equilibrium condition, mathematical stability and entropy production

In what follows we found the main expressions in terms of γ, and the dimensionless quantities E, Ω_{ξ_0} and Ω_{Λ_0}, that arises from the near equilibrium condition, the mathematical stability and the entropy production.

4.2.1 Near equilibrium condition

As it was previously discussed, in the Eckart's theory it is necessary to fulfill the near equilibrium condition (1). Following Maartens [63],and according to Equations (2.3) and (2.4)

$$\frac{\ddot{a}}{a} = -\frac{1}{6}\left[\rho + 3\left(p + \Pi\right)\right] + \frac{\Lambda}{3}. \tag{4.8}$$

From the above expression, the condition to have an accelerated expansion driven only by the negativeness of the bulk viscous pressure Π, (namely, $\ddot{a} > 0$ and taking $\Lambda = 0$), is

$$-\Pi > p + \frac{\rho}{3}. \tag{4.9}$$

This last result implies that the viscous stress is greater than the equilibrium pressure p of the fluid, i.e., the near equilibrium condition is not fulfilled because in order to obtain an accelerated expansion the fluid has to be far from equilibrium. This situation could change if a positive CC is included [64, 65]. In this case the condition $\ddot{a} > 0$ in Eq. (4.8) leads to

$$-\Pi > \frac{-2\Lambda}{3} + p + \frac{\rho}{3}, \tag{4.10}$$

i.e., the near equilibrium condition could be fulfilled in some regime, because from Eq. (4.10) the viscous stress not necessarily is greater than the equilibrium pressure p. The near equilibrium condition given by Eq. (1) can be rewritten in terms of the dimensionless parameters, using the expression $\Pi = -3H\xi$, which is the viscous pressure in the Eckart's theory [53], and the EoS of the DM component, obtaining

$$l = \left| \frac{E(t)\Omega_{\xi_0}}{\gamma - 1} \right| \ll 1. \tag{4.11}$$

From the above equation is clear that, for a CDM component with $\gamma = 1$, it is not possible to satisfy the near equilibrium condition, and only for some kind of WDM with $\gamma > 1$ it is possible to find some constraints for the parameters of the models that can fulfill this condition. On the other hand, note that in the above expression the solution given by $E(t)$ drives the behavior of l as a function of the cosmic time t. In this sense, and since $E(t)$ is a decreasing function of time, the constraints on Ω_{ξ_0} are more restrictive as we look forward.

4.2.2 Mathematical stability

In this section we explore the mathematical stability of the solution in order to find possible new constraint upon the main free parameters of the model. To do so, we investigate the behavior of a perturbed solution of the form

$$E_\delta(\tau) = E(\tau) + h(\tau), \;\; |h(\tau)| \ll 1, \tag{4.12}$$

where $E(\tau)$ corresponds to the unperturbed solution and $h(\tau)$ is a small perturbation function. Introducing Eq. (4.12) in Eq. (4.2), we obtain the following differential equation, in our dimensionless notation, for $h(\tau)$ (see appendix E Eq. (E.2))

$$\dot{h} - \frac{9\Omega_{\xi_0}}{2}\left(E^2 - \frac{2\gamma}{3\Omega_{\xi_0}}E - \frac{\Omega_{\Lambda_0}}{3}\right)h = 0. \tag{4.13}$$

The above expression is a differential equation for the perturbation function $h(t)$ and must satisfy, in order to the solution $H(t)$ be mathematically stable, that $h \to 0$ when $t \to \infty$.

4.2.3 Entropy production

The First law of thermodynamics is

$$TdS = dU + pdV, \tag{4.14}$$

where T, S, U, V are the temperature, entropy, the total thermal energy, and the three dimensional volume of the cosmic fluid. The total thermal energy of the fluid and the physical three dimensional volume of the Universe are given respectively by $U = \rho V$ and $V = V_0 a^3$ (where V_0 is the volume at the present time). For a DM, the energy density is given by $\rho = n\left(mc^2 + 3k_B T\right)$, and for the particular case of CDM, the energy density is associated with the rest mass of DM given by $\rho = nmc^2$, in full units. With these, we get from Eq. (4.14) the Gibbs equation [140]

$$dS = -\left(\frac{\rho + p}{Tn^2}\right) dn + \frac{d\rho}{Tn}, \tag{4.15}$$

where $n = N/V$ is the number of particle density. The following integrability condition must hold on the thermodynamical variables ρ and n

$$\left[\frac{\partial}{\partial \rho}\left(\frac{\partial S}{\partial n}\right)_{\rho}\right]_{n} = \left[\frac{\partial}{\partial n}\left(\frac{\partial S}{\partial \rho}\right)_{n}\right]_{\rho}, \tag{4.16}$$

then, we considered the thermodynamic assumption in which the temperature is a function of the number of particles density and the energy density, i.e., $T(n, \rho)$. With this, the above integrability condition become in [138, 140]

$$n\frac{\partial T}{\partial n} + (\rho + p)\frac{\partial T}{\partial \rho} = T\frac{\partial p}{\partial \rho}. \tag{4.17}$$

We study the case of a perfect fluid and a viscous fluid separately in order to compare our result with the model without viscosity.

For a perfect fluid the particle 4-current is taken to be $n^{\alpha}_{;\alpha} = 0$, which, together with the conservation equation, gives the expressions for the particle density and the energy density, respectively

$$\dot{n} + 3Hn = \frac{\dot{N}}{N} \;\; = \;\; 0, \tag{4.18}$$

$$\dot{\rho} + 3H(\rho + p) \;\; = \;\; 0. \tag{4.19}$$

Assuming that the energy density depends on the temperature and the volume, i.e., $\rho(T, V)$ [138], we have the following relation:

$$\frac{d\rho}{da} = \frac{\partial\rho}{\partial T}\frac{dT}{da} + \frac{3V}{a}\frac{\partial\rho}{\partial V}. \tag{4.20}$$

Using Eqs. (4.19), (4.20), and the EoS, it can be shown (as in [138]) that the barotropic temperature from (4.17) is given by

$$\frac{T}{T_0} = \frac{\rho}{\rho_0}a^3 = \left(\frac{\rho}{\rho_0}\right)^{\frac{\gamma-1}{\gamma}}. \tag{4.21}$$

This last expression was previously found by Marteens in [140], using the method of characteristics. Additional to this, from Eq. (4.15), together with Eqs. (4.18), (4.19), and the EoS, we have $dS = 0$ or, consequently, $dS/dt = 0$, which imply that there is no entropy production in the cosmic expansion, i.e. the fluid is adiabatic.

For a viscous fluid an average 4-velocity is chosen in which there is no particle flux [111]; so, in this frame, the particle 4-current is taking again as $n^{\alpha}_{;\alpha} = 0$ and the equation (4.18) is still valid. On the other hand, from Eckart's theory, we have the conservation Eq. (2.6), which together with the Eq. (4.18) and the EoS, Eq. (4.15) give us the follow expression for the entropy production [138, 139], written in a dimensionless form,

$$\frac{dS}{d\tau} = \frac{3E^2\Omega_{\xi_0}\rho}{nT}, \tag{4.22}$$

written in a dimensionless form. Therefore, the entropy production in the viscous expanding universe is always positive and we recovered the behavior of a perfect fluid when $\Omega_{\xi_0} = 0$.

4.3 Study of the exact solution

In this section we study the exact solution (4.6) under the condition (4.7) in terms of the fulfillment of the near equilibrium condition, the mathematical stability, and the positiveness of the entropy production at the same time.

For that end, we focus our analysis in two defined late-time epochs of validity for the solution: (i) the actual time $\tau = 0$ for which $E = 1$, and (ii) the very late-times $\tau \to \infty$ for which $E = \sqrt{\Omega_{\Lambda_0}}$. We will extend the analysis for $0 < \tau < \infty$ and for $\tau < 0$. It is important to mention that the asymptotic behaviour of the solution (4.6), given by the de Sitter solution (4.5) when the condition (4.7) holds, leads to a universe dominated only by the CC in which the dissipative WDM is diluted due to the universe expansion, as can be seen by evaluating the Eq. (4.5) in the Friedmann equation Eq. (2.2), which leads to $\rho = 0$. Therefore, in this asymptotic behaviour we do not have a dissipative fluid to study the near equilibrium condition and the entropy production. Nevertheless, we can study these two conditions as an asymptotic regime of the solution while $\rho \to 0$.

4.3.1 Near equilibrium condition of the exact solution

Note that the near equilibrium condition (4.11), considering that $1 < \gamma \leq 2$, can be rewritten as

$$E(\tau) \ll \frac{\gamma - 1}{\Omega_{\xi_0}} = \frac{\gamma}{\Omega_{\xi_0}} - \frac{1}{\Omega_{\xi_0}}, \tag{4.23}$$

expression that tells us that, as a long as $E = H/H_0$ be much smaller than $(\gamma - 1)/\Omega_{\xi_0}$, the solution must be near the thermodynamic equilibrium. This, opens the possibility that the solution is able to fulfill this condition considering that $E(t)$, given by Eq. (4.6), decrease asymptotically to the de Sitter solution (4.5) as $\tau \to \infty$ under the condition (4.7), as we can be seen in Fig. 4.1. Furthermore, at the actual time $\tau = 0$, the condition (4.23) leads to

$$\Omega_{\xi_0} \ll \gamma - 1, \tag{4.24}$$

and for the very late times $\tau \to \infty$ leads to

$$\sqrt{\Omega_{\Lambda_0}} \ll \frac{\gamma - 1}{\Omega_{\xi_0}} = \frac{\gamma}{\Omega_{\xi_0}} - \frac{1}{\Omega_{\xi_0}}. \tag{4.25}$$

The fulfillment of the condition (4.24) implies the fulfillment of the condition (4.25), because $0 < \Omega_{\Lambda_0} \leq 1$ and from the condition (4.24) we get $1 \ll (\gamma - 1)/\Omega_{\xi_0}$. Also, note that, the fulfilment of the condition (4.24) implies the fulfilment of the condition (3.37).

In Summary, the fulfillment of the condition (4.24) implies the fulfillment of the near equilibrium condition from $0 \leq \tau < \infty$ and the condition (3.37) for which the solutions (3.35) behave as the de Sitter solution at very late times. It is important to note that, the fulfilment of the near equilibrium condition depends only on the values of γ and Ω_{ξ_0} and not in the values of Ω_{Λ_0}, with the characteristic that for a value of γ more closer to 1 (CDM) a smaller value of Ω_{ξ_0} must be considered. Even more, for $1 < \gamma \leq 2$ we can see that $\Omega_{\xi_0} < 1$. On the other hand, the condition (4.24) is independent of the behaviour of the solution because the election $E(\tau = 0) = 1$ is arbitrary, but, this not implies that the condition (4.25) be always fulfilled because this condition depends on the behaviour of the solution. In this sense note that, if we do not satisfy the condition (3.37) in Eq. (4.11), then E diverges and the solution will be far from near equilibrium in a finite time in the Big-Rip scenario.

For $\tau < 0$, it is still possible to fulfill the near equilibrium condition (4.23), as we mentioned above, while E be much smaller than $(\gamma - 1)/\Omega_{\xi_0}$. In Fig. 4.2 we depict the near equilibrium condition (4.11) as a function of Ω_{ξ_0} and E for the fixed values of $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. The red zone represents the values for which we are far from the near equilibrium and the green line represents the near equilibrium condition when $\Omega_{\xi_0} = 0.001$. Note that for these last values we are far from the equilibrium when $\Omega_{\xi_0} \geq 0.002$ ($l \geq 1$) for all E, and for $\Omega_{\xi_0} = 0.001$ we are far from the near equilibrium when $E \geq 2$, i. e., we are far from the near equilibrium from a time $\tau = -0.6001$ which is roughly equivalent to 8.63389 Gyrs backward in time (0.630211 times the life of theΛCDM universe), according to the Eq. (4.6).

From Fig. 4.2 we can see that when E grows, we can make Ω_{ξ_0} more closer to zero in order to fulfill the near equilibrium condition. For the solution (4.6) under the condition (4.7) this means that for $\tau < \infty$, for which E grows, we can stay in the near equilibrium while Ω_{ξ_0} be small enough to satisfy the condition (4.23). This is due to the election of the dissipation of the form $\xi = \xi_0 \rho$, because the dissipate pressure $\Pi = -3H\xi$ behaves as $\rho^{3/2}$ and the equilibrium pressure behaves as ρ and, therefore, considering that in this case ρ grows to the past, then the dissipative pressure grows more quickly than the equilibrium pressure and ξ_0 acts as a modulator of this growth.

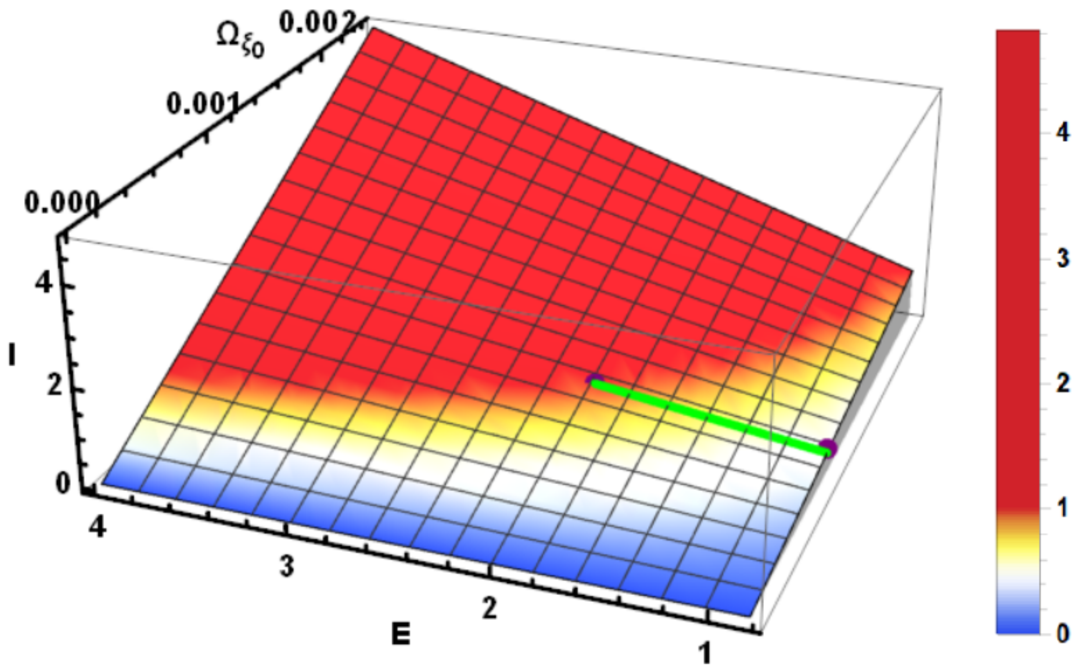

Figure 4.2: Behavior of l, given by Eq. (4.11), for $0 \leq \Omega_{\xi_0} \leq 0.02$ and $\sqrt{\Omega_{\Lambda_0}} \leq E \leq 4$. We also consider the fixed values of $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. The green line represent the near equilibrium condition when $\Omega_{\xi_0} = 0.001$ and the red zone represent the values for which we are far from the near equilibrium condition ($l > 1$).

4.3.2 Mathematical stability of the exact solution

To analyze the mathematical stability we use the Eqs. (4.2) and (4.13), changing the integration variable from t to E, obtaining in our dimensional notation the expression (see appendix E, Eq. (E.8))

$$\frac{dh}{dE} = \frac{\left(\Omega_{\Lambda_0}\Omega_{\xi_0} + 2\gamma E - 3E^2\Omega_{\xi_0}\right) h}{\left(E^2 - \Omega_{\Lambda_0}\right)\left(\gamma - E\Omega_{\xi_0}\right)}, \tag{4.26}$$

whose integration leads to (see appendix E, Eq. (E.14))

$$h(E) = C\left(\gamma - E\Omega_{\xi_0}\right)\left(E^2 - \Omega_{\Lambda_0}\right), \tag{4.27}$$

being C and integration constant. From this equation we can see that the the perturbation function is zero when $E = \pm\sqrt{\Omega_{\Lambda_0}}$ and when $E = \gamma/\Omega_{\xi_0}$. Furthermore, using the second derivative criteria we can see that

$$E_{min} = \frac{\gamma - \sqrt{\gamma^2 + 3\Omega_{\xi_0}^2 \Omega_{\Lambda_0}}}{3\Omega_{\xi_0}}, \tag{4.28}$$

and

$$E_{max} = \frac{\gamma + \sqrt{\gamma^2 + 3\Omega_{\xi_0}^2 \Omega_{\Lambda_0}}}{3\Omega_{\xi_0}}, \tag{4.29}$$

are the points where the function (4.27) have a relative minimum and maximum, respectively. Note that the argument in the square root of Eqs. (4.28) and (4.29) are always positive. Therefore, the perturbation function h is a decreasing function when $-\infty < E < E_{min}$ and $E_{max} < E < \infty$, and an increasing function when $E_{min} < E < E_{max}$.

Considering that the solution (4.6) is a decreasing function with the time, which tends to γ/Ω_{ξ_0} when $\tau \to -\infty$ [106] and to $\sqrt{\Omega_{\Lambda_0}}$ when $\tau \to \infty$, this last one as a long as we fulfill the condition (4.7), we can conclude that for $\Omega_{\xi_0} \neq 0$, value for which the point E_{min} given by the Eq. (4.28) is negative, the perturbative function is bounded above by $h(E_{max})$ and is zero when $\tau \to \pm\infty$, i. e., the solution (4.6) is mathematically stable. On the other hand, when $\Omega_{\xi_0} \to 0$, the Eq. (4.27) becomes

$$h(E) \to C\gamma \left(E^2 - \Omega_{\Lambda_0}\right), \tag{4.30}$$

function that has only a relative minimum for $E = \sqrt{\Omega_{\Lambda_0}}$. Therefore, in this case the perturbation function h is stable only for $E \to \sqrt{\Omega_{\Lambda_0}}$, i. e., for $\tau \to \infty$ (very late-times) and unstable for $E \to \infty$. This is an expected behavior because the solution (4.6) in this case tends to a solution with a singularity towards the past very similar to theΛCDM model, which coincides when $\gamma = 1$.

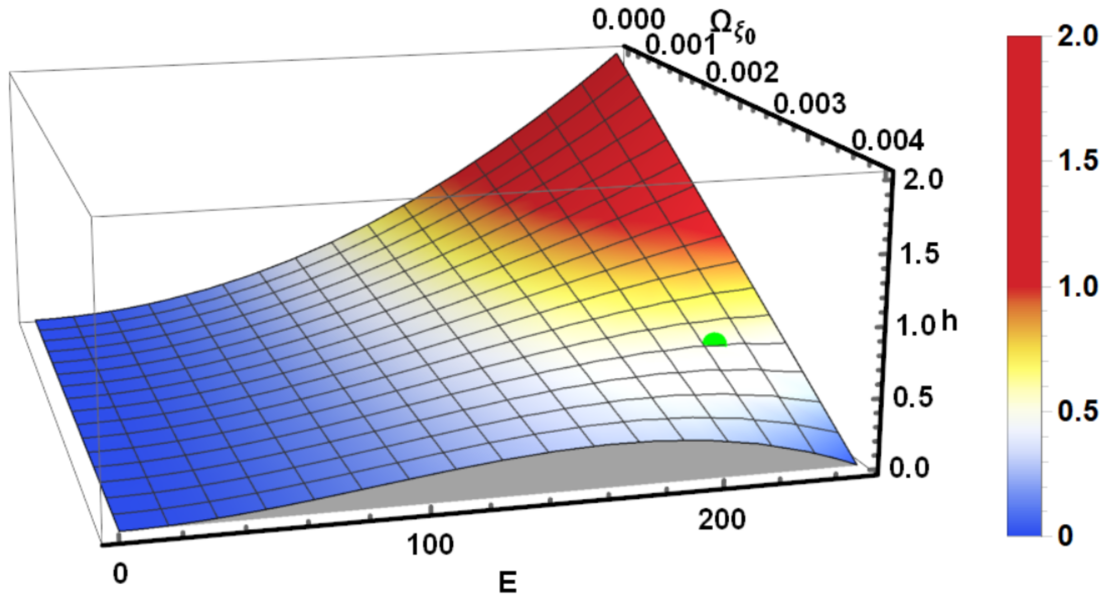

Figure 4.3: Behavior of h, given by Eq. (4.27), for $0 \leq \Omega_{\xi_0} \leq 0.1$ and $\sqrt{\Omega_{\Lambda_0}} \leq E \leq 10$. We also consider the fixed values of $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. The green point represent the value for $h(Emax)$ for the particular case of $\Omega_{\xi_0} = 0.003$

The behaviour in which the exact solution (4.6) becomes mathematically unstable when $\Omega_{\xi_0} \to 0$ seems to be not compatible with the near equilibrium condition (4.23). In particular, for $E = 1$ ($\tau = 0$) from equation (4.27) we can see that $h(E) \gg 1$ for a certain values of C and Ω_{Λ_0}, because from the condition (4.24) $1 \ll \gamma - \Omega_{\xi_0}$. Despite this fact we can always ensure the mathematically stability and the fulfillment of the near equilibrium condition for this solution at late times because the fulfillment of the condition (4.24) implies the fulfillment of the condition (4.7), therefore, the solution (4.6) tends to the de Sitter solution (4.5) for which the perturbation function (4.27) tends to zero. Even more, as a long as $\Omega_{\xi_0} \neq 0$, then the perturbation function remains bounded above and the solution is mathematically stable.

It is important to note that if we do not fulfill the condition (4.7), then we have a

Big Rip singularity in a future finite time which is mathematically unstable.

In Fig. 4.3 we present the behavior of perturbation function h, given by Eq. (4.27), as a function of E and Ω_{ξ_0}, for the particular values of $\gamma = 1.002$ and $\Omega_{\Lambda_0} = 0.69$. We use as a initial condition the arbitrary value of $h(E = 1) = 1 \times 10^{-5}$, for which $C = 1 \times 10^{-5}/((\gamma - \Omega_{\xi_0})(1 - \Omega_{\Lambda_0}))$.

4.3.3 Entropy production of the exact solution

In order to obtain the entropy production of the dissipative fluid from the Eq. (4.22), we need to find their temperature from the Eq. (4.17). Rewriting the conservation Eq. (2.6) in the form

$$\frac{d\rho}{da} = -\frac{3\rho}{a}\left(\gamma - 3H\xi_0\right), \tag{4.31}$$

we can rewrite Eq. (4.20) as

$$\rho\left(\gamma - 3H\xi_0\right) = -\frac{a}{3}\frac{\partial\rho}{\partial T}\frac{dT}{da} - V\frac{\partial\rho}{\partial V}. \tag{4.32}$$

Then, from Eq. (4.17), remembering that in the Eckart's theory $p \rightarrow p + \Pi$ [53], we have

$$n\frac{\partial T}{\partial n} + \rho\left(\gamma - 3H\xi_0\right)\frac{\partial T}{\partial\rho} = T\left[(\gamma - 1) - 3H\xi_0 - 3\xi_0\rho\frac{\partial H}{\partial\rho}\right], \tag{4.33}$$

expression with together Eq. (4.32) leads to

$$\frac{dT}{T} = -3\frac{da}{a}\left[(\gamma - 3H\xi_0) - 1 - 3\xi_0\rho\frac{\partial H}{\partial\rho}\right]. \tag{4.34}$$

Thus, using Eqs. (2.2) and (4.31) we get, from Eq. (4.34), the following expression (see appendix F , Eq. (F.8))

$$\frac{dT}{T} = \frac{d\rho}{\rho}\left[1 - \frac{\left(\frac{2}{3}\sqrt{3\left(\rho+\Lambda\right)} + \xi_0\rho\right)}{\frac{2}{3}\sqrt{3\left(\rho+\Lambda\right)}\left(\gamma - \sqrt{3\left(\rho+\Lambda\right)}\xi_0\right)}\right], \tag{4.35}$$

which has the following solution in our dimensionless notation (see appendix F, Eq. (F.19))

$$\ln\left(\frac{T}{T_0}\right)=\ln\left(\frac{\rho}{\rho_0}\right)+\frac{2\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\left[\operatorname{arctanh}\left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E}\right)-\operatorname{arctanh}\left(\sqrt{\Omega_{\Lambda_0}}\right)\right]}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}$$
$$\frac{-\gamma\ln\left(\frac{\rho}{\rho_0}\right)+\left[\gamma(2+\gamma)-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right]\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}. \tag{4.36}$$

On the other hand, integrating Eq. (4.31) with the help of Eq. (2.2), we obtain in our dimensionless notation the expression (see apendix F, Eq. (F.25))

$$\begin{aligned}\ln a^3 &= \frac{2\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\left[\operatorname{arctanh}\left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E}\right)-\operatorname{arctanh}\left(\sqrt{\Omega_{\Lambda_0}}\right)\right]}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}\\ &+ \frac{-\gamma\ln\left(\frac{\rho}{\rho_0}\right)+2\gamma\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)},\end{aligned} \tag{4.37}$$

from which we can express the Eq. (4.36) in terms of the scale factor as follows:

$$\ln\left(\frac{T}{T_0}\right)=\ln\left(\frac{\rho}{\rho_0}\right)+\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)+\ln a^3. \tag{4.38}$$

Hence, the temperature of the dissipative fluid as a function of the scale factor is given by

$$T=T_0\left(\frac{\rho}{\rho_0}\right)\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)a^3, \tag{4.39}$$

which reduced to the expression for the temperature of a perfect fluid (4.21) when $\Omega_{\xi_0}=0$.

Note that it is possible to obtain an expression for the temperature of the dissipative fluid that do not depends explicitly on ρ, by combining the Eqs. (4.37) and (4.39), which leads to

$$T=T_0 a^{\frac{3}{\gamma}\left(\gamma-\gamma^2+\Omega_{\xi_0}^2\Omega_{\Lambda_0}\right)}\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)^3\left[\left(\frac{E+\sqrt{\Omega_{\Lambda_0}}}{E-\sqrt{\Omega_{\Lambda_0}}}\right)\left(\frac{1-\sqrt{\Omega_{\Lambda_0}}}{1+\sqrt{\Omega_{\Lambda_0}}}\right)\right]^{\left(\frac{\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}}{\gamma}\right)}, \tag{4.40}$$

and from which we can see that the temperature is always positive in two cases: (i) when $\gamma - E\Omega_{\xi_0} > 0$ and $\gamma - \Omega_{\xi_0} > 0$ or (ii) when $\gamma - E\Omega_{\xi_0} < 0$ and $\gamma - \Omega_{\xi_0} < 0$. Therefore, if we fulfill the near equilibrium condition (4.23), then we obtain a positive expression for the temperature, since the condition (i), and from the same condition, we also fulfill the condition (4.7) from which the solution (4.6) tends asymptotically to the future at the usual de Sitter solution. Note that the condition (ii) implies that the fluid is far from the near equilibrium and the solution (4.6) has a Big-Rip singularity.

On the other hand, considering that the solution (4.6) is a decreasing function with time when the condition (4.7) holds, then the cubic term in the Eq. (4.40) is also a decreasing function; thus, considering that $E(\tau \to \infty) \to \sqrt{\Omega_{\Lambda_0}}$, a decreasing temperature with time requires that

$$\frac{a^{\frac{3}{\gamma}\left(\gamma-\gamma^2+\Omega_{\xi_0}^2\Omega_{\Lambda_0}\right)}}{\left(E-\sqrt{\Omega_\Lambda}\right)^{\left(\frac{\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}}{\gamma}\right)}} \to 0, \tag{4.41}$$

which is only possible, considering that $a(\tau \to \infty) \to \infty$, when the exponent of the power law for the scale factor is negative, i. e., if

$$\Omega_{\Lambda_0} < \frac{\gamma}{\Omega_{\xi_0}}\frac{(\gamma-1)}{\Omega_{\xi_0}}, \tag{4.42}$$

which is always true because $0 < \Omega_{\Lambda_0} \leq 1$ and the fulfilment of the condition (3.37) implies that $1 < \gamma/\Omega_{\xi_0}$, as well as the fulfillment of the condition (4.24) implies that $1 \ll (\gamma-1)/\Omega_{\xi_0}$. It is important to note that for CDM,which means that, if $\gamma = 1$, then the exponent of the power law for the scale factor is always positive and the temperature is an increasing function with time, which represents a contradictory behavior for an expanding universe.

In the Fig. 4.4 we present the numerical behaviour of the temperature T of the dissipative fluid, given by the Eq. (4.39), as a function of the scale factor a. We rewrite E as a function of the energy density ρ from Eq. (2.2) and we use the expression for ρ given by the Eq. (F.25). For the free parameters we use the values of $T_0 = 1$, $\Omega_{\xi_0} = 0.001$, $\Omega_{\Lambda_0} = 0.69$, and $\gamma = 1.002$. We also present the behaviour of the temperature when $\gamma = 1$. It is important to mention that the difference between the initial value of the temperature for WDM case and his final value is 0.0102895 for a scale factor that is 3.6 times more bigger than the actual size of the universe, this is the result to be close to the near equilibrium condition, that makes that temperature decrease very slowly to zero.

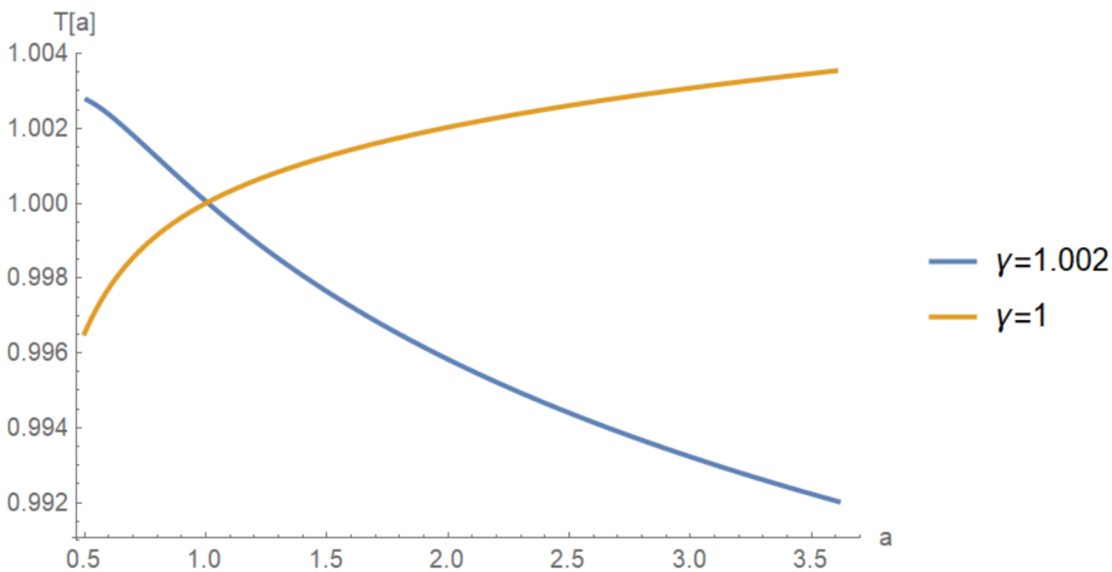

Figure 4.4: Numerical behavior of T, given by Eq. (4.39), for $0.5 \leq a \leq 3.5$. We also consider the fixed values of $T_0 = 1$, $\Omega_{\xi_0} = 0.001$, and $\Omega_{\Lambda_0} = 0.69$.

Now, with the temperature of the dissipative fluid given by Eq. (4.39), we can calculate the entropy from Eq. (4.22), which in our dimensionless notation, and using also that from Eq. (4.18) $n = n_0 a^{-3}$, takes the following form:

$$\frac{dS}{d\tau} = \frac{3E^2\Omega_{\xi_0}\rho}{nT} = \frac{3E^2\rho_0\Omega_{\xi_0}\left(\gamma - \Omega_{\xi_0}\right)}{n_0 T_0\left(\gamma - E\Omega_{\xi_0}\right)}. \tag{4.43}$$

The positiveness of the entropy production depends on the same cases as the positiveness of the temperature of the dissipative fluid, therefore, the fulfillment of the near equilibrium condition (which is only compatible for WDM component), implies the positiveness of the entropy production. Note that, for the asymptotic de Sitter solution the entropy production goes to a constant but, in this case, $\rho \to 0$ (which implies a null Ω_{ξ_0}) and then we have a null entropy production. On the other hand, as well as the temperature of the dissipative fluid, the entropy production goes to infinity when we do not satisfy the condition (3.37) in a finite time in the Big-Rip singularity.

In Fig. 4.5 we show the numerical behaviour of the entropy production, given by the Eq. (4.43) as a function of time τ for $\Omega_{\xi_0} = 0.001$, $\Omega_{\Lambda_0} = 0.69$, and $\gamma = 1.002$. We also show the behavior of the entropy production when $\Omega_{\xi_0} = 2.4$. For this last one value, the near equilibrium condition is not satisfied, since this value enter in to contradiction with Eq. (3.37), and the entropy production diverge in a finite time given by Eq. (3.36)

$$\tau_s = \frac{2\Omega_{\xi_0} \log\left[\left(\frac{1-\sqrt{\Omega_{\Lambda_0}}}{1+\sqrt{\Omega_{\Lambda_0}}}\right)^{\frac{\gamma}{2\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}}} \left(1-\Omega_{\Lambda_0}\right)^{\frac{1}{2}} \left(\frac{-\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)\right]}{3\left(\gamma^2 - \Omega_{\xi_0}^2\Omega_{\Lambda_0}\right)}, \tag{4.44}$$

which according to our parameters, this is equal to $\tau_s = 0.346571$, which is roughly equivalent to 4.98626Gyrs from the present time.

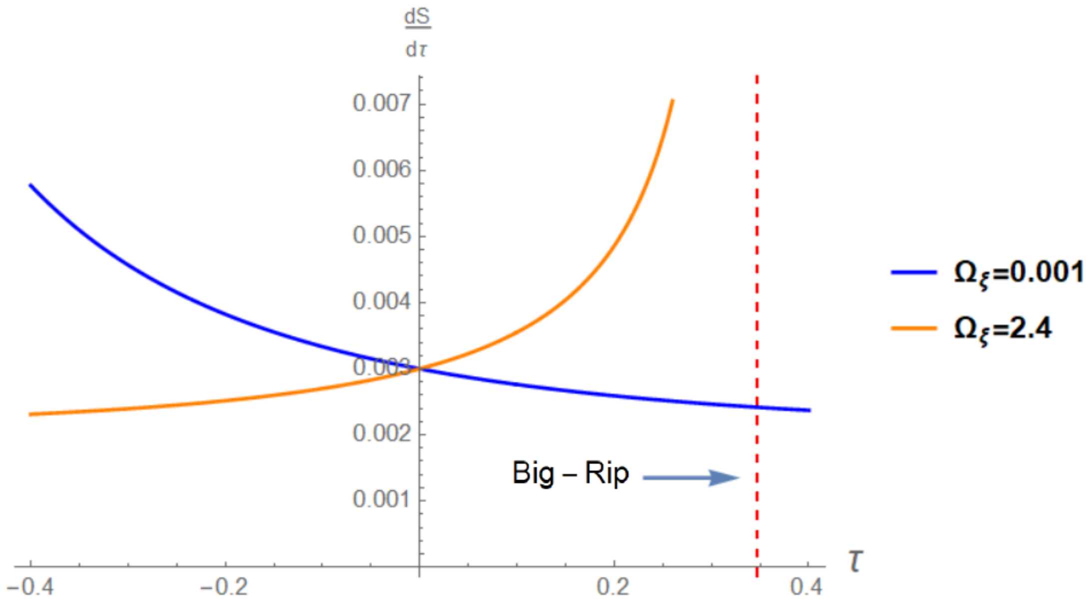

Figure 4.5: Numerical behavior of $dS/d\tau$, given by Eq. (4.43), for $-0.4 \leq \tau \leq 0.4$. We also consider the fixed values of $\Omega_{\Lambda_0} = 0.69$ and $\gamma = 1.002$. The red dashed line represent the time τ_s, given by Eq. (3.36), in which the Big-Rip singularity occurs.

In summary, the condition given by Eq. (4.24) together with the condition $\Omega_{\xi_0} < 1$, for the present time, describe a viscous WDM model that is compatible with the near equilibrium condition, and presents a proper physical behavior of the temperature (a decreasing function with the scale factor), and entropy production (without future Big-Rip singularity).

In this sense, all this previous thermodynamics analysis will help us to define the best prior definition for our cosmological parameters γ and Ω_{ξ_0}, in order to constraints with the cosmological data. Accordingly, our model has two more free parameters than the standard ΛCDM model, which appear from the possibility of a more general model of the DM component, that takes into account a warm nature with a non-perfect fluid description.

4.3.4 Cosmological constraints

In this section, we shall constraint the free parameters of the viscous ΛWDM model studied in this paper with the type Ia supernovae (SNe Ia) data coming from the Pantheon sample [148], which consists in 1048 data points in the redshift range $0.01 \leq z \leq 2.3$; and the observational Hubble parameter data (OHD) compiled by Magaña *et al.* [149], which consists in 51 data points in the redshift range $0.07 \leq z \leq 2.36$. To do so, we compute the best-fit parameters and their respective confidence regions with the affine-invariant Markov Chain Monte Carlo (MCMC) method [150], implemented in the pure-Python code *emcee* [151], by setting 30 chains or “walkers”.

As a convergence test, we compute every 50 steps the autocorrelation time of the chains τ_{corr}, provided by the *emcee* module. If the current step is larger than $50\tau_{corr}$, and if the values of τ_{corr} changed by less than 1%, then we will consider that the chains are converged and the code is stopped. The first $5\tau_{corr}$ steps are discarded as “burn-in” steps. This convergence test is complemented with the mean acceptance fraction, which should be between 0.2 and 0.5 [151], and can be modified by the stretch move provided by the *emcee* module.

Since we are implementing a Bayesian statistical analysis, we need to construct the Gaussian likelihood

$$\mathcal{L} = \mathcal{N} \exp\left(-\frac{\chi_I^2}{2}\right). \tag{4.45}$$

Here, $\mathcal{N}$ is a normalization constant, which does not influence in the MCMC analysis, and χ_I^2 is the merit function, where I stands for each data set considered, namely, SNe Ia, OHD, and their joint analysis in which $\chi^2_{joint} = \chi^2_{SNe} + \chi^2_{OHD}$.

The merit function for the OHD data is constructed as

$$\chi^2_{OHD} = \sum_{i=1}^{51} \left[\frac{H_i - H_{th}(z_i, \theta)}{\sigma_{H,i}}\right]^2, \tag{4.46}$$

where H_i is the observational Hubble parameter at redshift z_i with an associated error $\sigma_{H,i}$, all of them provided by the OHD sample, H_{th} is the theoretical Hubble parameter at the same redshift, and θ encompasses the free parameters of the model under study. It is important to mention that in our MCMC analysis we consider the value of the Hubble parameter at the current time, H_0, as a free parameter, which is written as $H_0 = 100\frac{km/s}{Mpc}h$, with h dimensionless and for which we consider the Gaussian prior $G(0.7403, 0.0142)$, according to the value of H_0 obtained by A. G. Riess *et al.* [20].

On the other hand, the merit function for the SNe Ia data is constructed as

$$\chi^2_{SNe} = \sum_{i=1}^{1048} \left[\frac{\mu_i - \mu_{th}(z_i, \theta)}{\sigma_{\mu,i}} \right]^2, \tag{4.47}$$

where μ_i is the observational distance modulus of each SNe Ia at redshift z_i with an associated error $\sigma_{\mu,i}$, μ_{th} is the theoretical distance modulus for each SNe Ia at the same redshift, and θ encompasses the free parameters of the model under study. Following this line, the theoretical distance modulus can be obtained, for a flat FLRW space-time, from the expression

$$\mu_{th}(z_i, \theta) = 5 \log_{10} \left[\frac{d_L(z_i, \theta)}{Mpc} \right] + \bar{\mu}, \tag{4.48}$$

where $\bar{\mu} = 5 \left[\log_{10}(c) + 5 \right]$, c is the speed of light given in units of km/s, and d_L is the luminosity distance given by

$$d_L(z_i, \theta) = (1 + z_i) \int_0^{z_i} \frac{dz'}{H(z', \theta)}. \tag{4.49}$$

In the Pantheon sample the distance estimator is obtained using a modified version of the Tripp's formula [152], with two nuisance parameters calibrated to zero with the BEAMS whit Bias Correction (BBC) method [153]. Hence, the observational distance modulus for each SNe Ia is given by

$$\mu_i = m_{B,i} - \mathcal{M}, \tag{4.50}$$

where $m_{B,i}$ is the corrected apparent B-band magnitude of a fiducial SNe Ia at redshift z_i, all of them provided by the pantheon sample [2], and $\mathcal{M}$ is a nuisance parameter which must be jointly estimated with the free parameters θ of the model under study. Therefore, we can rewrite the merit function (4.47) in matrix notation (denoted by bold symbols) as

$$\chi^2_{SNe} = \mathbf{M}(z, \theta, \mathcal{M})^\dagger \mathbf{C}^{-1} \mathbf{M}(z, \theta, \mathcal{M}), \tag{4.51}$$

where $[\mathbf{M}(z, \theta, \mathcal{M})]_i = m_{B,i} - \mu_{th}(z_i, \theta) - \mathcal{M}$, and $\mathbf{C} = \mathbf{D}_{stat} + \mathbf{C}_{sys}$ is the total uncertainties covariance matrix, being $\mathbf{D}_{stat} = diag(\sigma^2_{m_B,i})$ the statistical uncertainties of m_B and $\mathbf{C}_{sys}$ the systematic uncertainties in the BBC approach [3].

Finally, to marginalize over the nuisance parameters $\bar{\mu}$ and $\mathcal{M}$, we define $\bar{\mathcal{M}} = \bar{\mu} + \mathcal{M}$, and the merit function (4.51) is expanded as [154]

$$\chi^2_{SNe} = A(z, \theta) - 2B(z, \theta)\bar{\mathcal{M}} + C\bar{\mathcal{M}}^2, \tag{4.52}$$

where

$$A(z, \theta) = \mathbf{M}(z, \theta, \bar{\mathcal{M}} = 0)^\dagger \mathbf{C}^{-1} \mathbf{M}(z, \theta, \bar{\mathcal{M}} = 0), \tag{4.53}$$

$$B(z, \theta) = \mathbf{M}(z, \theta, \bar{\mathcal{M}} = 0)^\dagger \mathbf{C}^{-1} \mathbf{1}, \tag{4.54}$$

$$C = \mathbf{1}\mathbf{C}^{-1}\mathbf{1}. \tag{4.55}$$

Therefore, by minimizing the expanded merit function (4.52) with respect to $\bar{\mathcal{M}}$, it is obtained $\bar{\mathcal{M}} = B(z, \theta)/C$, and the expanded merit function reduced to

$$\chi^2_{SNe} = A(z, \theta) - \frac{B(z, \theta)^2}{C}, \tag{4.56}$$

[2]Currently available online in the 2022 GitHub repository *https://github.com/dscolnic/Pantheon*. The corrected apparent B-band magnitude $m_{B,i}$ for each SNe Ia with their respective redshifts z_i and errors $\sigma_{m_B,i}$ are available in the document *lcparam_full_long.txt*.

[3]Currently available online in the 2022 GitHub repository *https://github.com/dscolnic/Pantheon* in the document *sys_full_long.txt*.

which depends only on the free parameters of the model under study.

It is important to mention that the expanded and minimized merit function (4.56) provides the same information as the merit function (4.51) since the best-fit parameters minimize the merit function, and their corresponding value can be used as an indicator of the goodness of the fit independently of the data set used: the smaller the value of χ^2_{min} is, the better is the fit. In this sense, in principle the value of χ^2_{min} obtained for the best fit parameters can be reduced by adding free parameters to the model under study, resulting in overfitting.

Hence, we compute the Bayesian criterion information (BIC) [155] to compare the goodness of the fit statistically. This criterion adds a penalization in the value of χ^2_{min} that depends on the total number of free parameters of the model, θ_N, according to the expression

$$BIC = \theta_N \ln{(n)} + \chi^2_{min}, \tag{4.57}$$

where n is the total number of data points in the corresponding data sample. So, when two different models are compared, the one most favored by the observations statistically corresponds to the one with the smallest value of BIC. In general, a difference of $2-6$ in BIC is evidence against the model with higher BIC, a difference of $6-10$ is strong evidence, and a difference > 10 is very strong evidence.

Since in the merit function of the two data sets, the respective model is considered thought the Hubble parameter as a function of the redshift (see Eqs. (4.46) and (4.49)), then we numerically integrate Eq. (2.7) with $m=1$, which can be rewritten, considering that $\dot{z} = -(1+z)H$, as

$$\frac{dH}{dz} = \frac{1}{2(1+z)}\left[3\gamma H - 3\xi_0\left(3H^2 - \Lambda\right) - \frac{\Lambda\gamma}{H}\right], \tag{4.58}$$

using as initial condition $H(z=0) = H_0 = 100\frac{km/s}{Mpc}h$, and taking into account that $\xi_0 = \Omega_{\xi_0}/(3H_0)$ and $\Lambda = 3H_0^2(1-\Omega_{m0})$; this last one derived from Eq. (2.2), which leads to $\Omega_{m0} + \Omega_{\Lambda 0} = 1$. Even more, for a further comparison, we also constraint the free parameters of the ΛCDM model, whose respective Hubble parameter as a function of the redshift is given by

$$H(z) = 100\frac{km/s}{Mpc}h\sqrt{\Omega_{m0}(1+z)^3 + 1 - \Omega_{m0}}. \tag{4.59}$$

Therefore, the free parameters of the viscous ΛWDM model are $\theta = \{h, \Omega_{m0}, \Omega_{\xi_0}, \gamma\}$, and the free parameters of the ΛCDM model are $\theta = \{h, \Omega_{m0}\}$. For the free parameters Ω_{m0}, Ω_{ξ_0}, and γ we consider the following priors: $\Omega_{m0} \in F(0,1)$, $\gamma \in G(1.00, 0.02)$, and $0 < \Omega_{\xi_0} < \gamma - 1$; where F stands for flat prior, and the prior for Ω_{ξ_0} is derived from the constraint given by Eq. (4.24).

In Table 4.1 we present the total steps, the mean acceptance fraction (MAF), and the autocorrelation time τ_{corr} of each free parameter, obtained when the convergence test is fulfilled during our MCMC analysis for both, the viscous ΛWDM and ΛCDM, models. The values of the MAF are obtained for a value of the stretch move of $a = 7$ for the ΛCDM model, and a value of $a = 3$ for the viscous ΛWDM model.

4.3.5 Results and discussion

The best-fit values of the ΛCDM and viscous ΛWDM models, obtained for the SNe Ia data, OHD, and in their joint analysis, as well as their corresponding goodness of fit criteria, are presented in Table 4.2. The uncertainties presented correspond to $1\sigma(68.3\%)$, $2\sigma(95.5\%)$, and $3\sigma(99.7\%)$ of confidence level (CL). In Figures 4.6 and 4.7 we depict the joint and marginalized credible regions of the free parameters space of the ΛCDM and viscous ΛWDM model, respectively. The admissible regions presented in the joint regions correspond to $1\sigma(68.3\%)$, $2\sigma(95.5\%)$, and $3\sigma(99.7\%)$ of confidence level (CL).

From the best-fit parameters presented in Table 4.2 it is possible to see that there is no remarkable differences between the best-fit values of h and Ω_{m0} obtained for the ΛCDM model and the viscous ΛWDM model. This is an expected

Table 4.1: Final values of the total number of steps, mean acceptance fraction (MAF), and autocorrelation time τ_{corr} for each model free parameters, obtained when the convergence test described in Section 4.3.4 is fulfilled for a MCMC analysis with 30 chains or "walkers". The values of the MAF are obtained for a value of the stretch move of $a = 7$ for the ΛCDM model, and a value of $a = 3$ for the viscous ΛWDM model.

			τ_{corr}			
Data	Total steps	MAF	h	Ω_{m0}	Ω_{ξ_0}	γ
ΛCDM model						
SNe Ia	1050	0.370	16.5	17.5	$\cdots$	$\cdots$
OHD	1000	0.367	14.9	17.1	$\cdots$	$\cdots$
SNe Ia+OHD	800	0.364	15.8	15.4	$\cdots$	$\cdots$
viscous ΛWDM model						
SNe Ia	2700	0.385	45.8	44.3	49.6	51.9
OHD	2450	0.377	39.6	45.5	48.2	48.9
SNe Ia+OHD	2700	0.379	43.0	45.9	50.5	53.3

Table 4.2: Best-fit values and goodness of fit criteria for the viscous ΛWDM model with free parameters h, Ω_{m0}, Ω_{ξ_0}, and γ; and for the ΛCDM model with free parameters h and Ω_{m0}, obtained in the MCMC analysis described in the Section 4.3.4 for the SNe Ia data, OHD, and in their joint analysis. The uncertainties correspond to $1\sigma(68.3\%)$, $2\sigma(95.5\%)$, and $3\sigma(99.7\%)$ of confidence level (CL), respectively. The best-fits values for the ΛCDM model are used for the sake of comparison with the viscous ΛWDM model.

	Best-fit values				Goodness of fit	
Data	h	Ω_{m0}	$\Omega_{\xi_0}(\times 10^{-2})$	γ	χ^2_{min}	BIC
ΛCDM model						
SNe Ia	$0.740^{+0.014\,+0.028\,+0.043}_{-0.013\,-0.028\,-0.040}$	$0.299^{+0.022\,+0.045\,+0.064}_{-0.022\,-0.042\,-0.058}$	$\cdots$	$\cdots$	1026.9	1040.8
OHD	$0.720^{+0.009\,+0.018\,+0.026}_{-0.009\,-0.017\,-0.025}$	$0.241^{+0.014\,+0.027\,+0.038}_{-0.014\,-0.027\,-0.037}$	$\cdots$	$\cdots$	28.6	36.5
SNe Ia+OHD	$0.710^{+0.008\,+0.016\,+0.023}_{-0.008\,-0.016\,-0.022}$	$0.259^{+0.012\,+0.024\,+0.034}_{-0.012\,-0.022\,-0.036}$	$\cdots$	$\cdots$	1058.3	1072.3
Viscous ΛWDM model						
SNe Ia	$0.741^{+0.014\,+0.029\,+0.044}_{-0.014\,-0.028\,-0.044}$	$0.293^{+0.023\,+0.048\,+0.066}_{-0.022\,-0.043\,-0.064}$	$0.980^{+1.232\,+2.954\,+4.318}_{-0.716\,-0.943\,-0.979}$	$1.023^{+0.015\,+0.031\,+0.045}_{-0.011\,-0.019\,-0.021}$	1026.9	1054.7
OHD	$0.721^{+0.009\,+0.018\,+0.026}_{-0.010\,-0.020\,-0.029}$	$0.237^{+0.017\,+0.037\,+0.053}_{-0.017\,-0.033\,-0.045}$	$1.026^{+1.250\,+2.641\,+3.763}_{-0.738\,-0.988\,-1.019}$	$1.023^{+0.014\,+0.030\,+0.048}_{-0.012\,-0.019\,-0.022}$	28.5	44.2
SNe Ia+OHD	$0.709^{+0.009\,+0.017\,+0.026}_{-0.009\,-0.017\,-0.027}$	$0.261^{+0.015\,+0.030\,+0.047}_{-0.015\,-0.030\,-0.041}$	$1.633^{+1.381\,+2.871\,+4.186}_{-1.059\,-1.544\,-1.627}$	$1.026^{+0.015\,+0.031\,+0.045}_{-0.013\,-0.021\,-0.024}$	1056.9	1084.9

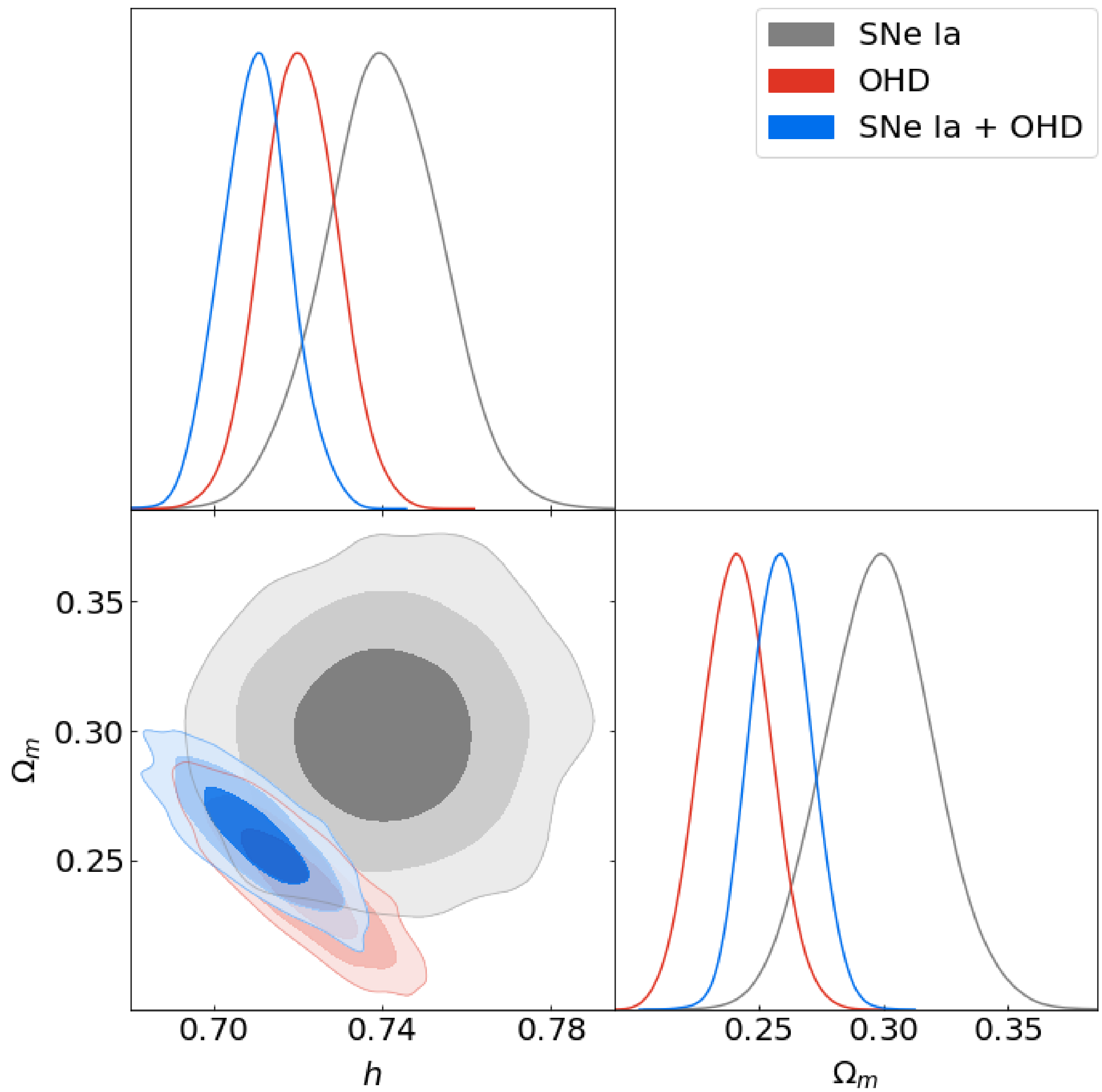

Figure 4.6: Joint and marginalized regions of the free parameters h and Ω_{m0} for the ΛCDM model, obtained in the MCMC analysis described in the Section 4.3.4. The admissible regions presented in the joint regions correspond to $1\sigma(68.3\%)$, $2\sigma(95.5\%)$, and $3\sigma(99.7\%)$ of confidence level (CL), respectively. The best-fit values for each model free parameter are shown in Table 4.2.

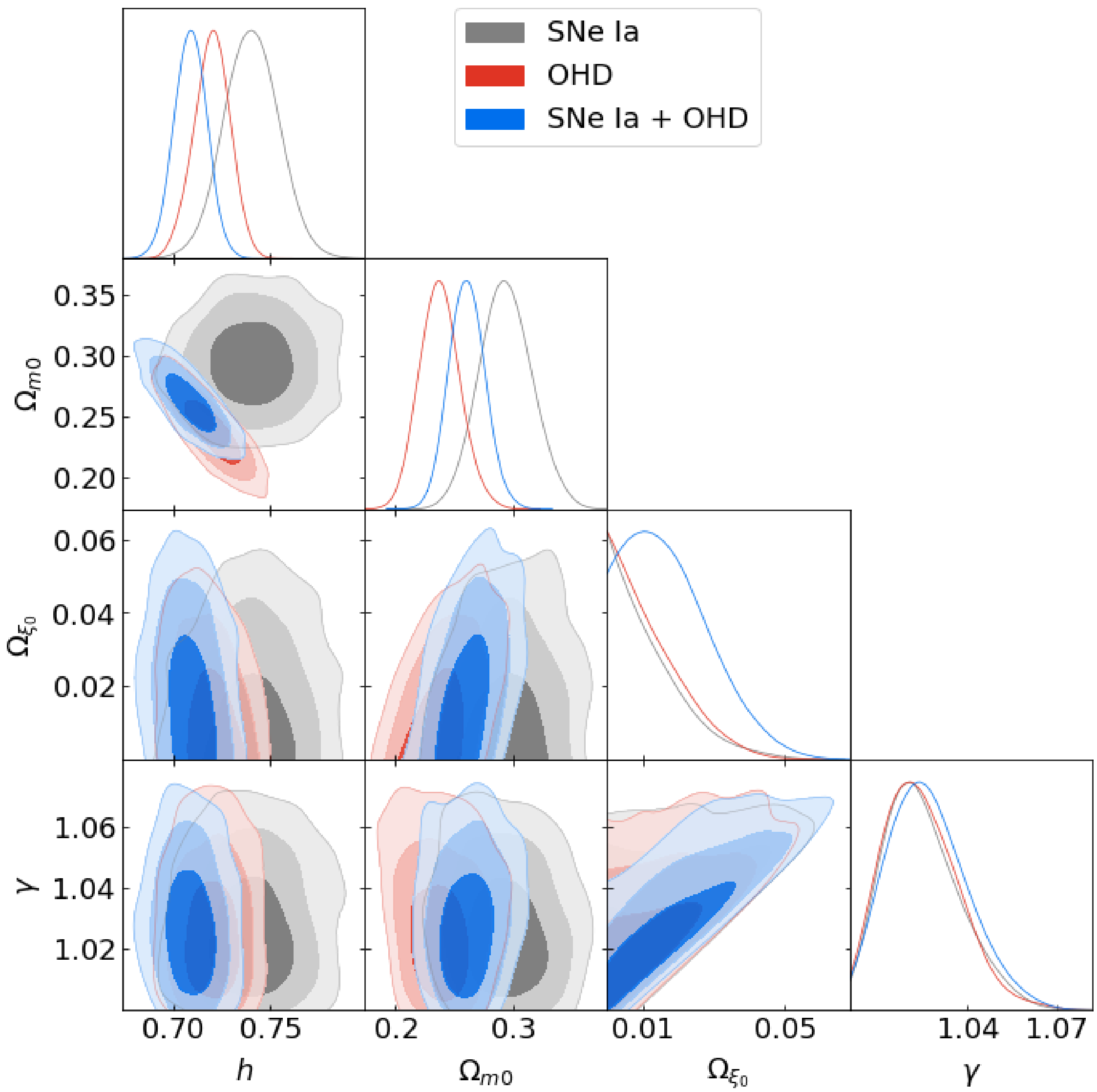

Figure 4.7: Joint and marginalized regions of the free parameters h, Ω_{m0}, Ω_{ξ_0}, and γ for the viscous ΛWDM model, obtained in the MCMC analysis described in the Section 4.3.4. The admissible regions presented in the joint regions correspond to $1\sigma(68.3\%)$, $2\sigma(95.5\%)$, and $3\sigma(99.7\%)$ of confidence level (CL), respectively. The best-fit values for each model free parameter are shown in Table 4.2.

behavior due to the similarity between the two models at late-times, as well to the past, considering the best-fit values obtained for Ω_{ξ_0}. From the point of view of the goodness of fit criteria, we can conclude that the two models are able to describe the SNe Ia, OHD, and SNe Ia+OHD data, with very similar values of χ^2_{min}, again due to the similarity of the behavior of the two models, especially at late times.

A remarkable result is that the viscous ΛWDM model exhibits a slightly lower value of χ^2_{min} for the SNe Ia+OHD data than the ΛCDM model, despite the fact that the ΛWDM model has a greater value of BIC than the ΛCDM model. This translate into a better fit for the ΛWDM model because the two extra free parameters of the viscous ΛWDM model are a consequence of a more general description of the DM component, which takes into account a warm nature and a non-perfect fluid description, suggest by previous investigations that use this alternatives to face tensions of the standard model, and they are not been added by hand in order to force a better fit.

Even more, the ΛCDM model assumes beforehand a CDM (one less free parameter) and all their matter components are describes as perfect fluids (another less free parameter) which leads to a good fit of the combined SNe Ia+OHD data with less free parameters but, the price to pay is the problems mentioned above that the ΛCDM model experience today. Therefore, we have an alternative model which goes beyond to the standard ΛCDM model by considering a dissipative WDM, which has the capability to describe the SNe Ia and OHD data in the same way as the ΛCDM model, together with a more general description of DM nature.

By the other hand, the best fit values contrasted with the combined SNe Ia + OHD data, for γ and Ω_{ξ_0} are given by $1.02555^{+0.04453}_{-0.02424}$ and $0.01633^{+0.04185}_{-0.01627}$ respectively at 3σ CL. Note that, both deviations around the mean value satisfy the near equilibrium condition Eq. (4.24), which means, that these values are compatible with our previous theoretical thermodynamics conclusion ($\Omega_{\xi_0} < 1$ and $\gamma \neq 1$). For the small values of Ω_{ξ_0} and γ given by the data, we are far from near equilibrium condition at $H = H_0 21.61855$ according to Eq. (4.11) , which means that, we are far from near equilibrium at a redshift of $z = 11.63228$ according to Eq. (4.58), by then, we can ensure the near equilibrium condition for the actual data measurement at $z \sim 2.3$.

Also, is important to mention that, with the data measurement we can obtain the actual size of dissipation for our ΛWDM model. Note that, according to our dimensionless expression, $\Omega_{\xi_0} = 3\xi_0 H_0$, ξ_0 has dimension of time, and the bulk viscosity is given by $\xi = \xi_0\rho$, which writing in dimensionless full ($c \neq 1$), is given by $\xi = \xi_0\varepsilon$, being ε the energy density of matter (that has the same dimension of pressure according to the EoS), then ξ would have the viscous units of $Pa \times s$. Follow the cosmological constraint, the maximum value for Ω_{ξ_0} leads to consider $\Omega_{\xi_0} < 0.05818$ (a generalization of our theoretical constraint $\Omega_{\xi_0} < 1$), by then, using the definition of Ω_{ξ_0} we have the follow constraint on bulk viscosity

$$\xi < \varepsilon \times \frac{0.05818}{3H_0}. \tag{4.60}$$

If we introduce the critical density given by $\varepsilon_0 = \frac{3c^2H_0^2}{8\pi G}$, and since our best fit values for the Hubble parameter and matter density are $H_0 = 70.9238 \frac{\text{km/s}}{\text{Mpc}} = (4.3507 \times 10^{17} s)^{-1}$ and $\Omega_m = 0.26073$, respectively, then the restriction over the upper limit of bulk viscosity would be

$$\xi < 1.87082 \times 10^6 Pa \times s. \tag{4.61}$$

For this particular case of values, if we consider the deviation around the mean values the upper limits remains of the order of $\xi \lesssim 10^6 Pa \times s$. Note that from the previous thermodynamic analysis ($\Omega_{\xi_0} < 1$) and using the value of H_0 and Ω_m given by [3, 20] we will find that

$$\xi_0 < 3.628 \times 10^7 Pa \times s, \tag{4.62}$$

which is a similar value with respect to the previous value found by B. D. Normann and I. Brevik in [156], where they consider a model with a viscous DM component with bulk viscosity of the form $\xi = \xi_0\rho^{\frac{1}{2}}$, and DE component given by the CC, they also suggest (see also [157]) that a value $\xi_0 \sim 10^6 Pa \times s$ for the present viscosity is reasonable.Furthermore, other investigations suggest the same orders of magnitude [147, 156–158], but since these models are not identical, some discrepancies are expected with our results.

Chapter 5

Conclusions and final discussions

We have discussed throughout this work the late and early times behavior of the exact solutions of viscousΛCDM models,looking for the first part of the investigation the conditions to have future and past singularities, following the classification given in [83, 84], and we have also found the possibility of solutions describing regular universes. In the late time model we consider a universe filled with dissipative CDM and CC and in the early time model we consider a universe filled with dissipative radiation and CC, taking into consideration two different expression for the dissipation, a constant bulk viscosity and a bulk viscosity proportional to the energy density.

We extend this study for a dissipative fluid model with a negative CC. In the Table 5.3 we summarize the early and late time singalirities obtained in each solutions and in Table 5.1 we summarize the asymptotic early and late behaviour without singularities found for these models.

For a positive CC in the late time behaviour a remarkable result of the solution with $m = 1$ and $\Omega_\xi < 1$ is that the solution behaves at late times like the de Sitter model, regardless of the viscosity value. In this sense this solution is suitable to constraint with the cosmological data, knowing that it evolves very close to theΛCDM model.

For a negative CC in the late time behaviour a remarkable result for the solution with $m = 1$ is that the dissipation in the DM component can drive the accelerated expansion and even a future Big-rip singularity, avoiding the big crunch singularity, that occurs for a flat DM filed universe with negative CC.

It is important to mention that within the context of singularities in phantom DE, the little rip singularity are discussed in the literature [105, 159, 160] under the context of having a universe in which the DE density increases without bound and the universe never reaches a finite-time for singularity. In our work these type of singularities don't appear because we are not considering phantom DE and unlike this we have asymptotic de Sitter-like behaviors with values of $E = 1$ as can be seen the Table (5.1). In the same way, let us note that our Big-rip Type I singularities also occurs with phantom-like behavior with a parameter of state given by Eq. (3.42), but this type of phantom occur in the context of our total fluid composed of dissipative DM and CC. Our results show that it is possible to extend the classification of Big-rip singularity to models where the phantom EoS is effective and not necessarily appears in phantom DE models.

For a positive CC in the early time behavior a remarkable result is that we only have universes without singularities. A special case appears for the solution with $m = 1$ which represents scenarios without singularity as we discussed in section 3.3. For this particular solution, beyond not having singularity, is that it's behaviour is very similar to the standard model for very small values of viscosity, in addition to being different from other singularity-free models [88, 89, 131, 132]. This behaviour is independent of the sign of the CC. In [58] a similar behaviour was obtained without the inclusion of the CC. In our solution a CC is considered and the soft Big-bang is characterized by having a zero scale factor at a very past time, which is different from the obtained in [88].

For a negative CC in the early time behaviour a remarkable result is that the singularities only appears in the case with $m = 0$, and the constraints in the parameters show that the singularity required values of Ω_ξ that depend on the values of the CC. So, despite the fact that in early times its contribution it is very small, its presence is required for the existence of this singularity.

Our results indicate that the inclusion of dissipation in theΛCDM model leads to solutions where the Big Rip singularities appears without a phantom DE and the avoidance of Big Bang singularities is also possible. Therefore, the dissipation mechanism, which is a more realistic description of cosmic fluid, can alleviate the theoretical problems of phantom DE and initial singularities, and also we can obtain solutions whose behaviour is very similar to the standardΛCDM model.

Table 5.1: Classification of the asymptotic behaviour for early and late times

$m = 0$

Solution	**Late-Time**	**Early-Time**	**Condition**
m=0 and $\Delta_0 > 0$	$\dfrac{\frac{\Omega_\xi}{\gamma} + \sqrt{\bar{\Delta}_0}}{2}$		$1 < \gamma < 2, \Omega_{\xi_0} > 0, -\left(\dfrac{\Omega_\xi}{2\gamma}\right)^2 < \Omega_\Lambda < 0$
		$\dfrac{\frac{\Omega_\xi}{\gamma} - \sqrt{\bar{\Delta}_0}}{2}$	$1 < \gamma < 2, \Omega_{\xi_0} > 0, -\left(\dfrac{\Omega_\xi}{2\gamma}\right)^2 < \Omega_\Lambda < 0$
m=0 and $\Delta_0 = 0$	$E_{ds} = 1$	$E_{ds} = 1$	$\Omega_\xi = 2\gamma, \Omega_\Lambda = -1$
	$E_{ds} = \frac{\Omega_\xi}{2}$		$\Omega_\xi < 2, \Omega_\Lambda = -\left(\dfrac{\Omega_\xi}{2}\right)^2$

Table 5.2: Classification of the asymptotic behaviour for early and late times
$m = 1$

Solution	Late-Time	Early-Time	Condition
m=1 and $\Omega_\Lambda > 0$	$E_{ds} = 1$	$E_{ds} = 1$	$\Omega_\xi = 1$
	$E_{ds} = \sqrt{\Omega_\Lambda}$		$\Omega_\xi < 1$
		Soft-Big-Bang $E_{ds} = \frac{4}{3\Omega_\xi}$	$\Omega_\xi < \frac{4}{3}$
m=1 and $\Omega_\Lambda < 0$	$E_{ds} = 1$	$E_{ds} = 1$	$\Omega_\xi = 1$
		Soft-Big-Bang $E_{ds} = \frac{4}{3\Omega_\xi}$	$\Omega_\xi < \frac{4}{3}$

Table 5.3: Classification of the early and late times singularities.

Solution	Late-Time	Early-Time	Condition
m=0 and $\Delta_0 = 0$	**Type 0B (Big-Crunch)**		$\Omega_\xi > 2, \Omega_\Lambda = -\left(\frac{\Omega_\xi}{2}\right)^2.$
		Type 0A (Big-Bang)	$\Omega_\xi < \frac{8}{3}, \Omega_\Lambda = -\left(\frac{3\Omega_\xi}{8}\right)^2.$
m=0 and $\Delta_0 < 0$	**Type 0B (Big-Crunch)**		$\Omega_\Lambda < -\left(\frac{\Omega_\xi}{2}\right)^2,$
		Type 0A (Big-Bang)	$\Omega_\Lambda < -\left(\frac{3\Omega_\xi}{8}\right)^2.$
m=1 and $\Omega_\Lambda > 0$	**Type I (Big-Rip)**		$\Omega_\xi > 1,\ \gamma_{eff} < 0$
m=1 and $\Omega_\Lambda < 0$	**Type 0B (Big-Crunch)**		$\Omega_\xi < 1.$
	Type I (Big-Rip)		$\Omega_\xi > 1, \gamma_{eff} < 0.$

We also have discussed throughout this work the near equilibrium condition, entropy production, and cosmological constraint of a cosmological model filled with a dissipative WDM for the second part of this research, where the bulk viscosity is proportional to the energy density, and a positive CC, which is described by an exact solution previously published in [106].

As an additional comment, it is important to mention that the analysis of this same solution for early times where viscosity is present in the radiation fluid, is available in [161] and is currently under review; this third investigation shows that the universe without initial singularity has a temperature that becomes infinite in an infinite time contrary to the standard ΛCMD model where the temperature grows to infinity in a finite time, additionally, the entropy becomes constant in this viscous universe as long it becomes infinitely hot and is compressed into the infinite past.

Assuming the condition given by Eq. (4.7) this solution behaves very similar to the standard cosmological model for small values of Ω_{ξ_0} as we can see in Fig. 4.1, avoiding the appearance of a future singularity in a finite time (Big-Rip).

We have shown that the presence of the CC together with a small viscosity from the expression (4.11), and considering the condition given by the Eq. (4.7), leads to a near equilibrium regime for the WDM component.

The WDM component has a temperature which decreases very slowly with the cosmic expansion as a result of being close to the near equilibrium condition, contrary to the non physically behavior found for the dust case ($\gamma = 1$), where the temperature increase with the scale factor. Besides, we have shown that the second law of thermodynamics is fulfilled as long we satisfied the conditions (4.7) and (4.23) (near equilibrium condition). On the contrary, the entropy production would diverge in a finite time if a Big-Rip singularity occurs.

To fulfill the two criteria discussed in this thesis, we need to have then $\Omega_{\xi_0} < 1$ and a WDM component with a EoS satisfying the constraint given in Eq.(4.24).

It is important to mention that in our WDM model we need to have $\Omega_{\xi_0} \ll \gamma - 1$.

For small values of Ω_{ξ_0} (in particularly $\Omega_{\xi_0} = 0.001$), the model enter into agreement with some previous results found, for example in [162], where cosmological bounds on the EoS for the DM were found, and the inclusion of the CC is considered. The bounds for a constant EoS are $-1.50 \times 10^{-6} < \omega < 1.13 \times 10^{-6}$ ($\omega = \gamma - 1$) if there is no entropy production and $-8.78 \times 10^{-3} < \omega < 1.86 \times 10^{-3}$ if the adiabatic speed of sound vanishes, both at 3σ of confidence level. Another example can be found in [163] where, using WMAP+BAO+HO observations, the EoS at the present time is given by $\omega = 0.00067^{+0.00011}_{-0.00067}$.

Highlighting again we have to mention that the asymptotic behavior of the exact solution at the infinite future, given by (4.5), corresponds to the usual de Sitter solution, which indicates that this solution describes a dissipative WDM that could reproduce the same asymptotic behavior of the standard model. As long as the exact solution tends to this value, the near equilibrium condition Eq. (4.1) can be satisfied. Of course, the de Sitter solution has a constant temperature, according to Eq. (4.17) and, since in this case we have a null density and null pressure of the fluid, the entropy production is zero, according to Eq. (4.22)

We have shown in this thesis that, previously to any constraining from the cosmological data, the study of thermodynamics consistences required by the Eckart approach, such as the near equilibrium condition and entropy production, leads to important constraints on the cosmological parameter, such as the given one by Eq. (4.24), which implies the necessity of WDM, in agreement with some previous results found in [162–164]. On the other hand, the constrain (4.7) tells us that Big-Rip singularities are avoided at late times, if the near equilibrium condition is preserved. Even though, the exact solution explored behaves very similar to the standard model, and open the possibility of a more realistic fluid description of the DM containing dissipation processes, within the Eckart's framework, giving us physically important clues about the EoS of this component (γ) and the size of dissipation involved (Ω_{ξ_0}).

In this sense, this thermodynamics analysis had the aim of finding a model that satisfies the near equilibrium condition, together with a proper behavior of the temperature and entropy production, helping us to find the best prior definition for the cosmological constraints discussed in Sec.4.3.4 .

The result of the cosmological constraint are shown in Table 4.2, as well as their corresponding goodness of fit criteria and the joint analysis for the SNe Ia and OHD data. The difference in the BIC value of 12.6 between the two models is a reflection of considering a more general theory with two more free parameters (γ and Ω_{ξ_0}), than the standard ΛCDM model (γ fixed to 1 and a perfect DM component), obtaining also a slightly better fit for the combined SNe Ia + OHD data for the ΛWDM model, being in this sense a more realistic model that describes in the same way the SNe Ia and OHD data like the standard cosmological model does.

With the cosmological values Ω_{ξ_0} and γ at 3σ CL obtained with the combined SNe Ia + OHD data, we can conclude that the near equilibrium condition is full satisfies even for a redshift of $z = 11$ according to Eq. (4.58), and then, and proper physical behavior of the temperature (decreasing function with the scale factor), and entropy production (without future Big-Rip singularity), is obtained. Even more, our data analysis suggest that the actual value of bulk viscosity has an upper limit of the order of $10^6 Pa \times s$ in agreement with some previous investigation. It is also important to mention that, the model theoretically (and even from the cosmological data) does not rule out values smaller than this, since the space of values obtained from the theoretical ground at the present time is given by $0 < \Omega_{\xi_0} < 1$, and allows in principle, get smaller values, that can be more acceptable from the viscous hydrodynamic point of view.

Finally, the main contribution of this second study is to show that the inclusion of a CC allows to fulfill the near equilibrium condition of a dissipative WDM component, which is not possible if the accelerated expansion is due only to the negativeness of the bulk pressure, renouncing to a unified DM model but gaining a solution which behaves very close to the standard model, with the same asymptotic behavior, and also describing the combined SNe Ia + OHD data in the same way as the ΛCDM. Besides, the fulfillment of the near equilibrium condition implies a positive entropy production. Therefore, our exact solution describes a physically viable model for a dissipative WDM component, which is supported by many investigations that have extended the possibilities for the DM nature to face tensions of the standard model.

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Appendices

Appendix A

Friedmann equations and study of perfect and non-perfect fluids

A.1 Friedmann equations

From Friedman Robertson-Walker metric Eq. (1.2), we can write the components of $g_{\mu\nu}$ as

$$g_{ij} = a^2 \left(\delta_{ij} + k \frac{x^i x^j}{1 - k\mathbf{x}^2} \right) \;\; g_{i0} = 0 \;\; g_{00} = -1. \tag{A.1}$$

The affine connection is defined as $\Gamma^{\mu}_{\nu\kappa}$

$$\Gamma^{\mu}_{\nu\kappa} = \frac{1}{2} g^{\mu\lambda} \left[\frac{\partial g_{\lambda\nu}}{\partial x^{\kappa}} + \frac{\partial g_{\lambda\kappa}}{\partial x^{\nu}} - \frac{\partial g_{\nu\kappa}}{\partial x^{\lambda}} \right], \tag{A.2}$$

we calculate the non-zero components of the affine connection and we will have

$$\begin{aligned}
\Gamma^0_{ij} &= \frac{1}{2}g^{00}\left[\frac{\partial g_{0i}}{\partial x^j}+\frac{\partial g_{0j}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^0}\right]+\frac{1}{2}g^{0k}\left[\frac{\partial g_{ki}}{\partial x^j}+\frac{\partial g_{kj}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right],\\
&= \frac{1}{2}g^{00}\left[\frac{\partial g_{0i}}{\partial x^j}+\frac{\partial g_{0j}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^0}\right],\\
&= \frac{1\partial g_{ij}}{\partial x^0},\\
&= a\dot{a}\left[\delta_{ij}+\frac{kx^ix^j}{1-k\mathbf{x}^2}\right],\\
&= a\dot{a}\tilde{g}_{ij}.
\end{aligned} \tag{A.3}$$

Where $\tilde{g}_{ij}$ is defined as the purely spatial metric and $\tilde{g}^{ij}$ is the reciprocal of the 3×3 matrix $\tilde{g}_{ij}$ which in general is different from the components ij of the 4×4 reciprocal matrix $g_{\mu\nu}$.

$$\begin{aligned}
\Gamma^i_{0j} &= \frac{1}{2}g^{i0}\left[\frac{\partial g_{0j}}{\partial x^j}+\frac{\partial g_{0j}}{\partial x^0}-\frac{\partial g_{0j}}{\partial x^0}\right]+\frac{1}{2}g^{ik}\left[\frac{\partial g_{0k}}{\partial x^0}+\frac{\partial g_{kj}}{\partial x^0}-\frac{\partial g_{0j}}{\partial x^0}\right],\\
&= \frac{1}{2}g^{ik}\left[\frac{\partial g_{0k}}{\partial x^j}+\frac{\partial g_{kj}}{\partial x^0}-\frac{\partial g_{0j}}{\partial x^k}\right],\\
&= \frac{1}{2}g^{ik}2a\dot{a}\tilde{g}_{ij},\\
&= \frac{1}{2}\left(a^{-2}\tilde{g}^{ij}\right)2a\dot{a}\tilde{g}_{ij},\\
&= \frac{\dot{a}}{a}\delta_{ij}.
\end{aligned} \tag{A.4}$$

$$\Gamma^i_{jl} = \frac{1}{2}\tilde{g}^{im}\left[\frac{\partial\tilde{g}_{jm}}{\partial x^l}+\frac{\partial\tilde{g}_{lm}}{\partial x^j}-\frac{\partial\tilde{g}_{jl}}{\partial x^m}\right] = \Gamma^{\tilde{\mu}}_{\nu\kappa}. \tag{A.5}$$

In this last expression $\Gamma^{\tilde{\mu}}_{\nu\kappa}$ represent the purely spatial component of the connection. Now, from the momentum energy equation for an ideal fluid given by (A.47) we have

$$T^{00} = \rho(t),\;\; T^{0i} = 0,\;\; T^{ij} = \tilde{g}^{ij}(x)a^{-2}(t)p(t). \tag{A.6}$$

We must verify the momentum conservation law $T^{\nu\mu};\mu = 0$, for the case $T^{i\mu};\mu$ is automatically satisfied since the FRW metric is invariant under Lorentz transformations, therefore the term to verify is

$$
\begin{aligned}
0 &= T^{0\mu};\mu = \frac{\partial T^{0\mu}}{\partial x^{\mu}} + \Gamma^{0}_{\mu\nu}T^{\nu\mu} + \Gamma^{\mu}_{\mu\nu}T^{0\nu}, && \text{(A.7)}\\
0 &= \frac{\partial T^{00}}{\partial t} + \Gamma^{0}_{ij}T^{ij} + \Gamma^{i}_{i0}T^{00}, && \text{(A.8)}\\
& && \text{(A.9)}
\end{aligned}
$$

where we will have

$$\boxed{\frac{d\rho}{dt} + \frac{3\dot{a}}{a}(p+\rho) = 0}. \tag{A.10}$$

We consider the Einstein field equations given by

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} - \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}, \tag{A.11}$$

where $R_{\mu\nu}$ is the Ricci tensor given by

$$R_{\mu\nu} = \frac{\partial \Gamma^{\lambda}_{\lambda\nu}}{\partial x^{\nu}} - \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\lambda}} + \Gamma^{\lambda}_{\mu\sigma}\Gamma^{\sigma}_{\nu\lambda} - \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma}. \tag{A.12}$$

The expression (A.11) has ten independent components, considering that each index goes from 0 to 4. But given the freedom of choice, we have of the four coordinates of space-time, the independent equations are reduced in number to six. These equations are the basis of the mathematical formulation of general relativity. Also, R is known as the Ricci curvature scalar, where if we consider the contraction over the two indices of Eq. (A.11) we find that the curvature scalar is related to the trace of the momentum energy tensor and the CC by

$$R - 2R - 4\Lambda = -\frac{8\pi G}{c^4}T, \tag{A.13}$$

from this last relationship, we can isolate R and we can write the expression (A.11) equivalently as

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^{\lambda}_{\lambda}\right). \tag{A.14}$$

Previously we showed that the affine connection with two or three time-like indices are zero, so the no null-components of the Ricci tensor Eq. (A.12) are

$$R_{ij} = \frac{\partial \Gamma^{\kappa}_{\kappa i}}{\partial x^j} - \left[\frac{\partial \Gamma^{\kappa}_{ij}}{\partial x^{\kappa}} + \frac{\partial \Gamma^{0}_{ij}}{\partial t}\right] + \left[\Gamma^{0}_{i\kappa}\Gamma^{\kappa}_{j0} + \Gamma^{\kappa}_{i0}\Gamma^{0}_{j\kappa} + \Gamma^{l}_{i\kappa}\Gamma^{\kappa}_{jl}\right] - \left[\Gamma^{\kappa}_{ij}\Gamma^{l}_{\kappa l} + \Gamma^{0}_{ij}\Gamma^{l}_{0l}\right], \tag{A.15}$$

$$R_{00} = \frac{\partial \Gamma^{i}_{i0}}{\partial t} + \Gamma^{i}_{0j}\Gamma^{j}_{0i}, \tag{A.16}$$

Now we use the non-zero equations of the connection to order (A.3)-(A.5), in order to obtain the components of the Ricci tensor, in this way the elements we need are

- $\frac{\partial \Gamma^{0}_{ij}}{\partial t} = \tilde{g}_{ij}\frac{d}{dt}\left(a\dot{a}\right)$,
- $\Gamma^{0}_{ij}\Gamma^{\kappa}_{j0} = \tilde{g}_{ij}\dot{a}^2$,
- $\Gamma^{0}_{ij}\Gamma^{l}_{0l} = 3\tilde{g}_{ij}\dot{a}^2$
- $\frac{\partial \Gamma^{i}_{i0}}{\partial t} = 3\frac{d}{dt}\left(\frac{\dot{a}}{a}\right)$,
- $\Gamma^{i}_{0j}\Gamma^{j}_{i0} = 3\left(\frac{\dot{a}}{a}\right)$.

With these last results we can rewrite the equations (A.15) and (A.16) as follows

$$\begin{aligned} R_{ij} &= \tilde{R}_{ij} - 2\dot{a}^2\tilde{g}_{ij} - a\ddot{a}\tilde{g}_{ij}, && \text{(A.17)} \\ R_{00} &= 3\frac{d}{dt}\frac{\dot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 = 3\frac{\ddot{a}}{a}, && \text{(A.18)} \end{aligned}$$

where $\tilde{R}_{ij}$ represents the purely spatial part of the Ricci tensor given by

$$\tilde{R}_{ij} = \frac{\partial \Gamma^{\kappa}_{\kappa i}}{\partial x^{j}} - \frac{\partial \Gamma^{\kappa}_{ij}}{\partial x^{\kappa}} + \Gamma^{l}_{i\kappa}\Gamma^{\kappa}_{jl} - \Gamma^{l}_{ij}\Gamma^{\kappa}_{\kappa l}. \tag{A.19}$$

The purely spatial part of the affine connection it can be obtained from (A.5) giving us $\Gamma^{\kappa}_{ij} = kx^{\kappa}\tilde{g}_{ij}$, with this the expression (A.19) will be

$$\tilde{R}_{ij} = \frac{\partial \Gamma^{l}_{li}}{\partial x^{j}} - \frac{\partial \Gamma^{l}_{ji}}{\partial x^{l}} = k\delta_{ij} - 3k\delta_{ij} = -2k\delta_{ij}, \tag{A.20}$$

note that the spatial part of the metric $\tilde{g}_{ij}$ is just δ_{ij} so the above can be written as

$$\tilde{R}_{ij} = -2k\tilde{g}_{ij}. \tag{A.21}$$

And in this way we will finally have for (A.17) the following expression

$$R_{ij} = -\left[2k + 2\dot{a}^{2} + a\ddot{a}\right]], \tag{A.22}$$

then, we only need to develop the right-hand side of Einstein's equation (A.14), to do so we will again use our definitions given by Eq. (A.6) of the form

$$T_{00} = \rho, \;\; T_{i0} = 0 \;\; T_{ij} = a^{2}p\tilde{g}_{ij}, \tag{A.23}$$

with this we calculate the right-hand side of (A.14) and we will get

$$T_{ij} - \frac{1}{2}\tilde{g}_{ij}a^{2}\left(T^{k}_{k} + T^{0}_{0}\right) = a^{2}p\tilde{g}_{ij} - \frac{1}{2}a^{2}\tilde{g}_{ij}\left(3p - \rho\right) = \frac{1}{2}\left(\rho - p\right)a^{2}\tilde{g}_{ij}. \tag{A.24}$$

The purely temporary part will be

$$T_{00} + \frac{1}{2}\left(T^{k}_{k} + T^{0}_{0}\right)\rho + \frac{1}{2}\left(3p - \rho\right) = \frac{1}{2}\left(\rho + 3p\right). \tag{A.25}$$

Finally we will obtain from Einstein field equation, the pure spacial and time solutions given respectively by

$$-\left[2k + 2\dot{a}^{2} + a\ddot{a}\right]] + \Lambda a^{2} = -\frac{4\pi G a^{2}}{c^{4}}\left(\rho - p\right), \tag{A.26}$$

$$3\frac{\ddot{a}}{a} - \Lambda = -\frac{4\pi G}{c^4}\left(3p + \rho\right), \tag{A.27}$$

we rearrange the previous expressions a bit and we will have

$$2k + 2\dot{a}^2 + 2\ddot{a} = \frac{4\pi G a^2}{c^4}\left(\rho - p\right) + \Lambda a^2, \tag{A.28}$$

$$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^4}\left(3p + \rho\right) + \frac{\Lambda}{3}}. \tag{A.29}$$

The equation (A.29) is known as the acceleration equation and the equation (A.28) can be divided by the scale factor a and we get

$$\frac{2k}{a^2} + \frac{2\dot{a}^2}{a^2} + \frac{\ddot{a}}{a} = \frac{4\pi G}{c^4}\left(\rho - p\right) + \Lambda, \tag{A.30}$$

we can now substitute the expression (A.29) into (A.30), which gives us

$$\frac{2k}{a^2} + \frac{2\dot{a}^2}{a^2} - \frac{4\pi G}{3c^4}\left(3p + \rho\right) + \frac{\Lambda}{3} = \frac{4\pi G}{c^4}\left(\rho - p\right) + \Lambda, \tag{A.31}$$

finally we order the previous expression

$$\begin{aligned}
2\frac{\dot{a}^2}{a^2} &= \frac{4\pi G}{c^2}\left(\rho - p + \frac{3p}{3} - \frac{\rho}{3}\right) + \Lambda - \frac{2k}{a^2} - \frac{\Lambda}{3}, \\
\frac{2\dot{a}^2}{a^2} &= \frac{4\pi G}{c^4}\left(\frac{4\rho}{3}\right) + \frac{2\Lambda}{3} - \frac{2k}{a^2}, \\
\frac{\dot{a}^2}{a^2} &= \frac{8\pi G}{3c^4}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}. \qquad \text{(A.32)}
\end{aligned}$$

Where we now define $\dot{a}^2/a^2 = H(t)$ which is known as the Hubble constant, in this way we will finally have

$$\boxed{H^2 = \frac{8\pi G}{3c^4}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}}. \tag{A.33}$$

A.2 Energy momentum tensor for a perfect fluid

First suppose that we are in a frame of reference (distinguished by a tilde) in which the fluid is at rest at some particular position and time. At this space-time point, the perfect fluid hypothesis tells us that the energy-momentum tensor takes the form characteristic of spherical symmetry

$$\tilde{T}^{ij} = p\delta_{ij}, \tag{A.34}$$

$$\tilde{T}^{i0} = T^{0i} = 0, \tag{A.35}$$

$$\tilde{T}^{00} = \rho. \tag{A.36}$$

The coefficients p and ρ are called the pressure and the proper energy density, respectively. If we go into a reference frame at rest in the laboratory, and suppose that the fluid in this frame appears to be moving (at the given space-time point) with velocity $\boldsymbol{v}$. The connection between the comoving coordinates $\tilde{x}^{\beta}$ and the lab coordinates x^{α} is then

$$x^{\alpha} = \Lambda^{\alpha}_{\beta}(\boldsymbol{v})\tilde{x}^{\beta}, \tag{A.37}$$

with $\Lambda(\boldsymbol{v})$ the "boost´´ defined by [111]:

$$\Lambda^{0}_{0} = \gamma, \tag{A.38}$$

$$\Lambda^{i}_{0} = \gamma v_i, \tag{A.39}$$

$$\Lambda^{i}_{j} = \delta_{ij} + \boldsymbol{v}_i\boldsymbol{v}_j\frac{\gamma - 1}{\boldsymbol{v}^2}, \tag{A.40}$$

$$\Lambda^{0}_{j} = \gamma v_j, \tag{A.41}$$

where

$$\gamma = \left(1 - \boldsymbol{v}^2\right)^{-\frac{1}{2}}. \tag{A.42}$$

$T^{\alpha\beta}$ is a tensor, so in the lab frame it is

$$T^{\alpha\beta} = \Lambda^{\alpha}_{\gamma}(\boldsymbol{v})\Lambda^{\beta}_{\delta}(\boldsymbol{v})\tilde{T}^{\gamma\delta}, \tag{A.43}$$

or explicitly

$$
\begin{aligned}
T^{ij} &= p\delta_{ij} + (p+\rho)\frac{v_i v_j}{1-\boldsymbol{v}^2}, && \text{(A.44)}\\
T^{i0} &= (p+\rho)\frac{v_i}{1-\boldsymbol{v}^2}, && \text{(A.45)}\\
T^{00} &= \frac{(\rho+p)\,\boldsymbol{v}^2}{1-\boldsymbol{v}^2}. && \text{(A.46)}
\end{aligned}
$$

To check that this is a tensor, we note that Eqs. (A.44)-(A.46) can be integrated into a single equation

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)\,u^\alpha u^\beta, \qquad \text{(A.47)}$$

where u^α is the velocity four-vector

$$
\begin{aligned}
\boldsymbol{u} &= \frac{d\boldsymbol{x}}{d\tau} = \left(1-v^2\right)^{-\frac{1}{2}}\boldsymbol{v}, && \text{(A.48)}\\
u^0 &= \frac{dt}{d\tau} = \left(1-\boldsymbol{v}^2\right)^{-\frac{1}{2}}, && \text{(A.49)}
\end{aligned}
$$

normalized so that

$$u_\alpha u^\alpha = -1. \qquad \text{(A.50)}$$

Apart from energy and momentum, a fluid will in general carry one or more conserved quantities, such as the charge, the number of baryons minus the number of anti-baryons. Let us consider one such conserved quantity, and refer to it for brevity as the “particle number”. If n is the particle number density in a Lorentz frame that moves with the fluid at a given space-time point, then in this frame the particle current four-vector at this point is

$$\tilde{N}^0 = n \qquad \tilde{N}^i = 0. \qquad \text{(A.51)}$$

In any other Lorentz frame, in which the fluid at this point moves with velocity $\boldsymbol{v}$, the particle current is related to Eq. (A.51) by the “boost” $\Lambda(\boldsymbol{v})$

$$
\begin{aligned}
N^i &= \Lambda^i_\beta(\boldsymbol{v})\tilde{N}^\beta = \Lambda^i_0\tilde{N}^0 = \left(1-\boldsymbol{v}^2\right)^{-\frac{1}{2}} v^i n, && \text{(A.52)}\\
N^0 &= \Lambda^0_\beta(\boldsymbol{v})\tilde{N}^\beta = \Lambda^0_0\tilde{N}^0 = \left(1-\boldsymbol{v}^2\right)^{-\frac{1}{2}} n, && \text{(A.53)}
\end{aligned}
$$

or more directly

$$N^{\alpha} = nu^{\alpha}. \tag{A.54}$$

A.3 Energy momentum tensor for a non-perfect fluids

We suppose that the presence of weak space-time gradients in an non-perfect fluid has the effect of modifying the energy-momentum tensor and particle current vector by terms $\Delta T^{\alpha\beta}$ and ΔN^{α}, which are of first order in these gradients. Instead of (A.47) and (A.54), we then have

$$T^{\alpha\beta} = p\eta^{\alpha\beta} + (p+\rho)\, u^{\alpha}u^{\beta} + \Delta T^{\alpha\beta}, \tag{A.55}$$

$$N^{\alpha} = nu^{\alpha} + \Delta N^{\alpha}. \tag{A.56}$$

Once we allow such correction terms, the general practice is to define ρ and n as the total energy density and particle number density in a comoving frame

$$T^{00} = \rho, \tag{A.57}$$

$$N^{0} = n. \tag{A.58}$$

It is necessary in a relativistic fluid to specify whether u^{α} is the velocity of energy transport or particle transport. In the approach of Landau and Lifshitz [165] u^{α} is taken to be the velocity of energy transport, so that T^{i0} vanishes in a comoving frame. In the approach of Eckart [53], u^{α} is taken to be the velocity of particle transport, so that it is N^{i} that vanishes in a comoving frame. The two approaches are pefectly equivalent, therefore considering the Eckart's approach we have for a comoving frame

$$N^i = 0. \tag{A.59}$$

A comparison of (A.57)-(A.59) with (A.55) and (A.56) shows that in a comoving frame, the dissipative terms $\Delta T^{\alpha\beta}$ and ΔN^{α} are subject to the constraints

$$\Delta T^{00} = \Delta N^0 = \Delta N^i = 0, \tag{A.60}$$

and therefore, in a general Lorentz frame

$$\begin{aligned} u^{\alpha} u^{\beta} \Delta T_{\alpha\beta} &= 0, && \text{(A.61)} \\ \Delta N^{\alpha} &= 0. && \text{(A.62)} \end{aligned}$$

All effects of dissipation thus show up as contributions to $\Delta T^{\alpha\beta}$. Our task is now to construct the most general possible dissipative tensor $\Delta T^{\alpha\beta}$ allowed by Eq. (A.61) and by the second law of thermodynamics.

To this end, let us calculate the entropy produced by the fluid motions, we start by contracting (which means doing a sum over repeated indices) the conservation law of energy and momentum for a perfect fluid Eq. (A.47)

$$0 = \frac{\partial}{\partial x^{\beta}} T^{\alpha\beta}(x) = \frac{\partial p}{\partial x_{\alpha}} + \frac{\partial}{\partial x^{\beta}} \left[(\rho + p)\, u^{\alpha} u^{\beta} \right], \tag{A.63}$$

with u_{α}, and using the relation

$$0 = \frac{\partial}{\partial x^{\beta}} \left(u_{\alpha} u^{\alpha} \right) = 2 u_{\alpha} \frac{\partial u^{\alpha}}{\partial x^{\beta}}, \tag{A.64}$$

we get [1]

$$\begin{aligned} 0 &= u_{\alpha} \frac{\partial T^{\alpha\beta}}{\partial x^{\beta}}, \\ 0 &= u^{\beta} \frac{\partial p}{\partial x^{\beta}} - \frac{\partial}{\partial x^{\beta}} \left[(\rho + p)\, u^{\beta} \right]. && \text{(A.65)} \end{aligned}$$

[1] A term of the form $u_{\alpha} \frac{\partial u^{\alpha}}{\partial x^{\beta}} u^{\beta} + u_{\alpha} u^{\alpha} \frac{\partial u^{\beta}}{\partial x^{\beta}}$ appears, and then, by Eq. (A.64) the first term is zero, and a minus sing appear from the second term according to restriction Eq. (A.50)

From Eq. (A.55) together with Eq. (A.62) we get also the follow conservation equation

$$0=\frac{\partial N^{\alpha}}{\partial x^{\beta}}=\frac{\partial}{\partial x^{\alpha}}\left(nu^{\alpha}\right), \tag{A.66}$$

with this we can express Eq. (A.65) as

$$\begin{aligned} 0 &= u^{\beta}\left[\frac{\partial p}{\partial x^{\beta}}-n\frac{\partial}{\partial x^{\beta}}\left(\frac{p+\rho}{n}\right)\right], && \text{(A.67)}\\ 0 &= -nu^{\beta}\left[p\frac{\partial}{\partial x^{\beta}}\left(\frac{1}{n}\right)+\frac{\partial}{\partial x^{\beta}}\left(\frac{\rho}{n}\right)\right]. && \text{(A.68)}\end{aligned}$$

The Second law of thermodynamics tells us that the pressure p, the energy density ρ, and the volume per particle $1/n$ may be expressed as functions of the temperature T and the entropy per particle $\sigma\kappa$, in such a way that

$$\begin{aligned} \kappa T d\sigma &= pd\left(\frac{1}{n}\right)+d\left(\frac{\rho}{n}\right),\\ \kappa T\frac{\partial\sigma}{\partial x^{\beta}}dx^{\beta} &= p\frac{\partial}{\partial x^{\beta}}\left(\frac{1}{n}\right)dx^{\beta}+\frac{\partial}{\partial x^{\beta}}\left(\frac{\rho}{n}\right)dx^{\beta},\\ \kappa T\frac{\partial\sigma}{\partial x^{\beta}} &= p\frac{\partial}{\partial x^{\beta}}\left(\frac{1}{n}\right)+\frac{\partial}{\partial x^{\beta}}\left(\frac{\rho}{n}\right), && \text{(A.69)}\end{aligned}$$

where κ is the Boltzmann's constant. If we substitute Eq. (A.69) in Eq. (A.67) we get

$$\begin{aligned} 0 &= -\quad nu^{\beta}\left(\kappa T\frac{\partial\sigma}{\partial x^{\beta}}\right),\\ 0 &= -\quad \kappa T\frac{\partial}{\partial x^{\alpha}}\left(n\sigma u^{\alpha}\right). && \text{(A.70)}\end{aligned}$$

where in the last step we use the conservation Eq. (A.66). By therefore, from (A.65) we can write the general result

$$u_{\alpha}\frac{\partial}{\partial x^{\beta}}\left[p\eta^{\alpha\beta}+\left(p+\rho\right)u^{\alpha}u^{\beta}\right]=-\kappa T\frac{\partial}{\partial x^{\alpha}}\left(n\sigma u^{\alpha}\right). \tag{A.71}$$

By therefore, if we consider a first correction term in the energy momentum tensor, in order to satisfy Eq. (A.65) the following expression must be satisfy

$$\frac{\partial}{\partial x^{\alpha}}\left(n\sigma u^{\alpha}\right)=\frac{1}{\kappa T}u_{\alpha}\frac{\partial}{\partial x^{\beta}}\Delta T^{\alpha\beta}, \tag{A.72}$$

or equivalent

$$\frac{\partial S^\alpha}{\partial x^\alpha} = -\frac{1}{T}\frac{\partial u_\alpha}{\partial x^\beta}\Delta T^{\alpha\beta} + \frac{1}{T^2}\frac{\partial T}{\partial x^\beta}u_\alpha \Delta T^{\alpha\beta}, \tag{A.73}$$

where

$$S^\alpha = n\kappa\sigma u^\alpha - T^{-1}u_\beta \Delta T^{\alpha\beta}. \tag{A.74}$$

The entropy density in a comoving frame is $n\kappa\sigma = s^0$, so we may interpret S^α as the entropy current four-vector, and Eq. (A.73) thus gives the rate of entropy production per unit volume. The second law of thermodynamics then requires that $\Delta T^{\alpha\beta}$ be a linear combination of velocity and temperature gradients, such that the right-hand side of (A.73) is positive for all possible fluid configurations. Note that this is only possible because we have included the second term in Eq. (A.74) without this term, $\partial S^\alpha/\partial x^\alpha$ would not be simply quadratic in first derivatives, and hence could not be positive for all fluid configurations.

It is convenient at this point to go over to a comoving frame, in which u^α is given by $u^0 = 1$ and $u^i = 0$ at a given space-time point P. From (A.67), it follows that in this frame, all gradients of u^0 vanish at P. Setting u^i,$\partial u^0/\partial x^\alpha$ and ΔT^{00} equal to zero in Eq. (A.73), we find that in a Lorentz frame comoving at P, the rate of entropy production per unit volume at P is

$$\frac{\partial S^\alpha}{\partial x^\alpha} = -\left(\frac{1}{T}\dot{U}_i + \frac{1}{T}\frac{\partial T}{\partial x^i}\right)\Delta T^{i0} - \frac{1}{T}\frac{\partial u_i}{\partial x^j}\Delta T^{ij}. \tag{A.75}$$

This last expression has to be positive for all possible fluid configurations, then we must define

$$\begin{aligned} \Delta T^{i0} &= -\chi\left(\frac{\partial T}{\partial x^i} + T\dot{U}_i\right), & \text{(A.76)}\\ \Delta T^{ij} &= -\eta\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3}\nabla\cdot\mathbf{u}\delta_{ij}\right) - \xi\nabla\cdot\mathbf{u}\delta_{ij}. & \text{(A.77)}\end{aligned}$$

with positive coefficients

$$\chi \geq 0, \;\; \eta \geq 0, \;\; \xi \geq 0, \tag{A.78}$$

so that (A.75) reads

$$\frac{\partial S^\alpha}{\partial x^\alpha} = \frac{\chi}{T^2} \quad (\nabla T + T\dot{u})^2 \tag{A.79}$$

$$+ \quad \frac{\eta}{2T}\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3}\delta_{ij}\nabla \cdot \mathbf{u}\right)\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} - \frac{2}{3}\delta_{ij}\nabla \mathbf{u}\right) \tag{A.80}$$

$$+ \quad \frac{\xi}{T}(\nabla \mathbf{u})^2 \geq 0. \tag{A.81}$$

Except for the relativistic correction $T\dot{\mathbf{u}}$ in (A.76), the form of (A.76) and (A.77) is the same as in the nonrelativistic theory of non-perfect fluid [165], and we therefore may identify χ,η and ξ as the coefficients of heat conduction, shear viscosity, and bulk viscosity.

It now only remains to translate our results from the forms of the four velocity and Eqs. (A.60), (A.76), (A.77), which are valid only in comoving frames, to forms valid in general Lorentz frames. To do so, let us define a shear tensor

$$W_{\alpha\beta} \equiv \frac{\partial u_\alpha}{\partial x^\beta} + \frac{\partial u_\beta}{\partial x^\alpha} - \frac{2}{3}\eta_{\alpha\beta}\frac{\partial u^\gamma}{x^\gamma}, \tag{A.82}$$

a heat-flow vector

$$Q_\alpha \equiv \frac{\partial T}{\partial x^\alpha} + T\frac{\partial U}{\partial x^\beta}u^\beta, \tag{A.83}$$

and a projection tensor on the hyperplane normal to u^α

$$H_{\alpha\beta} \equiv \eta_{\alpha\beta} + u_\alpha u_\beta. \tag{A.84}$$

It is straightforward to check that in a comoving Lorentz frame, our Eqs. (A.60),(A.76),(A.77) for $\Delta T^{\alpha\beta}$ are satisfied by the tensor

$$\Delta T^{\alpha\beta} = -\eta H^{\alpha\gamma}H^{\beta\delta}W_{\gamma\delta} - \chi\left(H^{\alpha\gamma}u^\beta + H^{\beta\gamma}u^\alpha\right)Q_\gamma - \xi H^{\alpha\beta}\frac{\partial u^\gamma}{\partial x^\gamma}. \tag{A.85}$$

Since this expression is Lorentz-invariant, and valid in a comoving Lorentz frame, it is valid in all Lorentz frames. In general the coefficients χT , η, and ξ might be expected on dimensional grounds to be of the order of the pressure, or the thermal energy density, times some sort of mean free time.

Appendix B

Singularities for a cosmological model with inhomogeneous equation of state

In this part of the appendix, we will present a model discussed in [83], with the aim of complete the discussion of the different types of singularities discussed in chapter 3.2. In [83] it is shown that, dealing with nonlinear equations of state one can see that other kind of singularities appears. The following notation corresponds to the original notation of the authors, where is consider the simple case of a linear EoS for matter content given by $p = \omega\rho$, then, for a non-phantom fluid ($\omega > -1$) one obtains a Big Bang singularity, while for a phantom fluid $\omega < -1$ the singularity is a future Type I (Big Rip). Hence, in order to obtain the other type of singularities the authors consider phantom fluids modeled by non-linear equations of state given by

$$p = -\rho - f(\rho), \tag{B.1}$$

where f is a positive function, and has the the following form $f(\rho) = A\rho^{\alpha}$ with $A > 0$. In this case, from the conservation equation $\dot{\rho} = -3H(\rho + p)$ and the Friedmann equation $H^2 = \kappa\rho/3$, where $\kappa = 8\pi G$, one obtains the following dynamical equation

$$\dot{\rho} = \sqrt{3\kappa} A \rho^{\alpha+\frac{1}{2}}, \tag{B.2}$$

whose solution is

$$\rho = \begin{cases} \left[\frac{\sqrt{3\kappa}A}{2}\left(t-t_0\right)\left(1-2\alpha\right)+\rho_0^{\frac{1}{2}-\alpha}\right]^{\frac{2}{1-2\alpha}} & \text{when } \alpha \neq \frac{1}{2}, \\ \rho_0 e^{\sqrt{3\kappa}A(t-t_0)} & \text{when } \alpha = \frac{1}{2}. \end{cases} \tag{B.3}$$

The authors, integrate the conservation equation , to obtain the scale factor using Eq. (B.3), resulting in

$$a = \begin{cases} a_0 \exp\left[\frac{1}{3A(1-\alpha)}\left(\rho^{1-\alpha}-\rho_0^{1-\alpha}\right)\right] & \text{when } \alpha \neq 1, \\ -a_0\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{3A}} & \text{when } \alpha = 1. \end{cases} \tag{B.4}$$

Then, we have the following different situations (see also [39]):

1. When $\frac{1}{2} < \alpha < 1$, one has future Type I singularities, since in this case ρ, p and a diverge at $t_s = t_0 - \frac{2}{\sqrt{3\kappa}A}\frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha} > t_0$.

2. When $\alpha < 0$ we have a past singularity of Type II, since the energy density vanishes for $t_s = t_0 - \frac{2}{\sqrt{3\kappa}A}\frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha} < t_0$, implying that the pressure diverges at $t = t_s$

3. When $\alpha > 1$, the energy density and the pressure diverge but the scale factor remains finite at $t = t_s$, implying that we have a future Type III singularity.

4. When $0 < \alpha < \frac{1}{2}$ there are two different cases:

   (a) $\frac{1}{1-2\alpha}$ is not a natural number. One has a past Type IV singularity at $t_s = t_0 - \frac{2}{\sqrt{3\kappa}A}\frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha} > t_0$, , since higher derivatives of H diverge at $t = t_s$

   (b) $\frac{1}{1-2\alpha}$ is a natural number. In that case there are not any singularites and the dynamics is defined from $t_s = t_0 - \frac{2}{\sqrt{3\kappa}A}\frac{\rho_0^{\frac{1}{2}-\alpha}}{1-2\alpha} > t_0$, , since higher derivatives of H diverge at $t = t_s$ to $t \to \infty$

Appendix C

Exact solutions

C.1 Differential equation for $H(t)$

In Eckart's theory the field equations in the presence of bulk viscosity are

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3} + \frac{\Lambda}{3}, \tag{C.1}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6}\left(\rho + 3P_{eff}\right) + \frac{\Lambda}{3}, \tag{C.2}$$

where the "point" represents a derivative with respect to cosmic time t, a is the scale factor, H is the Hubble constant, and P_{eff} is an effective pressure given by

$$P_{eff} = p + \Pi, \tag{C.3}$$

here viscous pressure is defined as

$$\Pi = -3H\xi. \tag{C.4}$$

The conservation equation is given by

$$\dot{\rho} + 3H(\rho + p + \Pi) = 0. \tag{C.5}$$

We insert (C.1) into (C.2) and we will get

$$\dot{H} + \frac{\rho}{3} + \frac{\Lambda}{3} = \frac{-1}{6}\left(\rho + 3P_{eff}\right) + \frac{\Lambda}{3}, \tag{C.6}$$

we use (C.3) and we will get

$$\begin{aligned}
\dot{H} + \frac{\rho}{3} &= -\frac{1}{6}\left(\rho + 3\left(p + \Pi\right)\right), \\
\dot{H} + \frac{\rho}{3} &= -\frac{1}{6}\left(\rho + 3\left(p - 3H\xi\right)\right), \\
\dot{H} + \frac{\rho}{3} &= -\frac{\rho}{6} - \frac{1}{2}\left(p - 3H\xi_0\rho^m\right), \\
\dot{H} + \frac{\rho}{3} + \frac{\rho}{6} &= -\frac{1}{2}\left(p - 3H\xi_0\rho^m\right), \\
\dot{H} + \frac{3\rho}{6} &= -\frac{p}{2} + \frac{3H\xi_0\rho^m}{2}, \\
\dot{H} + \frac{\rho}{2} &= -\frac{\left(\gamma - 1\right)\rho}{2} + \frac{3H\xi_0\rho^m}{2}, \\
2\dot{H} &+ \rho + \left(\gamma - 1\right)\rho - \frac{3H\xi_0\rho^m}{2} = 0.
\end{aligned} \tag{C.7}$$

From the equation (C.1) we have

$$\rho = 3H^2 - \Lambda \tag{C.8}$$

we substitute this previous result in (C.7) and we will get

$$\boxed{2\dot{H} + 3\gamma H^2 - 3\xi_0 H(3H^2 - \Lambda)^m - \Lambda\gamma = 0.} \tag{C.9}$$

We write this differential equation in a dimensionless way as follows, we divide (C.9) between $3H_0$ and we will obtain

$$\frac{2\dot{E}}{3} + \gamma E^2 - 3^m\xi_0 H_0^{2m-1} E\left(E^2 - \Omega_{\Lambda_0}\right)^m - \Omega_{\Lambda_0}\gamma = 0, \tag{C.10}$$

where we have adopted the following definitions

$$E = \frac{H(t)}{H_0}, \tag{C.11}$$

$$\Omega_\Lambda = \frac{\Lambda}{3H_0^2}, \tag{C.12}$$

$$T = H_0 t. \tag{C.13}$$

It is important to mention that, the “dot" here represents a derivative with respect to $T = H_o t$. If we define the following expression

$$\Omega_{\xi_0} = 3^m \xi_0 H_0^{2m-1}, \tag{C.14}$$

then, we can write (C.10) as follows

$$\boxed{\frac{2\dot{E}}{3} + \gamma E^2 - \Omega_{\xi_0} E \left(E^2 - \Omega_{\Lambda_0}\right)^m - \Omega_{\Lambda_0}\gamma = 0.} \tag{C.15}$$

C.2 Solution to the viscosity-free model, ΛCMD

Since we are interested in comparing our solutions of the equation (2.7) for different values of m with the standard model ΛCDM, we show below the solution for $H(t)$ and $a(t)$ with the initial conditions $H(t=0) = H_0$ and $a(t=0) = 1$, for the case without dissipation ($\xi = 0$). Therefore, we will obtain by solving the differential equation (2.7)

$$\begin{aligned}
2\dot{H} + 3\gamma H^2 - \Lambda\gamma &= 0, \\
2\dot{H} &= \gamma\Lambda - 3\gamma H^2 \\
\frac{2}{\gamma}\int \frac{dH}{\Lambda - 3H^2} &= t + C \\
\frac{2}{\gamma\Lambda}\int \frac{dH}{\left(1 - \frac{3}{\Lambda}H^2\right)} &= t + C.
\end{aligned} \tag{C.16}$$

To solve the integral (C.16) we make the following change of variable

$$\sqrt{\frac{3}{\Lambda}} H = \tanh\theta, \tag{C.17}$$

with which we have

$$dH = \sqrt{\frac{\Lambda}{3}} \operatorname{sech}^2(\theta)\, d\theta, \tag{C.18}$$

we substitute in Eq. (C.16) and we get

$$\begin{aligned}\frac{2}{\gamma}\sqrt{\frac{1}{3\Lambda}}\int\frac{\operatorname{sech}^2\theta d\theta}{1-\tanh^2\theta} &= t+C,\\ \frac{2}{\gamma}\sqrt{\frac{1}{3\Lambda}}\int d\theta = t+C,\\ \frac{2}{\gamma}\sqrt{\frac{1}{3\Lambda}}\theta = t+C. \end{aligned}\tag{C.19}$$

we use (C.17) in (C.19) and we will get

$$\begin{aligned}\frac{2}{\gamma}\sqrt{\frac{1}{3\Lambda}}\operatorname{arctanh}\left(\sqrt{\frac{3}{\Lambda}}H\right) &= t+C,\\ \operatorname{arctanh}\left(\sqrt{\frac{3}{\Lambda}}H\right) &= \frac{\gamma t\sqrt{3\Lambda}}{2}+\frac{\gamma\sqrt{3\Lambda}C}{2},\end{aligned}\tag{C.20}$$

we can write the above in a dimensionless way and we will obtain

$$\operatorname{arctanh}\left(\frac{E}{\sqrt{\Omega_\Lambda}}\right) = \frac{\gamma T3\sqrt{\Omega_\Lambda}}{2}+\frac{\gamma 3\sqrt{\Omega_\Lambda}C}{2},\tag{C.21}$$

then, we will have from (C.21)

$$E=\sqrt{\Omega_\Lambda}\tanh\left[\frac{3\gamma\sqrt{\Omega_\Lambda}}{2}(T+C)\right].\tag{C.22}$$

We calculate the constant C imposing as boundary condition that in $T=0$ we have $E=1$ and with this we will obtain from (C.22) the follows expression

$$\begin{aligned}1 &= \sqrt{\Omega_\Lambda}\tanh\left[\frac{3\gamma\sqrt{\Omega_\Lambda}C}{2}\right],\\ \frac{2}{3\gamma\sqrt{\Omega_\Lambda}}\operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right) = C,\end{aligned}\tag{C.23}$$

substituting this into (C.22) we will get

$$\boxed{E=\sqrt{\Omega_\Lambda}\tanh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}+\operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right).}\tag{C.24}$$

The next step is to find the scale factor from the previous expression, for this we remember the definition of E and we will have

$$E = \frac{H(t)}{H_0} = \frac{\left(\frac{\dot{a}}{a}\right)}{H_0} = \frac{da}{dt}\left(\frac{1}{a \cdot H_0}\right) = \frac{da}{dt \cdot H_0}\left(\frac{1}{a}\right) = \frac{da}{dT}\frac{1}{a}, \tag{C.25}$$

then by substituting this in (C.24) we will have

$$\int_1^a \frac{da}{a} = \int_0^T \sqrt{\Omega_\Lambda}\tanh\left[\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right] dT, \tag{C.26}$$

to solve the right-hand integral we make the following change of variable (the integral on the left side is trivial)

$$u = \frac{3\gamma\sqrt{\Omega_\Lambda}T}{2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right), \tag{C.27}$$

$$du = \frac{3\gamma\sqrt{\Omega_\Lambda}dT}{2},$$

$$\frac{2du}{3\gamma\sqrt{\Omega_\Lambda}} = dT, \tag{C.28}$$

We substitute this into (C.26) and we get

$$\ln a = \int_0^T \sqrt{\Omega_\Lambda}\tanh(u)\left(\frac{2du}{3\gamma\sqrt{\Omega_\Lambda}}\right),$$

$$\ln a = \frac{2}{3\gamma}\int_0^T \tanh(u)du,$$

$$\ln a = \frac{2}{3\gamma}\left(\ln(\cosh u)|_0^T\right),$$

$$\ln a = \frac{2}{3\gamma}\left(\ln\left(\cosh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)\right) - \ln\left(\cosh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)\right)\right),$$

$$\ln a = \frac{2}{3\gamma}\left(\ln\frac{\left(\cosh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)\right)}{\cosh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)}\right),$$

$$a(T) = \left(\frac{\left(\cosh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2} + \operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)\right)}{\cosh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)}\right)^{\frac{2}{3\gamma}}. \tag{C.29}$$

We can write in a simpler way if we rename

$$
\begin{aligned}
x &= \frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}, && \text{(C.30)}\\
y &= \operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right), && \text{(C.31)}
\end{aligned}
$$

and we will have from Eq. (C.29)

$$
\begin{aligned}
&a(T) = \left(\frac{\cosh\left(x+y\right)}{\cosh y}\right)^{\frac{2}{3\gamma}},\\
&a(T) = \left(\frac{\cosh x\cdot\cosh y+\sinh x\cdot\sinh y}{\cosh y}\right)^{\frac{2}{3\gamma}},\\
&a(T) = \left(\cosh x+\sinh x\cdot\tanh y\right)^{\frac{2}{3\gamma}},\\
&a(T) = \left(\cosh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}\right)+\sinh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}\right)\cdot\tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)\right)^{\frac{2}{3\gamma}},\\
&a(T) = \left(\cosh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}\right)+\sinh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}\right)\cdot\tanh\left(\operatorname{arctanh}\left(\frac{1}{\sqrt{\Omega_\Lambda}}\right)\right)\right)^{\frac{2}{3\gamma}},
\end{aligned}
$$

finally we will have

$$
\boxed{a(T) = \left(\cosh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}\right)+\frac{\sinh\left(\frac{3\gamma\sqrt{\Omega_\Lambda}T}{2}\right)}{\Omega_\Lambda}\right)^{\frac{2}{3\gamma}}}. \tag{C.32}
$$

C.3 Exact solution $m = 0$

From Eq. (2.7), for the case $m = 0$ we will have

$$
2\dot{H}+3\gamma H^2-3\xi_0 H-\Lambda\gamma = 0, \tag{C.33}
$$

we divide by 3γ and we will have

$$
\frac{2\dot{H}}{3\gamma}+H^2-\frac{\xi_0 H}{\gamma}-\frac{\Lambda}{3} = 0. \tag{C.34}
$$

From here, we solve the integral, for three different cases where (2.14) is greater than, equal to, and less than zero.

C.3.1 $\Delta_0 > 0$

From Eq. (C.34) we will have

$$\int \frac{dH}{H^2 - \frac{\xi_0 H}{\gamma} - \frac{\Lambda}{3}} = \frac{-3\gamma t}{2} + C, \tag{C.35}$$

we will complete squares in the denominator

$$\int \frac{dH}{H^2 - \frac{\xi_0 H}{\gamma} - \frac{\Lambda}{3} + \frac{\xi_0^2}{4\gamma^2} - \frac{\xi_0^2}{4\gamma^2}} = \frac{-3\gamma t}{2} + C, \tag{C.36}$$

we regroup terms and we will have

$$\int \frac{dH}{-\frac{1}{4}\left(\frac{4\Lambda}{3} + \left(\frac{\xi_0}{\gamma}\right)^2\right) + \left(H - \frac{\xi_0}{2\gamma}\right)^2} = \frac{-3\gamma t}{2} + C. \tag{C.37}$$

We observe that in the first factor the discriminant expression (2.14) is formed, we can also make the following change of variable $u = H - \xi_0/2\gamma$ and we will have

$$\int \frac{du}{-\frac{\Delta_0}{4} + u^2} = \frac{-3\gamma t}{2} + C, \tag{C.38}$$

now, we factor by $-4\Delta_0$

$$\frac{-1}{4\Delta_0}\int \frac{du}{1 - \frac{4u^2}{\Delta_0}} = \frac{-3\gamma t}{2} + C, \tag{C.39}$$

we do a trigonometric substitution

$$u = \sqrt{\frac{\Delta_0}{4}}\tanh\theta, \tag{C.40}$$

$$du = \sqrt{\frac{\Delta_0}{4}}\operatorname{sech}^2\theta d\theta. \tag{C.41}$$

With this we will have from Eq. (C.39)

$$\begin{aligned}
-\frac{4\sqrt{\Delta_0}}{2\Delta_0}\int \frac{\operatorname{sech}^2 d\theta}{1 - \tanh^2\theta} &= \frac{-3\gamma t}{2} + C,\\
\frac{-2}{\sqrt{\Delta_0}}\int d\theta &= \frac{-3\gamma t}{2} + C,\\
\frac{-2}{\sqrt{\Delta_0}}\theta &= \frac{-3\gamma t}{2} + C,
\end{aligned} \tag{C.42}$$

using (C.40) we will have

$$\frac{-2}{\sqrt{\Delta_0}}\,\mathrm{arctanh}\,\frac{2u}{\sqrt{\Delta_0}} = \frac{-3\gamma t}{2} + C. \tag{C.43}$$

We use our definition of $u = H - \xi_0/2\gamma$ and we will have

$$\begin{aligned}\frac{-2}{\sqrt{\Delta_0}}\,\mathrm{arctanh}\,\frac{2\left(H-\frac{\xi_0}{2\gamma}\right)}{\sqrt{\Delta_0}} &= \frac{-3\gamma t}{2} + C,\\ \frac{2\left(H-\frac{\xi_0}{2\gamma}\right)}{\sqrt{\Delta_0}} &= \tan\left(\frac{3\gamma\sqrt{\Delta_0}t}{4} + C\right),\end{aligned} \tag{C.44}$$

from the previous result, we obtain an expression for H

$$H = \frac{\sqrt{\Delta_0}}{2}\tanh\left(\frac{3\gamma\sqrt{\Delta_0}t}{4} + C\right) + \frac{\xi_0}{2\gamma}, \tag{C.45}$$

we calculate the constant C , considering that in $t = 0$ we have $H = H_0$ and we get

$$\begin{aligned}H_0 &= \frac{\sqrt{\Delta_0}}{2}\tanh\left(+C\right) + \frac{\xi_0}{2\gamma},\\ C &= \mathrm{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right),\end{aligned} \tag{C.46}$$

our exact solution will finally be

$$\boxed{H = \frac{\sqrt{\Delta_0}}{2}\tanh\left(\frac{3\gamma\sqrt{\Delta_0}t}{4} + \mathrm{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right)\right) + \frac{\xi_0}{2\gamma}.} \tag{C.47}$$

We will obtain the scale factor remembering that $H = \dot{a}/a$ and we will have

$$\int\frac{da}{a} = \int\left(\frac{\sqrt{\Delta_0}}{2}\tanh\left(\frac{3\gamma\sqrt{\Delta_0}t}{4} + \mathrm{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right)\right) + \frac{\xi_0}{2\gamma}\right)dt, \tag{C.48}$$

to solve the right-hand integral we make the following change of variable (the integral on the left side is trivial)

$$u = \frac{3\gamma\sqrt{\Delta_0}t}{4} + \mathrm{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right), \tag{C.49}$$

$$du = \frac{3\gamma\sqrt{\Delta_0}t}{4}dt. \tag{C.50}$$

And so we will have from Eq. (C.48)

$$
\begin{aligned}
\ln a &= \frac{\sqrt{\Delta_0}}{2}\left(\frac{4}{3\gamma\sqrt{\Delta_0}}\right)\int \tanh u du + \frac{\xi_0 t}{2\gamma} + C, \\
\ln a &= \frac{\sqrt{\Delta_0}}{2}\left(\frac{4}{3\gamma\sqrt{\Delta_0}}\right)\int \tanh u du + \frac{\xi_0 t}{2\gamma} + C, \\
\ln a &= \frac{2}{3\gamma}\ln\cosh u + \frac{\xi_0 t}{2\gamma} + C, \qquad \text{(C.51)}
\end{aligned}
$$

using (C.49) and canceling the logarithms we will have from the previous expression

$$
a = \cosh\left(\frac{3\gamma\sqrt{\Delta_0}t}{4} + \operatorname{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right)\right)^{\frac{2}{3\gamma}} \exp\left(\frac{\xi_0 t}{2\gamma}\right) C. \qquad \text{(C.52)}
$$

We find the constant C considering that in $t = 0$ we will have $a = 1$ and with this, the result will be

$$
\begin{aligned}
1 &= \cosh\left(\operatorname{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right)\right)^{\frac{2}{3\gamma}} C, \qquad \text{(C.53)} \\
C &= \cosh\left(\operatorname{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right)\right)^{-\frac{2}{3\gamma}}. \qquad \text{(C.54)}
\end{aligned}
$$

Therefore the scale factor for this solution, will be from Eq. (C.51)

$$
\boxed{a = \exp\left(\frac{\xi_0 t}{2\gamma}\right)\left\{\frac{\cosh\left(\frac{3\gamma\sqrt{\Delta_0}t}{4} + \operatorname{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right)\right)}{\cosh\left(\operatorname{arctanh}\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{\Delta_0}}\right)\right)}\right\}^{\frac{2}{3\gamma}}}. \qquad \text{(C.55)}
$$

C.3.2 $\Delta_0 = 0$

The case in which the discriminant is equal to zero implies a very particular value for the CC, from (2.14) we will have

$$
\frac{\Lambda}{3} = -\frac{\xi^2}{4\gamma^2}. \qquad \text{(C.56)}
$$

From (C.35), we substitute this last expression and we will have

$$\begin{aligned}
\int \frac{dH}{H^2 - \frac{\xi_0 H}{\gamma} + \frac{\xi_0^2}{4\gamma}} &= \frac{-3\gamma t}{2} + C, \\
\int \frac{dH}{H^2 - \frac{\xi_0 H}{\gamma} + \frac{\xi_0^2}{4\gamma^2}} &= \frac{-3\gamma t}{2} + C, \\
\int \frac{dH}{\left(H - \frac{\xi_0}{2\gamma}\right)^2} &= \frac{-3\gamma t}{2} + C, \qquad \text{(C.57)}
\end{aligned}$$

we make the change of variable $u = H - \xi_0/2\gamma$ and we will have from Eq. (C.57)

$$\begin{aligned}
\frac{1}{\left(H - \frac{\xi_0}{2\gamma}\right)} &= \frac{3\gamma t}{2} + C, \\
\left(H - \frac{\xi_0}{2\gamma}\right) &= \frac{1}{\frac{3\gamma t}{2} + C}, \qquad \text{(C.58)}
\end{aligned}$$

we calculate the constant C considering that at $t = 0$, we have $H(0) = H_0$

$$C = \frac{1}{\left(H_0 - \frac{\xi_0}{2\gamma}\right)}. \qquad \text{(C.59)}$$

we substitute in (C.58) and we will obtain

$$\begin{aligned}
H &= \frac{1}{-\frac{3\gamma t}{2\gamma} + \left(\frac{1}{\left(H_0 - \frac{\xi_0}{2\gamma}\right)}\right)} + \frac{\xi_0}{2\gamma}, \\
H &= \frac{H_0 - \frac{\xi_0}{2\gamma} + \frac{\xi_0}{2\gamma}\left(\frac{3\gamma t}{2}\left(H_0 - \frac{\xi_0}{2\gamma}\right) + 1\right)}{\frac{3\gamma t}{2}\left(H_0 - \frac{\xi_0}{2\gamma}\right) + 1}, \\
H &= \frac{H_0 + \frac{3t\xi_0}{4}\left(H_0 - \frac{\xi_0}{2\gamma}\right)}{\frac{3\gamma t}{2}\left(H_0 - \frac{\xi_0}{2\gamma}\right) + 1}, \qquad \text{(C.60)}
\end{aligned}$$

then, we will have the following solution for the Hubble parameter

$$\boxed{H = \frac{4H_0 + 3\xi_0\left(H_0 - \frac{\xi_0}{2\gamma}\right)t}{6\gamma t\left(H_0 - \frac{\xi_0}{2\gamma}\right) + 4}}. \qquad \text{(C.61)}$$

From Eq. (C.61) we calculate the scale factor

$$\begin{aligned}
\frac{\dot{a}}{a} &= \frac{4H_0 + 3\xi_0\left(H_0 - \frac{\xi_0}{2\gamma}\right)t}{6\gamma t\left(H_0 - \frac{\xi_0}{2\gamma}\right) + 4}, \\
\int \frac{da}{a} &= \int \left(\frac{4H_0 + 3\xi_0\left(H_0 - \frac{\xi_0}{2\gamma}\right)t}{6\gamma t\left(H_0 - \frac{\xi_0}{2\gamma}\right) + 4}\right) dt.
\end{aligned} \tag{C.62}$$

We do partial fractions on the right side and we will have

$$\ln a = \int \left(\frac{\xi_0}{2\gamma} + \frac{4\left(H - \frac{\xi_0}{2\gamma}\right)}{4 + 6\gamma\left(H - \frac{\xi_0}{2\gamma}\right)t}\right) dt, \tag{C.63}$$

the first integral on the right hand side is trivial and for the second factor we make the following change of variable

$$u = 4 + 6\gamma\left(H - \frac{\xi_0}{2\gamma}\right)t, \tag{C.64}$$

$$\frac{du}{6\gamma\left(H - \frac{\xi_0}{2\gamma}\right)} = dt, \tag{C.65}$$

with this we will have from Eq. (C.63)

$$\begin{aligned}
\ln a &= \frac{\xi_0}{2\gamma}t + \frac{2}{3\gamma}\int \frac{du}{u} + C, \\
\ln a &= \frac{\xi_0}{2\gamma}t + \frac{2}{3\gamma}\ln\left(4 + 6\gamma\left(H - \frac{\xi_0}{2\gamma}\right)t\right) + C.
\end{aligned} \tag{C.66}$$

For the constant C we consider that in $t = 0$ we will have $a = 1$ and in this way we will have $C = -\ln 4$, then the previous results becomes in

$$\ln a = \frac{\xi_0 t}{2\gamma} + \ln\left(1 + \frac{3\gamma}{2}\left(H_0 - \frac{\xi_0}{2\gamma}\right)\right). \tag{C.67}$$

Finally we cancel the logarithms and we will obtain

$$\boxed{a = \exp\left[\frac{\xi_0 t}{2\gamma}\right]\left[\frac{3\gamma t}{2}\left(H - \frac{\xi_0}{2\gamma}\right) + 1\right]^{\frac{2}{3\gamma}}.} \tag{C.68}$$

C.3.3 $\Delta_0 < 0$

It is possible to develop the integral (C.35) again, but now considering that the discriminant given by the expression (2.14) is negative, however it is easier if from Eq. (C.47) we make the following change $\Delta_0 \rightarrow -\Delta_0$ and we will get

$$\boxed{H = \frac{-\sqrt{|\Delta_0|}}{2}\tan\left(\frac{3\gamma\sqrt{|\Delta_0|}t}{4} - \arctan\left(\frac{2H_0 - \frac{\xi_0}{\gamma}}{\sqrt{|\Delta_0|}}\right)\right) + \frac{\xi_0}{2\gamma}}, \tag{C.69}$$

From Eq. (C.55), we will have a scale factor given by

$$\boxed{a(T) = \exp\left(\frac{\Omega_\xi}{2\gamma}T\right)\left\{\frac{\cos\left[\frac{3\gamma\sqrt{|\bar{\Delta}_0|}}{4}T - \arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right]}{\cos\left[\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right]}\right\}^{\frac{2}{3\gamma}}}, \tag{C.70}$$

C.4 Exact solution $m = 1$ with $\Lambda > 0$

From the equation (2.7) we will have for the case $m = 1$

$$\begin{aligned}
2\dot{H} + 3\gamma H^2 - 3\xi_0 H\left(3H^2 - \Lambda\right) - \Lambda\gamma &= 0, \\
2\dot{H} + \gamma\left(3H^2 - \Lambda\right) - 3\xi_0 H\left(3H^2 - \Lambda\right) &= 0, \\
2\dot{H} + \left(3H^2 - \Lambda\right)\left(\gamma - 3\xi_0 H\right) &= 0, \\
2\dot{H} + 3\gamma\left(H^2 - \frac{\Lambda}{3}\right)\left(1 - 3\frac{\xi_0}{\gamma}H\right) &= 0.
\end{aligned} \tag{C.71}$$

From this last result, we obtain the following integral to be solved

$$\int \frac{dH}{\left(H^2 - \frac{\Lambda}{3}\right)\left(1 - 3\frac{\xi_0}{\gamma}H\right)} = -\frac{3t\gamma}{2} + C, \tag{C.72}$$

We focus on the left-hand side and rewrite using partial fractions we get

$$\frac{1}{(\gamma^2-3\Lambda\xi^2)}\left[\int\frac{9\xi^2\gamma dH}{(\gamma-3H\xi_0)}+\int\frac{3\gamma^2 dH}{3H^2-\Lambda}+\int\frac{9H\gamma\xi dH}{(3H^2-\Lambda)}\right],$$
$$\frac{1}{\left(\gamma^2-3\Lambda\left(\xi\right)^2\right)}\left[\int\frac{9\xi^2\gamma dH}{(\gamma-3H\xi_0)}-\frac{3\gamma^2}{\Lambda}\int\frac{dH}{1-\frac{3H^2}{\Lambda}}+\int\frac{9H\xi dH}{(3H^2-\Lambda)}\right]. \tag{C.73}$$

We have 3 integrals to solve so we will make some changes of variables described below

first integral	**second integral**	**third integral**
$u = \gamma - 3H\xi_0$ $-\frac{du}{3\xi_0} = dH$	$H = \sqrt{\frac{\Lambda}{3}}\tanh\theta$ $\sqrt{\frac{3}{\Lambda}}dH = \operatorname{sech}^2\theta d\theta$	$v = 3H^2-\Lambda$ $\frac{dv}{6} = HdH$

Table C.1: Changes of variables for the solution (C.73)

We substitute these changes of variables in (C.73) and we will obtain

$$
\begin{aligned}
-\frac{3\gamma t}{2}+C &= \frac{1}{(\gamma^2-3\Lambda\xi^2)}\left[-\frac{9\xi_0^2\gamma}{3\xi_0}\int\frac{du}{u}-\frac{3\gamma^2}{\Lambda}\int\frac{\sqrt{\frac{\Lambda}{3}}\,\mathrm{sech}^2\,\theta d\theta}{\left(1-\tanh^2\theta\right)}+\frac{9\xi\gamma_0}{6}\int\frac{dv}{v}\right],\\
-\frac{3\gamma t}{2}+C &= \frac{1}{(\gamma^2-3\Lambda\xi^2)}\left[-3\xi_0\gamma\ln(u)-\frac{\gamma^2}{\sqrt{\frac{\Lambda}{3}}}\int\frac{\mathrm{sech}^2\,\theta d\theta}{\mathrm{sech}^2\,\theta}+\frac{3\xi_0\gamma}{2}ln(v)\right]\\
-\frac{3\gamma t}{2}+C &= \frac{1}{(\gamma^2-3\Lambda\xi^2)}\left[-3\xi_0\gamma\ln\left(\gamma-3H\xi_0\right)-\frac{\gamma^2}{\sqrt{\frac{\Lambda}{3}}}\int\theta d\theta+\frac{3\xi_0\gamma}{2}ln(3H^2-\Lambda)\right],\\
-\frac{3\gamma t}{2}+C &= \frac{1}{(\gamma^2-3\Lambda\xi^2)}\left[-3\xi_0\gamma\ln\left(\gamma-3H\xi_0\right)-\frac{\gamma^2}{\sqrt{\frac{\Lambda}{3}}}\,\mathrm{arctanh}\left(\frac{H}{\frac{\Lambda}{3}}\right)+\frac{3\xi_0\gamma}{2}\ln(3H^2-\Lambda)\right],\\
t+C &= \frac{1}{3\left(\gamma^2-3\Lambda\xi^2\right)}\left[6\xi_0\ln\left(\gamma-3H\xi_0\right)+2\frac{\gamma}{\sqrt{\frac{\Lambda}{3}}}\,\mathrm{arctanh}\left(\frac{H}{\frac{\Lambda}{3}}\right)-3\xi_0\ln(3H^2-\Lambda)\right].
\end{aligned}
\tag{C.74}
$$

We will write this expression in a dimensionless way, we will use the definition $\Omega_{\xi_0}=3\xi_0 H_0$, which corresponds to the case $m=1$ from (C.14). The constant C can be fixed considering that in $t=0$ we have $H(0)=H_0$ and we will obtain

$$
t=\frac{2\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\ln\left(\frac{\gamma-\Omega_\xi E}{\gamma-\Omega_\xi}\right)-\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\ln\left(\frac{E^2-\Omega_\Lambda}{1-\Omega_{\Lambda_0}}\right)}{3H_0\sqrt{\Omega_{\Lambda_0}}\left(\gamma^2-\Omega_\xi^2\Omega_{\Lambda_0}\right)}+\frac{2\gamma\,\mathrm{arctanh}(\frac{E}{\sqrt{\Omega_{\Lambda_0}}})}{3H_0\sqrt{\Omega_{\Lambda_0}}\left(\gamma^2-\Omega_\xi^2\Omega_{\Lambda_0}\right)}-\frac{2\gamma\,\mathrm{arctanh}(\frac{1}{\sqrt{\Omega_{\Lambda_0}}})}{3H_0\sqrt{\Omega_{\Lambda_0}}\left(\gamma^2-\Omega_{\xi_0}^2\Omega_{\Lambda_0}\right)}.
\tag{C.75}
$$

we rewrite using logarithms and use that $T=H_0 t$ and we will have

$$
\boxed{T=\frac{\Omega_{\xi_0}\sqrt{\Omega_{\Lambda_0}}\log\left(\frac{(1-\Omega_{\Lambda_0})(\gamma-E\Omega_{\xi_0})^2}{\left(E^2-\Omega_{\Lambda_0}\right)(\gamma-\Omega_{\xi_0})^2}\right)}{3\sqrt{\Omega_{\Lambda_0}}\left(\gamma^2-\Omega_\xi^2\Omega_{\Lambda_0}\right)}+\frac{\gamma\log\left(\frac{\left(\sqrt{\Omega_{\Lambda_0}}-1\right)\left(\sqrt{\Omega_{\Lambda_0}}+E\right)}{\left(\sqrt{\Omega_{\Lambda_0}}+1\right)\left(\sqrt{\Omega_{\Lambda_0}}-E\right)}\right)}{3\sqrt{\Omega_{\Lambda_0}}\left(\gamma^2-\Omega_\xi^2\Omega_{\Lambda_0}\right)}.}
\tag{C.76}
$$

C.5 Exact solution $m = 1$ with $\Lambda < 0$

From the expression (C.76) we make the change $\Omega_{\Lambda_0} \to -\left|\Omega_{\Lambda_0}\right|$, therefore we will have

$$T = \frac{i\Omega_{\xi_0}\sqrt{\left|\Omega_{\Lambda_0}\right|}\log\left(\frac{(1+\left|\Omega_{\Lambda_0}\right|)(\gamma-E\Omega_{\xi_0})^2}{\left(E^2+\left|\Omega_{\Lambda_0}\right|\right)(\gamma-\Omega_{\xi_0})^2}\right)}{3i\sqrt{\left|\Omega_{\Lambda_0}\right|}\left(\gamma^2+\Omega_\xi^2\left|\Omega_{\Lambda_0}\right|\right)} + \frac{\gamma\log\left(\frac{\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}-1\right)\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}+E\right)}{\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}+1\right)\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}-E\right)}\right)}{3i\sqrt{\left|\Omega_{\Lambda_0}\right|}\left(\gamma^2+\Omega_\xi^2\left|\Omega_{\Lambda_0}\right|\right)}, \tag{C.77}$$

we simplify the i and we will have

$$\boxed{T = \frac{\Omega_{\xi_0}\sqrt{\left|\Omega_{\Lambda_0}\right|}\log\left(\frac{(1+\left|\Omega_{\Lambda_0}\right|)(\gamma-E\Omega_{\xi_0})^2}{\left(E^2+\left|\Omega_{\Lambda_0}\right|\right)(\gamma-\Omega_{\xi_0})^2}\right)}{3\sqrt{\left|\Omega_{\Lambda_0}\right|}\left(\gamma^2+\Omega_\xi^2\left|\Omega_{\Lambda_0}\right|\right)} + \frac{i\gamma\log\left(\frac{\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}+1\right)\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}-E\right)}{\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}-1\right)\left(i\sqrt{\left|\Omega_{\Lambda_0}\right|}+E\right)}\right)}{3\sqrt{\left|\Omega_{\Lambda_0}\right|}\left(\gamma^2+\Omega_\xi^2\left|\Omega_{\Lambda_0}\right|\right)}}. \tag{C.78}$$

Appendix D

Solution for γ_{eff}

Here we will study the expression for γ_{eff}. To simplify the study we will make use of the following change of variable $\rho' = \rho + \Lambda$ and $p' = p - \Lambda$ in Eqs. (2.2)-(1.5) and we have

$$\begin{aligned} H^2 &= \frac{\rho'}{3}, && \text{(D.1)} \\ \dot{H} + H^2 &= -\frac{(\rho' + 3p')}{6}. && \text{(D.2)} \end{aligned}$$

The effect of viscosity is introduced from the equation (2.4) and for a barotropic fluid it is obtained

$$\gamma_{eff} = \gamma + \frac{\Pi}{\rho'}, \tag{D.3}$$

using (D.1) we get from the previous expression

$$\gamma_{eff} = \gamma + \frac{\Pi}{3H^2}, \tag{D.4}$$

We use Eq. (D.2) to find an expression for Π and we will get

$$\begin{aligned} \dot{H} + H^2 &= -\frac{(\rho' + 3p')}{6}, \\ \dot{H} + H^2 &= -\frac{1}{6}\left(3H^2 + 3P + 3\Pi\right), && \text{(D.5)} \end{aligned}$$

where in (D.5) we have used the equations (C.3)-(D.1), then we will have

$$\begin{aligned}
\dot{H} + H^2 &= -\frac{1}{2}\left(H^2 + p' + \Pi\right), \\
\dot{H} + H^2 &= -\frac{1}{2}\left(H^2 + (\gamma - 1)\,\rho' + \Pi\right), && \text{(D.6)} \\
\dot{H} + H^2 &= -\frac{1}{2}\left(H^2 + (\gamma - 1)\,3H^2 + \Pi\right), && \text{(D.7)} \\
2\dot{H} + 2H^2 &= -H^2 - 3\omega H^2 - \Pi - 2\dot{H}. && \text{(D.8)}
\end{aligned}$$

Where in (D.6) we have used the Eos, and in (D.7) we have used Eq. (D.1). With all this we finally have an expression for Π from Eq. (D.8)

$$\begin{aligned}
\Pi &= -3H^2 - 3(\gamma - 1)\,H^2 - \Pi - 2\dot{H}, \\
\Pi &= -3\gamma H^2 - 2\dot{H}, && \text{(D.9)}
\end{aligned}$$

We substitute (D.9) in Eq. (D.4) and we will get

$$\begin{aligned}
\gamma_{eff} &= \gamma + \frac{\Pi}{3H^2}, \\
\gamma_{eff} &= \gamma + \frac{-3\gamma H^2 - 2\dot{H}}{3H^2}, \\
\gamma_{eff} &= \gamma - \gamma - \frac{2\dot{H}}{3H^2}, \\
\gamma_{eff} &= -\frac{2\dot{H}}{3H^2}. && \text{(D.10)}
\end{aligned}$$

In a dimensionless way we get

$$\gamma_{eff} = \frac{-2}{3}\left(\frac{\dot{E}}{E^2}\right). \tag{D.11}$$

D.1 γ_{eff} for $m = 0$, and $\Delta_0 = 0$

Using Eq. (C.61) we have

$$\frac{dE}{dT} = \frac{3\Omega_\xi\left(1 - \frac{\Omega_\xi}{2\gamma}\right)}{4 + 6\gamma\left(1 - \frac{\Omega_\xi}{2\gamma}\right)T} - \frac{\left(4 + 3\Omega_\xi\left(1 - \frac{\Omega_\xi}{2\gamma}\right)T\right)\left(6\gamma\left(1 - \frac{\Omega_\xi}{2\gamma}\right)\right)}{\left(4 + 6\gamma\left(1 - \frac{\Omega_\xi}{2\gamma}\right)T\right)^2}, \tag{D.12}$$

$$E^2 = \frac{\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)^2}{\left(4 + 6\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)^2}. \tag{D.13}$$

Then, the result for γ_{eff} Eq. (D.11) will be

$$\begin{aligned}
\gamma_{eff} &= -\frac{2}{3}\left(\frac{\left(4 + 6\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right) 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right)}{\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)^2} - \frac{\left(6\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right)\right)}{\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)}\right), \\
&= -\frac{2}{3\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)}\left(\frac{\left(4 + 6\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right) 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right)}{\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)} - \left(6\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right)\right)\right), \\
&= -\frac{2}{3\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)^2}\Bigg(\left(4 + 6\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right) 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) - \\
&\left(6\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right)\right)\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)\Bigg), \\
&= -\frac{2}{3\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)^2}\Bigg(12\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) + 18\gamma\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right)^2 T - 24\gamma \left(1 - \frac{\Omega_\xi}{2\gamma}\right) - \\
&18\gamma\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right)^2 T\Bigg), \\
&= -\frac{2}{3}\left(\frac{-24\gamma \left(\frac{\Omega_\xi}{2\gamma} - 1\right)^2}{\left(4 + 3\Omega_\xi \left(1 - \frac{\Omega_\xi}{2\gamma}\right) T\right)^2}\right).
\end{aligned} \tag{D.14}$$

Finally we will have

$$\gamma_{eff} = \frac{16\gamma \left(\frac{\Omega_\xi}{2\gamma} - 1\right)^2}{\left(4 - 3\Omega_\xi T \left(\frac{\Omega_\xi}{2\gamma} - 1\right)\right)^2}. \tag{D.15}$$

D.2 γ_{eff} for $m=0$, and $\Delta_0<0$

Using Eq. (C.69) we have

$$\frac{dE}{dT}=-\frac{3\gamma\left|\Delta_0\right|}{8}\sec\left(\frac{3}{4}\gamma\left|\Delta_0\right|t-\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\left|\sqrt{\bar{\Delta}_0}\right|}\right)\right)^2, \tag{D.16}$$

$$E^2=\left(\frac{-\sqrt{|\bar{\Delta}_0|}}{2}\tan\left(\frac{3\gamma\sqrt{|\bar{\Delta}_0|}t}{4}-\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right)+\frac{\Omega_\xi}{2\gamma}\right)^2. \tag{D.17}$$

Then, the result for Eq. (D.11) is direct and we will have

$$\gamma_{eff}=-\frac{2\dot{E}}{3E^2}=\frac{\gamma\left|\bar{\Delta}_0\right|\sec\left(\arctan\left(\frac{3}{4}\gamma\sqrt{|\bar{\Delta}_0|}T-\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{|\bar{\Delta}_0|}}\right)\right)^2}{4\left(\frac{\Omega_\xi}{2\gamma}-\frac{\sqrt{|\bar{\Delta}_0|}}{2}\tan\left(\frac{3}{4}\gamma\sqrt{|\bar{\Delta}_0|}T-\arctan\left(\frac{2-\frac{\Omega_\xi}{\gamma}}{\sqrt{\Delta_0}}\right)\right)\right)^2}. \tag{D.18}$$

D.3 γ_{eff} for $m=1$

From expression Eq. (C.15) given by

$$\frac{2\dot{E}}{3}+\gamma E^2-\Omega_\xi\left(3H^2-\Lambda\right)^m-\Omega_{\Lambda_0}\gamma=0,$$

we isolate $\dot{E}$ and substitute in (D.11) and we will obtain

$$\gamma_{eff}=\frac{\gamma E^2-\Omega_\xi E\left(E^2-\Omega_\Lambda\right)^m-\Omega_\Lambda\gamma}{E^2}, \tag{D.19}$$

for $m=1$ and positive CC

$$\gamma_{eff}=\frac{\gamma E^2-\Omega_\xi E\left(E^2-\Omega_\Lambda\right)^1-\Omega_\Lambda\gamma}{E^2},$$

$$\gamma_{eff}=\gamma-\Omega_\xi E+\frac{\Omega_\xi\Omega_\Lambda\gamma}{E}-\frac{\Omega_\Lambda\gamma}{E^2}. \tag{D.20}$$

For $m = 1$ and negative CC

$$\gamma_{eff} = \gamma - \Omega_\xi - \frac{\Omega_\xi E^3 \left|\Omega_\Lambda\right| \gamma}{E^2} + \frac{\left|\Omega_\Lambda \gamma\right|}{E^2}, \tag{D.21}$$

in a Big-Crunch singularity $E \to -\infty$ [1] which leaves us for both expressions Eq. (D.20) and Eq. (D.21) $\gamma_{eff} \to \infty$

[1]For $m = 0$ we get from Eq. (D.19) $\gamma_{eff} = \gamma - \frac{\Omega_\xi}{E} - \frac{\Omega_\Lambda \gamma}{E^2}$ and in a Big-Crunch singularity $E \to -\infty$, by therefore we will get $\gamma_{eff} = \gamma$

Appendix E

Mathematical stability

Using Eq. (4.12) in (2.7) for the case $m = 1$ we will get the following differentials equation

$$2\dot{H}+2\dot{h}+3\gamma\left(H^2+2Hh\right)-3\xi_0 H\left[3\left(H^2+2Hh\right)-\Lambda\right]-3\xi_0 h\left[3\left(H^2+2Hh\right)-\Lambda\right]-\Lambda\gamma=0, \tag{E.1}$$

using the equality Eq. (2.7), we can reorder the previous expression as follows

$$\begin{aligned} 2\dot{h}+6\gamma Hh-18\xi_0 H^2 h-9\xi_0 H^2 h+3\xi_0\Lambda h &= 0, \\ 2\dot{h}+6\gamma Hh-27\xi_0 H^2 h+3\xi_0\Lambda h &= 0, \\ \dot{h}+\frac{27\xi_0 h}{2}\left(\frac{6\gamma H}{27\xi_0}-H^2+\frac{3\Lambda}{27}\right) &= 0, \\ \dot{h}-\frac{27\xi_0 h}{2}\left(H^2-\frac{2\gamma H}{9\xi_0}-\frac{\Lambda}{9}\right) &= 0, \end{aligned} \tag{E.2}$$

In principle, we must solve

$$h = h_i \exp\left[\frac{27\xi_0}{2}\int\left(H^2-\frac{2\gamma}{9\xi_0}H-\frac{\Lambda}{9}\right)dt\right]. \tag{E.3}$$

But we don't have an expression for $E(T)$, explicitly, therefore, we start our study from Eq. (E.2)

$$\dot{h}-\frac{27\xi_0}{2}\left(H(t)^2-\frac{2\gamma}{9\xi_0}H(t)-\frac{\Lambda}{9}\right)h=0, \tag{E.4}$$

and we change the integration variable from t to E, for this propose we use (C.15) for the case $m = 1$, which is given by

$$\begin{aligned}
\frac{2\dot{E}}{3} &+ \gamma E^2 - \Omega_{\xi_0} E \left(E^2 - \Omega_{\Lambda_0}\right) - \Omega_{\Lambda_0}\gamma = 0, \\
\frac{2\dot{E}}{3} &+ \gamma \left(E^2 - \Omega_{\Lambda_0}\right) - E\Omega_{\xi_0} \left(E^2 - \Omega_{\Lambda_0}\right) = 0, \\
\frac{2\dot{E}}{3} &+ \left(E^2 - \Omega_{\Lambda_0}\right) \left(\gamma - E\Omega_{\xi_0}\right) = 0,
\end{aligned}$$

therefore we have

$$\dot{E} = -\frac{3}{2} \left(E^2 - \Omega_{\Lambda_0}\right) \left(\gamma - E\Omega_{\xi_0}\right). \tag{E.5}$$

We write (E.4) in a dimensionless way and we will have

$$\frac{dh}{dT} - \frac{9\Omega_{\xi_0}}{2} \left(E^2 - \frac{2\gamma E}{3\Omega_{\xi_0}} - \frac{\Omega_{\Lambda_0}}{3}\right) h = 0, \tag{E.6}$$

to change the integration variable from t to E, we rewrite Eq. (E.6) as follows

$$\frac{dh}{dE}\dot{E} - \frac{9\Omega_{\xi_0}}{2} \left(E^2 - \frac{2\gamma E}{3\Omega_{\xi_0}} - \frac{\Omega_{\Lambda_0}}{3}\right) h = 0, \tag{E.7}$$

using (E.5) in the previous expression and sorting a bit we will get

$$\frac{dh}{dE} = \frac{\left(\Omega_{\Lambda_0}\Omega_{\xi_0} + 2\gamma E - 3E^2\Omega_{\xi_0}\right) h}{\left(E^2 - \Omega_{\Lambda_0}\right) \left(\gamma - E\Omega_{\xi_0}\right)}. \tag{E.8}$$

We can separate into partial fractions and we will get

$$\frac{dh}{dE} = \frac{\Omega_{\Lambda_0}}{E\Omega_{\xi_0} - \gamma} - \frac{2E}{\Omega_{\Lambda_0} - E^2}, \tag{E.9}$$

Therefore, we have to integrate

$$\int \frac{dh}{h} = \int \left(\frac{\Omega_{\Lambda_0}}{E\Omega_{\xi_0} - \gamma} - \frac{2E}{\Omega_{\Lambda_0} - E^2}\right) dE, \tag{E.10}$$

and we get

$$\ln h = \ln \left[\left(E\Omega_{\xi_0} - \gamma\right) \left(\Omega_{\Lambda} - E^2\right)\right] + C. \tag{E.11}$$

It is important to note that we must be careful with the initial condition that we will take before canceling the logarithms, because if we consider $E = 1$ as the initial condition, we will obtain

$$\ln h = \ln\left[\left(\Omega_{\xi_0} - \gamma\right)\left(\Omega_{\Lambda_0} - 1\right)\right] + C, \tag{E.12}$$

note that the second parenthesis inside the logarithm is always negative, because given the Friedmann equation (2.2) Ω_{Λ_0} cannot be greater than one, so again it is required that from the initial instant $T = 0$ we need to consider

$$\Omega_{\xi_0} < \gamma. \tag{E.13}$$

Otherwise, the result of the logarithm will be imaginary and we will not be able to speak of stability, which is reasonable considering that the condition (E.13) assures us a regular universe. Therefore, as long as we take into account (E.13) we can cancel the logarithms without problems and we will obtain

$$h(E) = C\left(E\Omega_{\xi_0} - \gamma\right)\left(\Omega_{\Lambda_0} - E^2\right). \tag{E.14}$$

Appendix F

Temperature

For the study of viscous fluids in the Eckart framework, the Eq. (4.17) is given by

$$n\frac{\partial T}{\partial n} + (\rho + P_{eff})\frac{\partial T}{\partial \rho} = T\frac{\partial P_{eff}}{\partial \rho}, \tag{F.1}$$

which will give us, using Eq. (2.4) and the Eos, the follows result

$$n\frac{\partial T}{\partial n} + \rho(\gamma - 3H\xi_0)\frac{\partial T}{\partial \rho} = T\left[(\gamma - 1) - 3H\xi_0 - 3\xi_0\rho\frac{\partial H}{\partial \rho}\right], \tag{F.2}$$

using Eq. (4.20) we will get the following

$$\begin{aligned} n\frac{\partial T}{\partial n} + \left(-\frac{a}{3}\frac{\partial \rho}{\partial T}\frac{dT}{da} - n\frac{\partial \rho}{\partial n}\right)\frac{\partial T}{\partial \rho} &= T\left[\gamma - 1 - 3H\xi_0 - 3\xi_0\rho\frac{\partial H}{\partial \rho}\right], \\ n\frac{\partial T}{\partial \rho}\frac{\partial \rho}{\partial n} - \frac{a}{3}\frac{dT}{da} - n\frac{\partial \rho}{\partial n}\frac{\partial T}{\partial \rho} &= T\left[\gamma - 1 - 3H\xi_0 - 3\xi_0\rho\frac{\partial H}{\partial \rho}\right], \\ -\frac{a}{3}\frac{dT}{da} &= T\left[\gamma - 1 - 3H\xi_0 - 3\xi_0\rho\frac{\partial H}{\partial \rho}\right], \\ \frac{dT}{T} &= -\frac{3da}{a}\left[\gamma - 1 - 3\xi_0\left(H + \rho\frac{\partial H}{\partial \rho}\right)\right]. \end{aligned} \tag{F.3}$$

From the conservation Eq. (4.31) we have

$$\frac{d\rho}{(\gamma - 3H\xi_0)\rho} = -\frac{3da}{a}, \tag{F.4}$$

we substitute this expression in (F.3) and we get

$$\begin{aligned}
\frac{dT}{T} &= \frac{d\rho}{\rho}\frac{\left[\gamma-1-3\xi_0\left(H+\rho\frac{\partial H}{\partial\rho}\right)\right]}{(\gamma-3H\xi_0)},\\
\frac{dT}{T} &= \frac{d\rho}{\rho}\left[\frac{(\gamma-3H\xi_0)}{(\gamma-3H\xi_0)}-\frac{1+3\xi_0\rho\frac{\partial H}{\partial}}{(\gamma-3H\xi_0)}\right],\\
\frac{dT}{T} &= \frac{d\rho}{\rho}\left[1-\frac{1+3\xi_0\rho\frac{\partial H}{\partial\rho}}{(\gamma-3H\xi_0)}\right].
\end{aligned} \tag{F.5}$$

Note that, if we consider $\xi_0 = 0$ in the previous expression we will get

$$\frac{dT}{T} = \frac{d\rho}{\rho}\left(\frac{\gamma-1}{\gamma}\right), \tag{F.6}$$

expression that corresponds to an ideal fluid [138]. To solve Eq. (F.5), we use Eq. (2.2) and we see that $\frac{\partial H}{\partial\rho} = \frac{1}{6H}$, by therefore, Eq. (F.5) will be

$$\begin{aligned}
\frac{dT}{T} &= \frac{d\rho}{\rho}\left[1-\frac{1+\frac{\xi_0\rho}{2H}}{(\gamma-3H\xi_0)}\right],\\
\frac{dT}{T} &= \frac{d\rho}{\rho}\left[1-\frac{2H+\xi_0\rho}{2H\,(\gamma-3H\xi_0)}\right],
\end{aligned} \tag{F.7}$$

We can express our result in terms of ρ using the Friedmann Eq. (2.2) and we get

$$\frac{dT}{T} = \frac{d\rho}{\rho}\left[1-\frac{\frac{2}{3}\sqrt{3\,(\rho+\Lambda)}+\xi_0\rho}{\frac{2}{3}\sqrt{3\,(\rho+\Lambda)}\left(\gamma-\sqrt{3\,(\rho+\Lambda)}\xi_0\right)}\right], \tag{F.8}$$

The first integral is trivial, so we focus in the second integral and we factor out constants to get

$$\frac{1}{2}\sqrt{3}\int\frac{\frac{2\sqrt{\Lambda+\rho}}{\sqrt{3}}+\xi_0\rho}{\rho\sqrt{\Lambda+\rho}\left(\gamma-\sqrt{3}\xi_0\sqrt{\Lambda+\rho}\right)}\,d\rho, \tag{F.9}$$

we substitute

$$x = \sqrt{\Lambda+\rho}, \tag{F.10}$$

$$dx = \frac{d\rho}{2\sqrt{\Lambda+\rho}} = \frac{d\rho}{2u}, \tag{F.11}$$

and we get from Eq. (F.9)

$$\sqrt{3}\int \frac{\xi_0\left(x^2-\Lambda\right)+\frac{2x}{\sqrt{3}}}{\left(x^2-\Lambda\right)\left(\gamma-\sqrt{3}\xi_0 x\right)}dx. \tag{F.12}$$

We use partial fraction to get from the last expression

$$\sqrt{3}\int \frac{6\Lambda\xi_0+2\sqrt{3}\gamma x}{3\left(\Lambda\left(3\Lambda{\xi_0}^2-\gamma^2\right)-x^2\left(3\Lambda{\xi_0}^2-\gamma^2\right)\right)}dx-\frac{\xi_0\left(3\Lambda\xi_0^2-\gamma^2-2\gamma\right)}{\gamma\left(\gamma^2-3\Lambda{\xi_0}^2\right)-\sqrt{3}\xi_0 x\left(\gamma^2-3\Lambda{\xi_0}^2\right)}dx$$

$$\frac{1}{\sqrt{3}\left(3\Lambda{\xi_0}^2-\gamma^2\right)}\int \frac{6\Lambda\xi_0+2\sqrt{3}\gamma x}{\Lambda-x^2}\,dx+\frac{\left(-3\sqrt{3}\Lambda{\xi_0}^3+\sqrt{3}\gamma^2\xi_0+2\sqrt{3}\gamma\xi_0\right)}{\left(\gamma^2-3\Lambda{\xi_0}^2\right)}\int \frac{1}{\gamma-\sqrt{3}\xi_0 x}\,dx. \tag{F.13}$$

From last expression the first integral would be

$$\frac{2}{\sqrt{3}\left(\gamma^2-3\Lambda{\xi_0}^2\right)}\int \frac{3\Lambda\xi_0+\sqrt{3}\gamma x}{x^2-\Lambda}\,du,$$

$$\frac{2}{\sqrt{3}\left(\gamma^2-3\Lambda{\xi_0}^2\right)}\int \left(\frac{\sqrt{3}\gamma x}{x^2-\Lambda}+\frac{3\Lambda\xi_0}{x^2-\Lambda}\right)\,dx, \tag{F.14}$$

follow the same change of variable we use in Table C.1 we would get (replacing Eq. (F.10))

$$\frac{\gamma\ln\left(\frac{\rho}{\rho_0}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}-\frac{2\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\left[\operatorname{arctanh}\left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E}\right)-\operatorname{arctanh}\left(\sqrt{\Omega_{\Lambda_0}}\right)\right]}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}. \tag{F.15}$$

For the second integral of Eq. (F.13), we will get

$$\frac{\left(-3\sqrt{3}\Lambda{\xi_0}^3+\sqrt{3}\gamma^2\xi_0+2\sqrt{3}\gamma\xi_0\right)}{\left(\gamma^2-3\Lambda{\xi_0}^2\right)}\int \frac{1}{\gamma-\sqrt{3}\xi_0 x}\,dx \tag{F.16}$$

we make the change of variable $v=\gamma-\sqrt{3}\xi_0 x$ and we will get in our dimensionless notation

$$\frac{\left(-3\sqrt{3}\Lambda{\xi_0}^3+\sqrt{3}\gamma^2\xi_0+2\sqrt{3}\gamma\xi_0\right)}{\left(\gamma^2-3\Lambda{\xi_0}^2\right)}\left(\frac{-\ln\left(\gamma-\sqrt{3}\xi_0 x\right)}{\sqrt{3}\xi_0}\right)\Big|_{x_i}^{x_f}, \tag{F.17}$$

we replace Eq. (F.10) and simplify the constants we get

$$-\frac{\left[\gamma(2+\gamma)-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right]\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}, \tag{F.18}$$

this can be rewritten using Eqs. (F.15) and (F.18) as

$$\ln\left(\frac{T}{T_0}\right)=\ln\left(\frac{\rho}{\rho_0}\right)+\frac{2\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\left[\operatorname{arctanh}\left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E}\right)-\operatorname{arctanh}\left(\sqrt{\Omega_{\Lambda_0}}\right)\right]}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}$$
$$\frac{-\gamma\ln\left(\frac{\rho}{\rho_0}\right)+\left[\gamma(2+\gamma)-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right]\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}. \tag{F.19}$$

We can also solve Eq. (F.4) using the Friedman Eq. (2.2) and we get in our dimensionless notation

$$\int\frac{d\rho}{\rho\left(\xi_0\sqrt{3(\rho+\Lambda)}-\gamma\right)}=3\ln a, \tag{F.20}$$

we use the same change of variables Eqs. (F.10) and (F.11) to get

$$2\int\frac{x}{\left(x^2-\Lambda\right)\left(\sqrt{3}\xi_0 x-\gamma\right)}\,dx. \tag{F.21}$$

From the last result, we use partial fraction to get

$$\frac{2}{\left(\gamma^2-3\Lambda\xi_0^2\right)}\int\frac{\sqrt{3}\Lambda\xi_0+\gamma x}{\Lambda-x^2}dx+\frac{2}{\left(\gamma^2-3\Lambda{\xi_0}^2\right)}\int\frac{\sqrt{3}\gamma\xi_0}{\sqrt{3}\xi_0 x-\gamma}dx, \tag{F.22}$$

the first integral would gives us in our dimensionless notation

$$\frac{2}{\left(\gamma^2-3\Omega_\Lambda\Omega_{\xi_0}^2\right)}\int\frac{\sqrt{3}\Lambda\xi_0+\gamma x}{\Lambda-x^2}dx=\frac{2\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\left[\operatorname{arctanh}\left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E}\right)-\operatorname{arctanh}\left(\sqrt{\Omega_{\Lambda_0}}\right)\right]}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}+\frac{-\gamma\ln\left(\frac{\rho}{\rho_0}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}, \tag{F.23}$$

where we use Eq. (F.10) and we make the integration from $E=1$ ($a_i=1$ and $\rho_i=\rho_0$) to arbitrary E ($a_f=a$ and $\rho_f=\rho$). The second integral in Eq. (F.22) would gives in our dimensionless notation

$$\frac{2}{\left(\gamma^2-3\Lambda{\xi_0}^2\right)}\int\frac{\sqrt{3}\gamma\xi_0}{\sqrt{3}\xi_0 x-\gamma}dx=\frac{2\gamma\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}, \tag{F.24}$$

then, from Eq. (F.22) using the result Eq. (F.23) and (F.24) we get the following result

$$\begin{aligned}
\ln a^3 &= \frac{2\Omega_{\xi_0}\sqrt{\Omega_\Lambda}\left[\operatorname{arctanh}\left(\frac{\sqrt{\Omega_{\Lambda_0}}}{E}\right)-\operatorname{arctanh}\left(\sqrt{\Omega_{\Lambda_0}}\right)\right]}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)} \\
&+ \frac{-\gamma\ln\left(\frac{\rho}{\rho_0}\right)+2\gamma\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)}{\left(\gamma^2-\Omega_{\Lambda_0}\Omega_{\xi_0}^2\right)}. \qquad \text{(F.25)}
\end{aligned}$$

There is a direct way to find the expression Eq. (4.39) or in other words, to express the temperature as a function of energy density, starting from Eq. F.5 we get[1]

$$\begin{aligned}
\frac{dT}{T} &= \frac{d\rho}{\rho}\left[1-\frac{1}{(\gamma-3H\xi_0)}-\frac{3\xi_0\rho\frac{\partial H}{\partial\rho}}{(\gamma-3H\xi_0)}\right]. \\
\int\frac{dT}{T} &= \int\frac{d\rho}{\rho}-\int\frac{d\rho}{\rho\,(\gamma-3H\xi_0)}-\int\frac{3\xi_0 d\rho\frac{\partial H}{\partial\rho}}{(\gamma-3H\xi_0)}. \\
\int\frac{dT}{T} &= \int\frac{d\rho}{\rho}-\int\frac{d\rho}{\rho\,(\gamma-3H\xi_0)}-\int\frac{3\xi_0 dH}{(\gamma-3H\xi_0)}. \qquad \text{(F.26)}
\end{aligned}$$

We express our last result in a dimensionless notation and using Eq. (F.4) we get

$$\int\frac{dT}{T}=\int\frac{d\rho}{\rho}+3\int\frac{da}{a}-\Omega_{\xi_0}\int\frac{dE}{(\gamma-E\Omega_{\xi_0})}, \qquad \text{(F.27)}$$

if we integrate from $E=1$ ($a_i=1$ and $\rho_i=\rho_0$) to arbitrary $E(a_f=a$ and $\rho_f=\rho)$ we get

$$\ln\left(\frac{T}{T_0}\right)=\ln\left(\frac{\rho}{\rho_0}\right)+3\ln a+\ln\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right), \qquad \text{(F.28)}$$

this give us the final result

$$\boxed{T=T_0\left(\frac{\rho}{\rho_0}\right)\left(\frac{\gamma-E\Omega_{\xi_0}}{\gamma-\Omega_{\xi_0}}\right)a^3.} \qquad \text{(F.29)}$$

[1] The third integral in the second line form a total derivative in H

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