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Hidden symmetries and nonlinear (super)algebras

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Abstract

The relevance of hidden symmetries is explored at the level of classical and quantum mechanics in a variety of physical systems related to conformal and superconformal invariance. Hidden symmetries, that correspond to nonlinear in momenta integrals of motion, generally lead to nonlinear algebras.

First, analyzing the $\mathfrak{sl}(2,\mathbb{R})$ symmetry, it is concluded that both the asymptotically free (at infinity) and the harmonically confined models are two different forms of dynamics described by the same symmetry algebra. A mapping between these two dynamics is constructed, and its applications are studied in one-, two- and three-dimensional systems.

Second, rational extensions of the conformal mechanics model of de Alfaro, Fubini and Furlan (AFF) are derived by employing the generalized Darboux transformation. In general, the obtained systems have an almost equidistant spectrum with some gaps inside, and their spectral properties imply the presence of hidden symmetries. The supersymmetric extensions of the AFF model are also studied, and the origin of the hidden bosonized superconformal symmetry of the quantum harmonic oscillator is established.

Finally, a three-dimensional generalization of the AFF system is considered. The model describes a particle with electric charge e in Dirac monopole background of magnetic charge g, and subjected to the central potential $\frac{m\omega^2}{2}r^2+\frac{\alpha}{2mr^2}$. When $\alpha=(eg)^2$, the classical trajectories are periodic for arbitrary initial conditions and at the quantum level, the spectrum acquires a peculiar degeneration. These characteristics are described by hidden symmetries, which can be obtained from the model without harmonic term by means of the mentioned mapping. A complementary spin-orbit coupling term gives rise to a supersymmetric extension of the system, characterized by superconformal symmetry. The spectrum-generating operators of the new model are shown to be nonlocal.

Keywords: Hidden symmetries; (Super-)Conformal symmetry; de Alfaro, Fubini and Furlan model; Harmonic oscillator; Supersymmetric quantum mechanics; Rationally extended systems; Darboux duality; Klein four-group; Dirac monopole.

Dedicatory

"... y la realidad plausible cae de pronto sobre mi...
me incorporo a medias enérgico, convencido, humano
y voy a escribir estos versos para convencernos de lo contrario..."

Álbaro de Campos (Fernando Pessoa), *La Tabaquería.* Extracto.

Dedicado a quienes me apoyaron (y soportaron) durante todo el proceso.

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Notations

Here we summarize some common notations used in the manuscript. In this Thesis we use $\hbar = c = 1$.

Geometry:

$g_{\mu\nu}$ and $\eta_{\mu\nu}$: The general metric tensor and Minkowski metric tensor.

$x_\mu = g_{\mu\nu} x^\nu = \sum_\nu g_{\mu\nu} x^\nu$ and $g_{\mu\nu} x^\mu x^\nu = \sum_{\mu,\nu} g_{\mu\nu} x^\mu x^\nu$: The Einstein summation convention.

ζ^μ: A Killing vector component.

$A \wedge B$ and d: The exterior product and the exterior derivative, respectively.

$\pounds_X T$: The Lie derivative of a tensor field T along the flow of the vector field X.

$i_X \omega \equiv \omega(X, \underbrace{\ldots\ldots\ldots}_{r-1 \text{ entries}})$: The contraction between a vector field and a differential r-form ω, which, in turns, is a differential $(r-1)$-form.

Classical mechanics:

$\mathcal{M}$: The configuration space.

$T\mathcal{M}_q$: The tangent space at $q \in \mathcal{M}$.

$T_*\mathcal{M}_q$: The cotangent space at $q \in \mathcal{M}$.

$T\mathcal{M}$: The tangent bundle.

$T_*\mathcal{M}$: The cotangent bundle.

q^i and $\dot{q}^i = \frac{dq^i}{dt}$: The generalized coordinates on $\mathcal{M}$ and its velocities.

$\mathcal{L}$, $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i}$ and H: The Lagrangian, the canonical momenta and the Hamiltonian.

$\omega = dq_i \wedge dp^i$: The symplectic two-form.

Supersymmetric quantum mechanics:

H: The quantum Hamiltonian.

L: A dimensionless quantum Hamiltonian.

ψ_*, $\widetilde{\psi}_*$: Two linearly independent eigenstates of L, with eigenvalue λ_* .

$W(\underbrace{\ldots\ldots\ldots}_{n \text{ entries}})$: The generalized Wronskian of n functions.

$\breve{L}$: A dimensionless supersymmetric partner of L.

$A^{\pm}$: The first order mutually conjugate intertwining operators.

$\mathbb{A}_n^{\pm}$: The higher order mutually conjugate intertwining operator.

$\Omega_*(x)\,,\breve{\Omega}_*(x)$: The Jordan states constructed by means of ψ_* and $\widetilde{\psi}_*$, respectively.

$\mathcal{H}$: A matrix-valued super-Hamiltonian operator.

$\mathcal{Q}_a$: A Supercharge.

$\mathcal{N}$: The number of supercharges.

Pauli matrices: $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$

$\Pi_{\pm} = \frac{1}{2}(1 \pm \sigma_3)$: Projectors to σ_3 subspaces.

Conformal mechanics:

H, D, and K: Generators of the $\mathfrak{so}(2,1)$ algebra.

$\mathcal{J}$ and $\mathcal{J}_{\pm}$: Generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra.

H_ν : The Hamiltonian of an asymptotically free conformal invariant system.

$\mathscr{H}_\nu$ and $\mathcal{C}_\nu^{\pm}$ The Hamiltonian of de Alfaro, Fubini and Furlan model and its ladder operators.

$\mathfrak{S}$: The conformal bridge transformation operator.

Rationally extended systems:

$\Delta_{\pm}$: The positive-negative Darboux scheme.

$A_{(\pm)}^{\pm}$: The self-conjugate intertwining operators of the positive-negative Darboux scheme.

$L_{(\pm)}$: The rationally extended system associated with the positive-negative Darboux scheme.

$\mathcal{A}^{\pm}$, $\mathcal{B}^{\pm}$, and $\mathcal{C}^{\pm}$: The spectrum-generating ladder operators of the ABC-type.

$\mathfrak{A}_i^{\pm}$, $\mathfrak{B}_i^{\pm}$, and $\mathfrak{C}_i^{\pm}$: The extended families of ladder operators of the ABC-type.

$\mathfrak{S}_z^{\pm}$, : The extended families of intertwining operators.

$\mathcal{U}_{0,z}^{(2\theta(z)-1)}$, and $\mathcal{I}_{N,z}^{(1-2\theta(z-N))}$: The extended subsets of generators of a nonlinear superalgebra.

Three-dimensional conformal mechanics in a monopole background:

$\nu = (eg)^2$: Here e and g are the particle's electric charge and the monopole's magnetic charges, respectively.

α : The coupling of the conformal mechanics potential.

$\boldsymbol{I}_1$, $\boldsymbol{I}_2$, $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$: Dynamical integrals for the case $\alpha = \nu^2$.

$\boldsymbol{J}$: The Poincaré vector integral.

$T^{(ij)}, T^{[ij]}$: Symmetric and anti-symmetric tensor integrals.

A charge-monopole superconformal model

$\boldsymbol{K} = \boldsymbol{J} + \frac{1}{2}\boldsymbol{\sigma}$: The total angular momentum.

$k = j \pm 1/2$: The eigenvalue of $\boldsymbol{K}^2$.

$\pm\omega\,\boldsymbol{\sigma}\cdot\boldsymbol{J}$: The spin-orbit coupling.

Θ, $\Theta^\dagger$, Ξ and $\Xi^\dagger$: Scalar intertwining operators.

$\mathcal{H}$ and $\breve{\mathcal{H}}$: Pauli type supersymmetric Hamiltonians in exact and spontaneously broken phase.

$\mathcal{Q}$, $\mathcal{Q}^\dagger$, $\mathcal{W}$, $\mathcal{W}^\dagger$: Nilpotent fermionic operators.

$\mathcal{R}$, $\mathcal{G}$ and $\mathcal{G}^\dagger$: The $R-$symmetry generators and the lowering and rising supersymmetric ladder operators.

$\mathcal{P}_\pm$: Projectors onto subspaces with fixed k.

$\mathcal{B}$ and $\mathcal{F}$: Generic bosonic and fermionic three-dimensional generators.

$\mathscr{B}$ and $\mathscr{F}$: Generic bosonic and fermionic one-dimensional generators.

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Introduction

Symmetries play a very important role in the construction of the fundamental theories that we have in physics nowadays. Examples of that are the general relativity and the Standard Model of particle physics, just to name a few. In this Thesis, we study hidden symmetries that control nontrivial aspects of classical dynamics, as well as spectral peculiarities in quantum and supersymmetric quantum mechanics models.

From a classical mechanics perspective, Noether's theorem reveals that behind the invariance of action under a symmetry transformation, there exists a conservation law. In general, the principle of least action assumes the existence of a Lagrangian $\mathcal{L}$, which in mechanics depends on the generalized coordinates and its velocities. Geometrically, these coordinates belong to a configuration space $\mathcal{M}$, which points are usually denoted by q, and their associated velocities are vectors that live on the tangent space $T\mathcal{M}_q$ at q , which in turns, are generated by the action of a particular tangent vector field. Then, naturally the Lagrangian is a function on the tangent bundle $T\mathcal{M} = \cup_{q\in\mathcal{M}}T\mathcal{M}_q$ of $\mathcal{M}$ [Nakahara (2003); Sundermeyer (2014)]. In this framework, symmetry is a one-parametric transformation generated by some conserved vector field. To compare transformations associated with two different vector fields, say $X = X^\mu\frac{\partial}{\partial q^\mu}$ and $Y = Y^\mu\frac{\partial}{\partial q^\mu}$, we compute the Lie derivative[1] of Y along the flow of X, denoted by $\pounds_X Y$, and it is not difficult to show that this operation reduces to the usual commutator between two vector fields $[X, Y] \in T\mathcal{M}$. This gives rise to a Lie algebra of vector fields on $T\mathcal{M}$ [Nakahara (2003)].

On the other hand, when we go to the Hamiltonian formalism, the dynamical variables considered now are the generalized coordinates and their canonical momenta $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$. One can show that under a general change of coordinates, p_i transform as the components of a vector in the cotangent space $T_*\mathcal{M}_q$ at q [Nakahara (2003)]. Then the phase space is naturally identified with the cotangent bundle $T_*\mathcal{M} = \cup_{q\in\mathcal{M}}T_*\mathcal{M}_q$ with local coordinates (q^i, p_i) on it [Arnold et al. (1989); Nakahara (2003); Sundermeyer (2014)]. Here, the symplectic form $\omega = dq^i \wedge dp_i$ encodes the Poisson bracket structure. Namely, with a given function $F = F(q, p)$ on the phase space, a Hamiltonian vector

[1] The Lie derivative evaluates the change of a tensor field (including scalar functions, vector fields and one-forms), along the flow defined by another vector field [Nakahara (2003)].

field,

$$X_F = \frac{\partial F}{\partial p^i}\frac{\partial}{\partial q_i} - \frac{\partial F}{\partial q^i}\frac{\partial}{\partial p_i}\,,$$

is associated, such that the contraction $i_{X_F}\omega \equiv \omega(X_F, .) = dF$. For two Hamiltonian vector fields X_F and X_G, it follows then that $\pounds_{X_F} X_G = X_{\{F,G\}}$ and $\pounds_{X_F} G = \{G, F\}$. If F is identified as the Hamiltonian of the system, then the last relation corresponds to the equation of motion for G [Arnold et al. (1989); Sundermeyer (2014)]. In this formalism, a symmetry transformation is a flow produced by a Hamiltonian vector field whose generating function in phase space is conserved in time.

As it is known, the Lie algebra mentioned above corresponds to a more abstract concept. A Lie group is a smooth manifold with an additional group structure, and any Lie group gives rise to a Lie algebra, which is its tangent space at the identity [Nakahara (2003); Gilmore (2006)]. When a group "acts" on some target space (that could be the same group manifold), an explicit form of its elements is required. This leads us to the representation theory. In Hamiltonian classical mechanics, the target space is $T_*\mathcal{M}$, the Lie algebra generators are identified with the Hamiltonian vector fields, and the group action corresponds to Hamiltonian flows. In the case of quantum theory, we look, in accordance with the celebrated Wigner theorem [Wigner (1931, 2012); Weinberg (1995)], for irreducible unitary representations of the quantum symmetry group of the system, and target space is the Hilbert space generated by eigenstates of the quantum Hamiltonian operator. In fact, the "algebraic" approach claims that the entire Hilbert space can be generated by the action of the symmetry operators on an arbitrary solution of the corresponding Schrödinger equation, i.e., the spectrum of the system is explained by symmetry.

Symmetries are intrinsic properties of the geometry that characterizes a given manifold. Suppose we have a space-time manifold with a metric structure $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. If ds^2 is invariant under a certain change of coordinates, we have an "isometry", which in accordance with the discussion above, is generated by a particular vector field, called Killing vector field [Nakahara (2003)]. We can ask for mechanical systems that respect the isometries of the space-time where they live, that gives rise to important physical consequences. For example, the construction of an action principle in Minkowski space that is invariant under the Poincaré group transformations $x^\mu \to y^\mu = \Lambda^\mu_\nu x^\nu + a^\mu$, where Λ^μ_ν are the Lorentz transformations, is just the same as to impose the relativity postulates. In this way, Poincaré invariant quantum field theories involve in their description field operators which provide certain representations of this symmetry group [Weinberg (2012); Sundermeyer (2014)]. The isometry condition for infinitesimal transformations $x^\mu \to x^\mu + \zeta^\mu$ corresponds to the Killing equation

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\zeta^\lambda + g_{\mu\lambda}\frac{\partial \zeta^\lambda}{\partial x^\nu} + g_{\lambda\nu}\frac{\partial \zeta^\lambda}{\partial x^\mu} = 0\,,$$

and for Poincaré transformations in Minkowski space its solutions are given by $\zeta^\mu = a^\mu + \omega^{\mu\nu}x_\nu$, where $\omega^{\mu\nu}$ is an antisymmetric matrix. To obtain the corresponding Killing vector fields we use the fact that Poincaré transformations admit the unitary representation $\exp\big(i(a^\mu T_\mu - \frac{1}{2}\omega^{\mu\nu}M_{\mu\nu})\big)$, where T_μ and $M_{\mu\nu}$ are our candidates for translations and Lorentz transformations generators, respectively. To identify them we should compare $y^\mu = x^\mu + \zeta^\mu$ with

$$\exp\Big(i(a^\nu T_\nu - \frac{1}{2}\omega^{\alpha\beta}M_{\alpha\beta})\Big)x^\mu \approx x^\mu + i(a^\nu T_\nu - \frac{1}{2}\omega^{\alpha\beta}M_{\alpha\beta})x^\mu \,,$$

which implies that $T_\mu = i\partial_\mu$ and $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu}$. Here $\Sigma_{\mu\nu}$ are operators that do not act on the coordinates, but their representations tell us about the spin of the corresponding fields.

The notion of Killing vectors is generalized to the so-called conformal Killing vectors, which are related to the coordinate changes so that $ds^2 \to \Omega(x)ds^2$, where $\Omega(x)$ is the conformal factor. Such transformations correspond, particularly, to dilatations $x^\mu \to cx^\mu$ and special conformal transformations $x^\mu \to (x^\mu - b^\mu x^2)/(1 - 2b^\nu x_\nu + b^2x^2)$ [Francesco et al. (1997)].

Conformal symmetry, as well as conformal field theories, have made an huge contribution on different aspects of physics, such as condensed matter, electrodynamics, and gravity, just to mention a few examples [Ginsparg (1988); Jackiw and Pi (2011)]. The two-dimensional case is special in this context. Indeed, consider the change of coordinates

$$x^1 \to x^1 + f_1(x^1, x^2)\,, \qquad x^2 \to x^2 + f_2(x^1, x^2)\,,$$

in flat space. This transformation can be shown to be of the conformal type if and only if $f_1(x^1, x^2)$ and $f_2(x^1, x^2)$ satisfy the Cauchy-Riemann equations, i.e., they are the real and imaginary parts of a holomorphic function. In the case of infinitesimal transformations, however, we can be less restrictive. To see this better, it is natural to take the complex coordinate $z = x^1 + ix^2$, together with its complex conjugate $\bar{z}$, and consider the infinitesimal transformation $z \to z + \varepsilon(z)$, where $\varepsilon(z)$ is assumed to be a meromorphic function which admits a Laurent expansion around $z = 0$. In this situation a (primary) field $\phi(z, \bar{z})$ infinitesimally transforms as $\delta\phi = -(\varepsilon\partial_z + \bar{\varepsilon}\partial_{\bar{z}})\phi$, from where we identify the symmetry generators $l_n = -z^{n+1}\partial_z$ and $\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}$, with $n \in \mathbb{Z}$. They produce a direct sum of two copies of the infinite-dimensional Witt algebra, while the global conformal group that maps the complex plane onto itself is obtained from the subalgebra $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$, which, in turn, is generated by $\{l_0,\, \bar{l}_0,\, l_\pm,\, \bar{l}_\pm\}$, [Francesco et al. (1997)]. Using these properties one can introduce a conformal field theory that does not even need a specific action principle. This corresponds to the so-called conformal bootstrap [Polyakov (1974)]. This type of theories, "minimal models", as they are often called [Belavin et al. (1984); Francesco et al. (1997)], appears in the study of critical points in the second-order phase transition phenomena, and their main advantage is the calculation of the correlation functions of 2 and 3 points, only by symmetry arguments. On the other hand, conformal theories in higher dimensions became popular after Maldacena's famous article [Maldacena (1999)], where a duality between a gravity theory in AdS (type IIB string theory in $AdS_5 \times S^5$) and a conformal field theory in the boundary ($\mathcal{N} = 4$ supersymmetric Yang-Mills) was shown. This AdS/CFT correspondence along with holographic techniques have found applications not only in black holes physics but also in other areas such as QCD [Ammon and Erdmenger (2015); Brodsky et al. (2015); Deur et al. (2015)].

Beyond the Standard Model it has been postulated supersymmetry, based on transformations that relate bosons and fermions [Weinberg (2000)]. These models refer to an action principle defined in the "super-space", which is a place where bosonic and fermion quantities (described by Grassmann's variables) live together. To overcome the Coleman-Mandula theorem: "space-time and internal symmetries cannot be combined in any but a trivial way", see [Pelc and Horwitz (1997)], the concept of symmetry is generalized to a $\mathbb{Z}_2$-graded algebra, or superalgebra, which is characterized by the supercommutator $[[A, B]]$,

- $[[A, B]] = [A, B]$ if A and B are bosonic generators,
- $[[A, B]] = [A, B]$ if one generator is bonosic while another is fermionic,
- $[[A, B]] = \{A, B\} = AB + BA$ if both generators are fermionic.

To discriminate between bosonic and fermionic objects it is necessary to introduce a grading operator Γ, $\Gamma^2 = 1$, that commutes with all bosonic generators and anti-commutes with fermionic ones. The conserved quantities that generate the supersymmetric transformations are called supercharges and are the fermionic operators. For the study of supersymmetry outside the framework of quantum field theory, the concepts of pseudo-classical mechanics [Berezin and Marinov (1975, 1976); Casalbuoni (1976)] and its quantum version, supersymmetric quantum mechanics [Witten (1981, 1982); Cooper et al. (1995)], were introduced. The latter has become an invaluable tool in the study of solvable potentials, and is closely related to the theory of integrable classical field systems and their solitonic and finite-gap type solutions [Matveev and Salle (1991)]. Details of this formalism are presented in the next chapter.

At this point it is clear that symmetries govern physics, and in this context, the notion of hidden symmetries becomes relevant [Cariglia (2014)]. To explain it, let us consider again classical mechanics. If, regardless of the initial conditions, it happens that the nature of the trajectories in some system is "special " (in a geometric sense), this should indicate on the presence of the hidden symmetries. Form the perspective of symmetry transformations, these objects mix the coordinate and velocity (momenta) variables in Lagrangian (Hamiltonian) formalism. At the quantum level, hidden symmetries can explain peculiar properties of the physical spectrum, such as a degeneration. Take, for example, the case of the Kepler-Coulomb problem, where we know that the system is invariant under rotations and that the particle trajectories, being conical sections, lie in the plane orthogonal to the angular momentum vector. We also know that the geometric properties are determined by the energy and the angular momentum itself, but there is one more special property, the orientation of the trajectory, which is given by the so-called Laplace-Runge-Lentz vector to be the second-order in canonical momenta quantity. This vector integral is also relevant at a quantum level because it explains the "accidental" degeneration in the spectrum of the hydrogen atom model [Pauli (1926)]. From now on, the nonlinear in canonical momenta integrals of motion different from Hamiltonian, like the mentioned Laplace-Runge-Lentz vector, will be called hidden symmetries. To study the geometric interpretation of these objects, which are usually related to Killing tensors and conformal Killing tensors [Cariglia (2014)], a good approach corresponds to the Eisenhart-Duval lift [Cariglia et al. (2018)], the procedure by which classical trajectories are identified with the null geodesics of a non-trivial geometry with two extra dimensions. Some other well known examples where these objects play a key role are the three-dimensional isotropic harmonic oscillator [Jauch and Hill (1940); Fradkin (1965)], the anisotropic harmonic oscillator [Bonatsos et al. (1994); de Boer et al. (1996)], the Higgs oscillator [Zhedanov (1992); Evnin and Rongvoram (2017)], nonlinear supersymmetry [Plyushchay (2019)] and a charged particle in a monopole background [Plyushchay and Wipf (2014); Inzunza et al. (2020b)].

The hidden symmetries satisfy nonlinear algebras in the general case. The first examples of nonlinear algebras introduced in field theory literature were the infinite W algebras [Zamolodchikov (1985)], which are necessary to study the nature of the infinite-dimensional groups that appear in two-dimensional conformal models. The listed above systems are examples of elementary models whose associated integrals of motions satisfy finite W algebras, which in turns, have played a relevant role in understanding of their infinite counterpart [de Boer et al. (1996)].

In the particular case of one-dimensional quantum mechanics, the supersymmetric algorithm allows us to build families of solvable potentials that have spectral peculiarities, perfectly encoded in hidden symmetries. A good example of this are the rational deformations of the harmonic oscillator, characterized by a potential of the form $x^2 - 2\ln(W(x))''$, where $W(x)$ is a regular polynomial on the real line [Krein (1957); Adler (1994)]. Systems of this nature find importance in the field of exceptional orthogonal polynomials, see for example [Dubov et al. (1994); Quesne (2012); Gómez-Ullate et al. (2013)]. The corresponding spectrum of this kind of systems is divided into g subsets of equidistant energy levels, isolated from each other. The first $(g-1)$ subsets, or bands, have a finite number of levels, while the last band has infinite number of equidistant discrete levels. In [Cariñena and Plyushchay (2017)], the spectrum-generating ladder operators for these systems were built, and they turned out to be higher order symmetry operators.

This Thesis reviews in a self-contained manner the results obtained within the framework of a three-years research project, in which we address the following problems:

a) Connection between different mechanical systems through symmetries

The $\mathfrak{so}(2,1)$ conformal algebra

$$[D,H]=iH\,,\qquad [D,K]=-iK\,,\qquad [K,H]=2iD\,,$$

describes different quantum systems with continuous spectrum, that is, H could represent the Hamiltonian of a free particle, Calogero models, monopole-charge system, etc. This algebra is isomorphic to the $\mathfrak{sl}(2,\mathbb{R})$ algebra,

$$[\mathcal{J}_0,\mathcal{J}_\pm]=\pm\mathcal{J}_\pm\,,\qquad [\mathcal{J}_-,\mathcal{J}_+]=2\mathcal{J}_0\,,$$

where $\mathcal{J}_0$ is a compact generator that represents the Hamiltonian of a confined system, such as the harmonic oscillator. We address the problem of establishing a mapping between these two forms of dynamics associated with conformal algebra. Such a transformation would be useful, particularly, for mapping conserved quantities that are easier to identify for one system than for the other.

b) Hidden and bosonized supersymmetry

In quantum mechanics, the reflection operator $\mathcal{R}$ is defined by $\mathcal{R}x=-x\mathcal{R}$ and $\mathcal{R}p=-p\mathcal{R}$. If we choose the supersymmetric grading operator Γ to be $\mathcal{R}$, we can construct bosonized supersymmetric systems [Plyushchay (1996, 2000a); Gamboa et al. (1999); Correa et al. (2007); Correa and Plyushchay (2007); Correa et al. (2008); Jakubský et al. (2010)] which do not employ fermionic degrees of freedom. We focus on the origin of the hidden bosonized superconformal symmetry of the harmonic oscillator in one dimension [de Crombrugghe and Rittenberg (1983); Balantekin et al. (1988); Cariñena and Plyushchay (2016a); Bonezzi et al. (2017)], that is, we build an unconventional supersymmetric system that, after nonlocal transformation of the Foldy-Wouthuysen type and a dimensional reduction [Jakubský et al. (2010)], produces the superalgebra we are looking for.

c) Hidden symmetries in rationally extended conformal mechanics

The simplest conformal invariant system that one can construct is

$$S=\int_{t_1}^{t_2}\left(\frac{1}{2}\dot{q}^2+\frac{g}{2q^2}\right)dt\,,\qquad q>0\,,$$

where g is a dimensionless constant that should be non-negative in classical mechanics and $g\geq -1/4$ at the quantum level. This model does not have a well-defined invariant ground state and to eliminate this deficiency, de Alfaro, Fubini, and Furlan used a particular coordinate and time change to transform the latter action into

$$S=\int_{\tau_1}^{\tau_2}\left(\frac{1}{2}\dot{y}^2+\frac{g}{2y^2}+\frac{\omega^2}{2}y^2\right)d\tau\,,\qquad x>0\,.$$

The corresponding Hamiltonian is compact and has a well-defined ground state at the quantum level, see [de Alfaro et al. (1976)]. This system, called the de Alfaro, Fubini and Furlan model (AFF), and its supersymmetric extensions [Akulov and Pashnev (1983); Fubini and Rabinovici (1984); Ivanov et al. (1989); Donets et al. (2000); Fedoruk et al. (2012)] have attracted a great attention over the years in a variety of fields such as particles dynamics in black hole backgrounds [Gibbons and Townsend (1999); Michelson and Strominger (1999); de Azcarraga et al. (1999); Britto-Pacumio et al. (2000); Galajinsky (2015)], cosmology [Duval et al. (1991); Pioline and Waldron (2003)], nonrelativistic AdS/CFT correspondence [Son (2008); Balasubramanian and McGreevy (2008); Barbon and Fuertes (2008); Chamon et al. (2011)], QCD confinement problem [Brodsky et al. (2015); Deur et al. (2015)], physics of Bose-Einstein condensates [Prain et al. (2010); Ohashi et al. (2017)] and anyon statistics [Leinaas and Myrheim (1977, 1988); Mackenzie and Wilczek (1988)]. We apply the generalized Darboux-Crum-Krein-Adler transformation (DCKA) [Moutard (1878, 1875); Darboux (1882); Crum (1955); Krein (1957); Adler (1994); Matveev and Salle (1991)] to the AFF model to construct rational deformations of this system. The objective is to follow the approach given in [Cariñena and Plyushchay (2017)] to find the ladder operators that generate spectrum of these systems.

d) Hidden symmetries in three-dimensional conformal mechanics

Consider a charged particle moving in a magnetic field generated by a Dirac monopole, i.e., in a monopole background [Sakurai (1994)], which is also subject to a central potential of the form $V(\boldsymbol{r}) = \frac{\alpha}{2m\boldsymbol{r}^2}$. In [Plyushchay and Wipf (2014)] it had already been shown that the system has hidden symmetries when $\alpha = (eg)^2$, where e and g are the particle's electric charge and the monopole magnetic charge, respectively. It was also shown that the system allows an $\mathcal{N} = 4$ supersymmetric extension. We investigate the possibility of obtaining hidden integrals of motion when the central potential is changed for $V(\boldsymbol{r}) = \frac{\alpha}{2m\boldsymbol{r}^2} + \frac{m\omega\boldsymbol{r}^2}{2}$, and we look for possible supersymmetric extensions. The results obtained from problem *a)* are used to investigate this problem.

The results of investigation of the listed problems were reported in the articles [Cariñena et al. (2018); Inzunza and Plyushchay (2018, 2019a,b); Inzunza et al. (2020a,b)].

The subsequent main part of the Thesis is organized as follows. In Chap. 1 we review the supersymmetric quantum mechanics formalism as well as the generalized Darboux transformations and their confluent extensions. In Chap. 2 we revisit the one-dimensional conformal mechanics model of de Alfaro, Fubini and Furlan [de Alfaro et al. (1976)], as well as its $\mathcal{N} = 2$ supersymmetric extension, leading us to the $\mathfrak{osp}(2,2)$ superconformal symmetry. In Chap. 3, based on [Inzunza et al. (2020a)], we consider the conformal bridge transformation and its applications to models in one and two dimensions. In Chap. 4, we explain the origin of the hidden bosonic superconformal symmetry of the harmonic oscillator [Inzunza and Plyushchay (2018)]. In Chap. 5 we review the results of ref. [Cariñena et al. (2018)], where rational extensions of the conformal mechanics model characterized by the potential $\frac{m(m+1)}{x^2}$ with $m = 1, 2, \ldots$, as well as its spectrum-generating ladder operators are constructed. In Chap. 6, following [Inzunza and Plyushchay (2019a)], we consider supersymmetric extensions of the rationally deformed system of Chap. 5, as well as its complete spectrum-generating nonlinear superalgebra. In Chap. 7 we exploit a discrete Klein four-group symmetry of the Schrödinger equation for the AFF model to generalize the construction of rationally extended systems and the spectrum-generating ladder operator sets for the case in which integer parameter m is replaced by a real number $\nu \geq -1/2$ [Inzunza and Plyushchay (2019b)]. In this case, the confluent Darboux transformations appear naturally. Chap. 8 and 9 are devoted to investigation, in the light of hidden symmetries, of the conformal mechanics in a monopole background as well as its supersymmetric extension, which is characterized by a three-dimensional realization of the $\mathfrak{osp}(2,2)$ superconformal symmetry [Inzunza et al. (2020b)]. The Thesis ends with its Conclusion and Outlook. In Appendix, some technical details are collected.

Chapter 1

Supersymmetric quantum mechanics

The application of supersymmetric ideas in nonrelativistic quantum mechanics has given us a better understanding of the problem of solvable potentials and its associated hidden symmetries. In this context, the main technique is the factorization method [Infeld and Hull (1951); Cooper et al. (1995)], which relates a particular quantum mechanical system with another one (the so-called superpartner). In the one-dimensional case, the formalism of construction of such operators (starting from a well known quantum system) receives the name of Darboux-Crum-Krein-Adler transformation [Moutard (1878, 1875); Darboux (1882); Crum (1955); Krein (1957); Adler (1994); Matveev and Salle (1991)]. An algorithmic procedure involves a given number of eigenstates of the original system, typically called "seed states", and in its confluent extension Jordan states are also considered [Schulze-Halberg (2013); Correa et al. (2015); Contreras-Astorga and Schulze-Halberg (2015)]. In this chapter we revisit these methods.

Generalization to higher spatial dimensions can be reformulated in different ways, see [Kirchberg et al. (2003); Ivanov et al. (2003); Kirchberg et al. (2005); Bellucci et al. (2005, 2006); Kozyrev et al. (2017)]. In this Thesis, we just consider the approach of a given Dirac Hamiltonian, whose square produces a supersymmetric Hamiltonian operator [Cooper et al. (1995)].

1.1 The one-dimensional case

In one-dimensional systems, the factorization method consists in introducing intertwining operators of the form

$$A=\sqrt{\frac{\hbar^2}{2m}}\frac{d}{dx}+W(x)\,,\qquad A^\dagger=-\sqrt{\frac{\hbar^2}{2m}}\frac{d}{dx}+W(x)\,,\tag{1.1.1}$$

which satisfy

$$AA^\dagger=H_+\,,\qquad A^\dagger A=H_-\,,\qquad H_\pm=-\tfrac{\hbar^2}{2m}\tfrac{d^2}{dx^2}+W(x)^2\pm\tfrac{\hbar}{\sqrt{2m}}W(x)'\,.\tag{1.1.2}$$

Here $W(x)$ is called superpotential and $H_\pm$ are the superpartner systems. Now, let us assume we know a function ψ_* such that $H_-\psi_* = 0$. This defines a nonlinear Riccati type equation for W

$$W(x)^2 - \frac{\hbar}{\sqrt{2m}}\frac{dW}{dx} = u(x)\,, \qquad u = (x) = \frac{\hbar^2}{2m}\frac{1}{\psi_*}\frac{d^2}{dx^2}\psi_*\,, \tag{1.1.3}$$

and a particular solution of which is

$$W(x) = -\frac{\hbar}{\sqrt{2m}}\frac{\psi_*'}{\psi_*} = -\frac{\hbar}{\sqrt{2m}}\ln(\psi_*)' \quad \Rightarrow \quad A = \frac{\hbar}{\sqrt{2m}}\left(\frac{d}{dx} - \ln(\psi_*)'\right) = \frac{\hbar}{\sqrt{2m}}\psi_*\frac{d}{dx}\frac{1}{\psi_*}\,, \tag{1.1.4}$$

which in turns implies $A\psi_* = 0$. This result allows us to conclude the following: For a given well known physical system we can select one of the two linear independent (formal) zero energy solutions to recognize the associated superpotential and use it to construct a new quantum mechanical system given by H_+. From the first equation in (1.1.4) it can be concluded that ψ_* must not have zeros in the domain of H_- to obtain a *priori*, a regular superpotential and a *posteriori*, a well defined superpartner in the same domain. If the selected state does not fulfill this condition we call the resulting system as a "virtual system" that makes no physical sense[1]. One typically refers to ψ_* as a seed state.

On the other hand, the action of operator A on other eigenfunctions of H_- produces eigenstates of H_+. To show this statement we use Eq. (1.1.2) to deduce the intertwining relations

$$AH_- = H_+A\,, \qquad A^\dagger H_+ = H_-A^\dagger\,. \tag{1.1.5}$$

Then, if ψ_λ is an eigenstate of H_- with eigenvalue λ we get

$$H_-\psi_\lambda = \lambda\psi_\lambda \qquad \Rightarrow \qquad H_+\left(A\psi_\lambda\right) = \lambda\left(A\psi_\lambda\right). \tag{1.1.6}$$

Of course these relations also work for the second linearly independent solution of the form

$$\widetilde{\psi}_\lambda = \psi_\lambda\int^x \frac{d\zeta}{(\psi_\lambda(\zeta))^2}\,, \tag{1.1.7}$$

which together with ψ_λ satisfies $W(\psi_\lambda, \widetilde{\psi_\lambda}) = 1$, where $W(.,.)$ is the Wronskian of two functions.

It is not difficult to show that operator $A^\dagger$ annihilates the state $A\widetilde{\psi_*} = 1/\psi_*$ which is one of the zero eigenvalue solutions of H_+[2]. Knowing this, one can say something about the spectrum of the latter Hamiltonian in correspondence with the behavior of the seed state. First, acting on physical states of H_-, operator A produces physical states of H_+ and second, if ψ_* is a physical

[1] Such virtual systems are useful in the context of higher order supersymmetry, see for example [Arancibia et al. (2013); Plyushchay (2017); Cariñena et al. (2018)].

[2] Note that with this method we only obtain one of the two linear independent solutions since $A\psi_* = 0$. To obtain the second linear independent solution we should extend the transformation by applying it to Jordan states.

state, then the spectrum of H_+ does not have this energy level. On the other hand, if the seed state is nonphysical, two things could happen: 1) $1/\psi_*$ is normalizable and system H_+ possesses an extra level and 2) $1/\psi_*$ is nonphysical and both systems are isospectral.

Finally, suppose we have a given number of differential operators denoted by I_i, each of them of a certain differential order d_i, which together with H_- span a symmetry algebra. In this context, it is not difficult to show the relation $[H_+, AI_iA^\dagger] = A[H_-, I_i]A^\dagger$, which means that when operator I_i is the integral of motion of H_-, then $A(I_i)A^\dagger$ (of differential order $d_i + 2$) is the integral for H_+ and the system is described (in the general case) by a certain nonlinear deformed algebra. In conclusion, the method not only serves to map states but also to obtain hidden integrals of motion of the generated system. This procedure is known as "Darboux-dressing".

We have the complete picture to extend our superpartner systems to supersymmetric quantum mechanics. We use our Hamiltonians and intertwiners to construct the 2×2 matrix operators

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}, \qquad \mathcal{Q}_1 = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \qquad \mathcal{Q}_2 = i\sigma_3\mathcal{Q}_1\,, \tag{1.1.8}$$

which satisfy the $\mathcal{N} = 2$[3] Poincaré superalgebra

$$[\mathcal{H}, \mathcal{Q}_a] = 0\,, \qquad \{\mathcal{Q}_a, \mathcal{Q}_b\} = 2\delta_{ab}\mathcal{H}\,, \tag{1.1.9}$$

with $\mathbb{Z}_2$ grading operator $\Gamma = \sigma_3$. In the case in which the state ψ_* (or $1/\psi_*$) is the physical ground state of H_- (H_+), then the spinor $(0, \quad \psi_*)^t$ (or $(1/\psi_*, \quad 0)^t$) is the supersymmetric invariant ground state of $\mathcal{H}$. Otherwise supersymmetry is spontaneously broken.

The method described in this paragraph is called the Darboux transformation and is the first step in an iterative process. In the next step we can produce a third new Hamiltonian by taking a seed state from H_+ and so on. The final form of the method after a several number of steps is called the "Darboux-Crum-Krein-Adler transformation" (DCKA transformation for short) whose details are explored in the following section.

1.2 DCKA transformation

Let us start with the equation

$$L\psi_\lambda = \lambda\psi_\lambda\,, \qquad L = -\frac{d^2}{dx^2} + V(x)\,, \tag{1.2.1}$$

corresponding to the eigenvalue problem of a Schrödinger type operator L. In this paragraph we treat Eq. (1.2.1) as a formal second order differential equation on some interval (a, b). Consider

[3] Here $\mathcal{N}$ indicates the number of true fermionic integrals.

now a set of solutions ψ_k corresponding to eigenvalues λ_k, $k = 1, \ldots, n$. We use them as seed states for our DCKA transformation and generate the new Schrödinger operator

$$\breve{L}\Psi_\lambda = \lambda\Psi_\lambda\,, \qquad \breve{L} = -\frac{d^2}{dx^2} + V(x) - 2\frac{d^2}{dx^2}\ln W(\psi_1, \ldots, \psi_n)\,. \tag{1.2.2}$$

If the set of the seed states is chosen in such a way that the generalized Wronskian of n functions

$$W(f_1(x), \ldots, f_n(x)) = \det\left(\frac{df_i(x)}{dx^{j-1}}\right)\,, \qquad i, j = 1 \ldots, n\,, \tag{1.2.3}$$

takes nonzero values on (a, b), then the potential of the generated system will also be nonsingular there. In general case, solutions of (1.2.2) are obtained from solutions of Eq. (1.2.1) as follows

$$\Psi_\lambda = \frac{W(\psi_1, \ldots, \psi_n, \psi_\lambda)}{W(\psi_1, \ldots, \psi_n)} = \mathbb{A}_n\psi_\lambda\,, \tag{1.2.4}$$

where $\mathbb{A}_n$ is the differential operator of order n defined recursively as

$$\mathbb{A}_n = A_n \ldots A_1\,, \qquad A_k = \mathbb{A}_{k-1}\psi_k\frac{d}{dx}\left(\frac{1}{\mathbb{A}_{k-1}\psi_k}\right)\,, \qquad k = 1, \ldots, n, \qquad \mathbb{A}_0 = 1\,. \tag{1.2.5}$$

Note that this operator is the natural generalization of (1.1.4) with $\hbar/\sqrt{2m} = 1$ and by the construction, $\ker\mathbb{A}_n = \text{span}\{\psi_1, \ldots, \psi_n\}$. Operator $\mathbb{A}_n$ and its Hermitian conjugate $\mathbb{A}_n^\dagger$ intertwine the operators L and $\breve{L}$,

$$\mathbb{A}_n L = \breve{L}\mathbb{A}_n\,, \qquad \mathbb{A}_n^\dagger\breve{L} = L\mathbb{A}_n^\dagger\,, \tag{1.2.6}$$

and satisfy relations

$$\mathbb{A}_n^\dagger\mathbb{A}_n = \prod_{k=1}^{n}(L - \lambda_k)\,, \qquad \mathbb{A}_n\mathbb{A}_n^\dagger = \prod_{k=1}^{n}(\breve{L} - \lambda_k)\,. \tag{1.2.7}$$

From the first equation in (1.2.7) one can find that $\ker\mathbb{A}_n^\dagger = \text{span}\{\mathbb{A}_n\widetilde{\psi}_1, \ldots, \mathbb{A}_n\widetilde{\psi}_n\}$. Similarly to (1.2.4), $\mathbb{A}_n^\dagger\Psi_\lambda = \psi_\lambda$ for $\Psi_\lambda \notin \ker\mathbb{A}_n^\dagger$, and

$$\mathbb{A}_n^\dagger(\widetilde{\mathbb{A}_n\widetilde{\psi}_k}) = \psi_k \in \ker\mathbb{A}_n\,.$$

Following the same approach as in the previous section, we can also use the pair L and $\breve{L}$ and their corresponding intertwining operators to construct an $\mathcal{N} = 2$ superextended system described by the 2×2 matrix Hamiltonian and the supercharges given by

$$\mathcal{H} = \begin{pmatrix} H_1 \equiv \breve{L} - \lambda_* & 0 \\ 0 & H_0 \equiv L - \lambda_* \end{pmatrix}, \qquad \mathcal{Q}_1 = \begin{pmatrix} 0 & \mathbb{A}_n \\ \mathbb{A}_n^\dagger & 0 \end{pmatrix}, \qquad \mathcal{Q}_2 = i\sigma_3\mathcal{Q}_1\,, \tag{1.2.8}$$

where λ_* is a constant associated with the energy levels of the seed states. These generators produce the (anti)commutation relations

$$[\mathcal{H},\mathcal{Q}_a]=0\,,\quad \{\mathcal{Q}_a,\mathcal{Q}_b\}=2\delta_{ab}P_n(\mathcal{H}+\lambda_*)\,, \tag{1.2.9}$$

which for $n=1$ correspond to an $\mathcal{N}=2$ Poincaré supersymmetry and for $n=2,3,\ldots$, we have a nonlinear deformation of the latter supersymmetry (here $P_n(\eta)$ represents a polynomial of order n in η). Examples of this kind of systems will be the main focus in Chap. 6.

The iterative nature of DCKA transformation allows us to derive some useful Wronskian identities for a given set of eigenstates. They are shown in Appendix A.

1.3 Jordan states and confluent Darboux transformation

Jordan states correspond to functions that are annihilated by a certain polynomial of the Schrödinger operator L [Correa et al. (2015)]. They were used, for example, in the construction of isospectral deformations of the harmonic oscillator [Cariñena and Plyushchay (2016b, 2017); Inzunza and Plyushchay (2018)], and also they can be used to construct solutions of the KdV equation [Correa and Fring (2016); Mateos Guilarte and Plyushchay (2017)]. These Jordan states will play a key role throughout this manuscript. This time we will focus our attention on building solutions of the fourth order differential equation $(L-\lambda_*)^2\chi_*=0$.

Let us take an eigenstate ψ_* with eigenvalue λ_* as a seed state of the Darboux transformation. The corresponding intertwining operators are

$$A_{\psi_*}=\psi_*\frac{d}{dx}\left(\frac{1}{\psi_*}\right)\,,\qquad A_{\psi_*}^{\dagger}=-\frac{1}{\psi_*}\frac{d}{dx}\psi_*\,. \tag{1.3.1}$$

According to Eq. (1.2.7), their product gives us the shifted Schrödinger operator $A_{\psi_*}^{\dagger}A_{\psi_*}=L-\lambda_*$, whose kernel is spanned by the linear independent states ψ_* and $\widetilde{\psi}_*$. The problem of constructing Jordan states reduces then to solving equations

$$A_{\psi_*}^{\dagger}A_{\psi_*}\Omega_*=(L-\lambda_*)\Omega_*=\psi_*\,,\qquad A_{\psi_*}^{\dagger}A_{\psi_*}\breve{\Omega}_*=(L-\lambda_*)\breve{\Omega}_*=\widetilde{\psi}_*\,. \tag{1.3.2}$$

Their solutions are given, up to a linear combination of ψ_* and $\widetilde{\psi}_*$, by particular solutions of respective inhomogeneous equations,

$$\Omega_*=\psi_*\int_a^x\frac{d\zeta}{\psi_*^2(\zeta)}\int_\zeta^b\psi_*^2(\eta)d\eta\,,\qquad \breve{\Omega}_*=\psi_*\int_a^x\frac{d\zeta}{\psi_*^2(\zeta)}\int_\zeta^b\psi_*(\eta)\widetilde{\psi}_*(\eta)d\eta\,. \tag{1.3.3}$$

Here the integration limits are chosen coherently with the region where the operator L is defined, and we have the relations

$$W(\psi_*,\Omega_*) = \int_x^b \psi_*^2 d\zeta\,, \qquad W(\psi_*,\breve{\Omega}_*) = \int_x^b \psi_* \widetilde{\psi}_* d\zeta\,, \tag{1.3.4}$$

which will be useful to produce nonsingular confluent Darboux transformations.

Let us inspect now the role of Jordan states (1.3.3) in DCKA transformation generated by a set of the seed states $\{\psi_n\}$. The intertwining operator (1.2.5) and Eqs. (1.2.6) and (1.3.2) give us the relations

$$\mathbb{A}_n\psi_* = (\breve{L}-\lambda_*)\mathbb{A}_n\Omega_*\,, \qquad \mathbb{A}_n\widetilde{\psi}_* = (\breve{L}-\lambda_*)\mathbb{A}_n\breve{\Omega}_*\,. \tag{1.3.5}$$

If the state ψ_* (or $\widetilde{\psi}_*$) is annihilated by $\mathbb{A}_n$, i.e., if the set of the seed states $\{\psi_n\}$ includes ψ_* (or $\widetilde{\psi}_*$), the function $\mathbb{A}_n\Omega_*$ (or $\mathbb{A}_n\breve{\Omega}_*$) will be an eigenstate of $\breve{L}$ with eigenvalue λ_* which is available to produce another Darboux transformation if we consider $\breve{L}$ as an intermediate system. Otherwise, the indicated function is a Jordan state of $\breve{L}$, and in correspondence with (1.3.3) we have

$$\mathbb{A}_n\Omega_* = (\mathbb{A}_n\psi_*)\int_a^x \tfrac{d\zeta}{(\mathbb{A}_n\psi_*)^2(\zeta)}\int_\zeta^b(\mathbb{A}_n\psi_*)^2(\eta)d\eta\,, \tag{1.3.6}$$

$$\mathbb{A}_n\breve{\Omega}_* = (\mathbb{A}_n\psi_*)\int_a^x \tfrac{d\zeta}{(\mathbb{A}_n\psi_*)^2(\zeta)}\int_\zeta^b(\mathbb{A}_n\psi_*)(\eta)\widetilde{\mathbb{A}_n\psi_*}(\eta)d\eta \tag{1.3.7}$$

up to a linear combination with $\mathbb{A}_n\psi_*$ and $\widetilde{\mathbb{A}_n\psi_*}$.

Having in mind that Jordan states appear naturally in the confluent generalized Darboux transformations [Correa et al. (2015)], one can consider directly a generalized Darboux transformation based on the following set of the seed states: $(\psi_1,\Omega_1,\ldots,\psi_n,\Omega_n)$. This generates a Darboux-transformed system which we denote by $\widehat{L}_{[2n]}$. The intertwining operator $\mathbb{A}_{2n}^{\Omega}$ as a differential operator of order $2n$ is built according to the same rule (1.2.5), but with the inclusion of Jordan states into the set of generating functions. By the construction, this operator annihilates the chosen $2n$ seed states, and one can show that

$$(\mathbb{A}_{2n}^{\Omega})^\dagger\mathbb{A}_{2n}^{\Omega} = \prod_{i=1}^n (L-\lambda_i)^2\,, \qquad \mathbb{A}_{2n}^{\Omega}(\mathbb{A}_{2n}^{\Omega})^\dagger = \prod_{i}^n(\widehat{L}_{[2n]}-\lambda_i)^2\,. \tag{1.3.8}$$

This, in particular, means that $\ker(\mathbb{A}_{2n}^{\Omega})^\dagger = \mathrm{span}\{\mathbb{A}_{2n}^{\Omega}\widetilde{\psi}_1, \mathbb{A}_{2n}^{\Omega}\breve{\Omega}_1,\ldots,\mathbb{A}_{2n}^{\Omega}\widetilde{\psi}_n,\mathbb{A}_{2n}^{\Omega}\breve{\Omega}_n\}$.

1.4 A three-dimensional example

Unlike the one-dimensional case, three-dimensional supersymmetric quantum mechanics does not have a unique generalization. Here, following [Cooper et al. (1995)], we begin with a charged massless Dirac particle in a four-dimensional Euclidian space. Assuming the presence of an external electromagnetic field, the Dirac's equation takes the form ($\hbar = e = c = 1$)

$$\gamma^\mu P_\mu \Psi = 0\,, \qquad P_\mu = -i\partial_\mu + A_\mu\,, \qquad \mu = 0,1,2,3\,, \tag{1.4.1}$$

where A_μ is the associated $U(1)$ gauge potential, the metric is just $\delta_{\mu\nu}$ and γ^μ are the Euclidean gamma matrices

$$\gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}, \qquad \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Gamma = \gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4.2}$$

Assuming that the gauge field does not depend on t, we can look for stationary solutions of the form $\Psi = e^{i\lambda t}\Phi(\boldsymbol{r})$. Expanding equation (1.4.1) in terms of this ansatz we get

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}\Phi = \mathcal{Q}_1\Phi\,, \qquad \mathcal{Q}_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}\cdot(\boldsymbol{\nabla} + i\boldsymbol{A}) - A_0 \\ -\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla} + i\boldsymbol{A}) - A_0 & 0 \end{pmatrix}. \tag{1.4.3}$$

By applying $\mathcal{Q}_1$ from the left, the Schrödinger equation $\lambda^2\Phi = \mathcal{H}\Phi$ is obtained, where

$$\mathcal{H} = (\boldsymbol{p} + \boldsymbol{A})^2 + A_0^2 + \Pi_+\,\boldsymbol{\sigma}\cdot(\boldsymbol{E} + \boldsymbol{B}) + \Pi_-\,\boldsymbol{\sigma}\cdot(\boldsymbol{E} - \boldsymbol{B})\,, \qquad \Pi_\pm = \frac{1}{2}(1 \pm \Gamma)\,, \tag{1.4.4}$$

is a Pauli Hamiltonian operator with $\boldsymbol{E} = -\boldsymbol{\nabla}A_0$ and $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$. The operator $\mathcal{Q}_1$, together with $\mathcal{Q}_2 = i\Gamma\mathcal{Q}_1$ and $\mathcal{H}$ produce a three-dimensional realization of the $\mathcal{N} = 2$ Poincaré supersymmetry with grading operator Γ. Furthermore, in the dual case $\boldsymbol{E} = \boldsymbol{B}$ (antidual case $\boldsymbol{E} = -\boldsymbol{B}$) the system possesses the nontrivial bosonic integral of motion $\boldsymbol{\mathcal{S}}^- = \Pi_-\,\boldsymbol{\sigma}$ ($\boldsymbol{\mathcal{S}}^+ = \Pi_+\,\boldsymbol{\sigma}$), and the commutation relations $[\mathcal{S}_i^-, \mathcal{Q}_a]$ ($[\mathcal{S}_i^+, \mathcal{Q}_a]$) with $a = 1, 2$, produce other 3 pairs of supercharges.

The system described by (1.4.4) has been studied properly in [Kirchberg et al. (2005)] where authors show that dual and anti-dual cases are the only ones that admit extensions of the Poincaré supersymmetry. In [Plyushchay and Wipf (2014)], the case of dual and anti-dual dyon (where the magnetic fields is due to a Dirac magnetic monopole) was considered, and it was shown that the system possesses the exceptional $\mathcal{N} = 4$ superconformal algebra $D(1, 2; \alpha)$ [Ivanov et al. (2003)].

1.5 Remarks

The tools considered in this chapter are going to be our principal methods for the rest of this Thesis. When we study one-dimensional potentials, our initial system for the DCKA transformation will always be a Newton-Hooke conformal invariant particle [Niederer (1973); Galajinsky (2010); Andrzejewski (2014); Galajinsky (2018)], the properties of which are described in Chaps. 2 and 3. Our principal target is to study the hidden symmetries of the nontrivial resulting systems and their supersymmetric extensions. This is the main content of Chaps. 4-7.

In Chap. 8 and 9 we study a three-dimensional generalization of the system introduced in Chap. 2, as well as its supersymmetric extensions. The resulting system will have superconformal symmetry that can be reinterpret in accordance with Sec. 1.4, but with a nontrivial gauge potential.

Chapter 2

One-dimensional conformal mechanics

As it was noted in the introduction, conformal invariance appears as a natural extension of the Poincaré symmetry of space-time, and involves the set of transformations that perform the change $g_{\mu\nu}dx^\mu dx^\nu \to \Omega(x)g_{\mu\nu}dx^\mu dx^\nu$, where $g_{\mu\nu}$ is the metric tensor and $\Omega(x)$ the conformal factor [Francesco et al. (1997); Sundermeyer (2014)]. The transformations that make this job (preservation of angles) are the space-time dilatations and the special conformal transformations. Some examples of space-time manifolds that allow this extension are the flat space (Minkowski), together with de Sitter (dS) and Anti de Sitter (AdS) spaces [Francesco et al. (1997); Nakahara (2003)].

The $\mathfrak{so}(2,1)$ conformal algebra is given by

$$[D,H]=iH\,,\qquad [D,K]=-iK\,,\qquad [K,H]=2iD\,, \tag{2.0.1}$$

being H, D and K the generators of time translations, dilatations and special conformal transformations, for details we recommend [Fedoruk et al. (2012)]. Taking the linear combinations

$$\mathcal{J}_0=\frac{1}{2}(\alpha^{-1}H+\alpha K)\,,\qquad \mathcal{J}_1=\frac{1}{2}(\alpha^{-1}H-\alpha K)\,,\qquad \mathcal{J}_2=D\,, \tag{2.0.2}$$

where α is a constant that compensates the dimensions of K and H, we obtain the Lorentz algebra in $(2+1)$-dimensional Minkowski space, with metric $\eta^{\mu\nu}=\text{diag}(-1,1,1)$, given by

$$[\mathcal{J}_\mu,\mathcal{J}_\nu]=-i\epsilon_{\mu\nu\rho}\mathcal{J}^\rho\,,\qquad \epsilon^{012}=1\,, \tag{2.0.3}$$

which, in turn, is isomorphic to the $\mathfrak{sl}(2\,,\mathbb{R})$ algebra, [Plyushchay (1993)],

$$[\mathcal{J}_0,\mathcal{J}_\pm]=\pm\mathcal{J}_\pm\,,\qquad [\mathcal{J}_-,\mathcal{J}_+]=2\mathcal{J}_0\,,\qquad \mathcal{J}_\pm=\mathcal{J}_1\pm i\mathcal{J}_2=\frac{1}{2\alpha}(H-\alpha^2K\pm i2\alpha D)\,. \tag{2.0.4}$$

This algebra has the automorphisms $\mathcal{J}_0\rightarrow\mathcal{J}_0\,,\,\mathcal{J}_\pm\rightarrow-\mathcal{J}_\pm\,$, and $\mathcal{J}_0\rightarrow-\mathcal{J}_0\,,\mathcal{J}_\pm\rightarrow-\mathcal{J}_\mp\,$, and the Casimir element is given by

$$\mathscr{F}=-\mathcal{J}_\mu\mathcal{J}^\mu=\mathcal{J}_0^2-\frac{1}{2}(\mathcal{J}_+\mathcal{J}_-+\mathcal{J}_-\mathcal{J}_+)=KH-D^2\,. \tag{2.0.5}$$

One of the objectives of this Thesis is to study models that have both this symmetry and some supersymmetric extensions of it. We also study possible nonlinear extensions of (super)conformal algebra, performed in terms of hidden symmetries.

This chapter is devoted to the analysis of classical and quantum conformal mechanical models. In Sec. 2.1 we review the theory behind the de Alfaro, Fubini, and Furlan (AFF) model, presented in [de Alfaro et al. (1976)], that looks for a well-defined one-dimensional quantum system with a conformal invariant ground state. In Sec. 2.2 we use the tools developed in Chap. 1 to construct the $\mathfrak{osp}(2,2)$ supersymmetric extension of the AFF model.

2.1 The de Alfaro, Fubini and Furlan model

Consider the one-dimensional system given by the action, [de Alfaro et al. (1976)],

$$I[q]=\int\mathcal{L}(q,\dot{q})dt\,,\quad \mathcal{L}=\frac{1}{2}\left(\dot{q}^2-\frac{g}{q^2}\right)\,, \tag{2.1.1}$$

where q takes values on the positive real line and has dimension $[q]=[\sqrt{t}]$, besides g is a dimensionless coupling constant which classically is assumed to be positive to avoid the "problem of fall to the center". This action could represent, for example, a Calogero model of two particles, but omitting the degree of freedom of the center of mass [Calogero (1969, 1972)].

On can show that the action (2.1.1) is invariant under time translations $t\rightarrow t+\alpha t$, space-time dilatations

$$x\rightarrow e^{\frac{\beta}{2}}x\,,\qquad t\rightarrow e^{\beta}t\,, \tag{2.1.2}$$

and the spacial conformal transformations

$$x\rightarrow\frac{x}{1-\gamma t}\,,\qquad t\rightarrow\frac{t}{1-\gamma t}\,, \tag{2.1.3}$$

where α, β and γ are parameters of the corresponding transformations. This symmetry is generated by the Hamiltonian H_g, the dilatations generator D, and generator of special conformal transformations K,

$$H_g = \frac{1}{2}(p^2 + \frac{g}{q^2}), \qquad D = \frac{1}{4}(qp + pq) - H_g t, \qquad K = \frac{1}{2}q^2 - 2Dt - H_g t^2, \tag{2.1.4}$$

where $p = \dot{q}$. These are the integrals of motion that satisfy the equation of the form $\frac{d}{dt}A = \frac{\partial A}{\partial t} + \{A, H\} = 0$ where $\{,\}$ denotes Poisson brackets. We often call objects of this type as "dynamical integrals", and in this case they obey the classical version of $\mathfrak{so}(2,1)$ algebra

$$\{D, H_g\} = H_g, \qquad \{D, K\} = -K, \qquad \{H_g, K\} = -2D, \tag{2.1.5}$$

and the Casimir invariant (2.0.5) takes the value $\mathscr{F} = \frac{1}{4}g$. The last relation in (2.1.4) gives us the solution of the corresponding Euler-Lagrange equation derived from (2.1.1),

$$q(t) = \sqrt{2(at^2 + 2bt + c)} = \sqrt{2\left(a(t + \frac{b}{a})^2 + \frac{\mathscr{F}}{a^2}\right)}, \tag{2.1.6}$$

where real-valued constants a, b and c correspond to the values of the integrals H_g, D and K, respectively (for a given initial configuration).

Note that in the case of $g = 0$, H_g takes the form of an object that looks like the Hamiltonian of a free particles, but is defined in the restricted domain $\mathbb{R}^+$. The notable difference between this system and the free particle H_f, which lives in $\mathbb{R}$, is that the latter has two additional integrals of motion, namely the momentum p and the Galileo boost generator $\chi = \tilde{q} - pt$, with $\tilde{q} \in \mathbb{R}$. They produce Heisenberg algebra and together with the generators $D_f = \frac{\chi P}{2}$ and $K_f = \frac{\chi^2}{2}$, leading to the Schrödinger symmetry [Niederer (1972); Duval and Horvathy (1994); Henkel and Unterberger (2003); Son (2008); Aizawa (2011)],

$$\{D_f, H_f\} = H_f, \qquad \{D_f, K_f\} = -K_f, \qquad \{H_f, K_f\} = -2D_f, \tag{2.1.7}$$

$$\{\chi, p\} = 1, \qquad \{H_f, p\} = \{K_f, \chi\} = 0 \qquad \{H_f, \chi\} = -p, \qquad \{K_f, p\} = \chi, \tag{2.1.8}$$

$$\{D_f, \chi\} = -\tfrac{1}{2}\chi, \qquad \{D_f, p\} = \tfrac{1}{2}p. \tag{2.1.9}$$

The model (2.1.1) has a problem at the quantum level, as we explain below: In the Schrödinger picture, the quantum version of the generators (2.1.4) are given by ($\hbar = 1$)

$$H_\nu = \frac{1}{2}\left(-\frac{d^2}{dq^2} + \frac{\nu(\nu+1)}{q^2}\right), \qquad D = \frac{1}{4i}\left(q\frac{d}{dq} + \frac{d}{dq}q\right), \qquad K = \frac{q^2}{2}, \tag{2.1.10}$$

where we have parameterized g as $\nu(\nu + 1)$. Obviously, the operators D and K are not integrals of motion, however they can be promoted to dynamical integrals, in the sense of the Heisenberg equation $\frac{dO}{dt} = \frac{\partial O}{\partial t} - i[O, H_\nu]$, by means of the unitary transformation

$$O \to {}_H O = e^{-iH_\nu t} O e^{iH_\nu t} \,. \tag{2.1.11}$$

This is the general recipe for moving from the Schrödinger picture to the Heisenberg picture, and in this last framework the operators D and K are changed for ${}_H D = D - H_\nu t$ and ${}_H K = K - 2Dt - H_\nu t^2$, respectively.

The Hamiltonian H_ν is self-adjoint for the cases in which $\nu \geq 0$ and admits self-adjoint extensions for $\nu \geq -1/2$, [Landau and Lifshitz (1965); Kirsten and Loya (2010)]. In these cases, H_ν has a continuous spectrum $E = \kappa^2/2$, with $\kappa \in \mathbb{R}$, in the domain $\{\psi \in L^2((0,\infty), dq) | \psi(0^+) = 0\}$ and the physical eigenstates are given by

$$\psi_\nu(q;\kappa) = \sqrt{q} J_{\nu+\frac{1}{2}}(\kappa q) \,, \tag{2.1.12}$$

where $J_\alpha(\zeta)$ is the Bessel function of the first kind

$$J_\alpha(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)} \zeta^{2n+\alpha} \,. \tag{2.1.13}$$

From here it is not difficult to show that the state $e^{i\alpha D}\psi_\nu(x;\kappa)$ corresponds to the energy $e^\alpha E$, which implies that the only scale-invariant solutions are those with zero energy eigenvalue, which in this case are given by the nonphysical solutions $q^{\nu+1}$ and $q^{-\nu}$, the first of which is not bounded at infinity and the second diverges when $q = 0$. This means that conformal symmetry is spontaneously broken at the quantum level.

To find a conformal invariant model with a well-defined ground state, the proposal in [de Alfaro et al. (1976)] is to consider the following change of the variables at the classical level

$$y(t) = \frac{q(t)}{\sqrt{u + vt + wt^2}} \,, \qquad d\tau = \frac{dt}{u + vt + wt^2} \,, \tag{2.1.14}$$

where $u > 0$, v and $w > 0$ are real constants with dimensions $[u] = 1$, $[v] = 1/t$ and $[w] = 1/t^2$, and $y > 0$. This is in fact related to a change of coordinates in an AdS$_2$ space, where t is not a good global coordinate, in contrast to τ [Michelson and Strominger (1999)]. Under the transformation (2.1.14), action (2.1.1) takes the form

$$\textstyle\int \mathcal{L}(y, y') d\tau + \frac{1}{4} \int d\tau \frac{d}{d\tau}[(v + 2wt(\tau)) q^2(t(\tau))] = I[y] + I_{surface} \,, \tag{2.1.15}$$

where $\mathcal{L}(y, y') = \frac{1}{2}(y'^2 - \frac{g}{y^2} - \omega^2 y^2)$, $y' = \frac{dy}{d\tau}$, and $\omega^2 = (4wu - v^2)/4$. Action $I[y] = \int \mathcal{L} d\tau$ is the so-called de Alfaro, Fubini and Furlan (AFF) model, from where we obtain the new time translation generator

$$\mathscr{H}_g = \frac{1}{2}\left(p^2 + \frac{g}{y^2} + \omega^2 y^2\right), \qquad p = y' . \tag{2.1.16}$$

The evolution parameter $\tau = \frac{1}{\omega}\text{acrtan}(\frac{v+2wt}{2\omega})$ varies in the finite interval $(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega})$, and new Hamiltonian (2.1.16) is conjugate to this good global time coordinate. As ω is a dimensionful parameter, $[\omega] = [1/t]$, (2.1.16) breaks the manifest scale invariance of the original system (2.1.1), and via such a basic mechanism the mass and length scales are introduced in holographic QCD (often referred to as "AdS/QCD") [Brodsky et al. (2015); Deur et al. (2015)].

In spite of the introduced scale, the action of the new system is conformal invariant as we will see now. The dilatation generator $\mathscr{D}$ and the conformal transformation generator $\mathscr{K}$ associated with the action $I[y]$ are given by the explicitly depending on time τ integrals

$$\mathscr{D} = \tfrac{1}{2}\left(yp\cos(2\omega\tau) + \left(\omega y^2 - \mathscr{H}_g\omega^{-1}\right)\sin(2\omega\tau)\right), \tag{2.1.17}$$

$$\mathscr{K} = \cos(2\omega\tau)\tfrac{y^2}{2} + \tfrac{\mathscr{H}_g}{\omega^2}\sin^2(\omega\tau) - \tfrac{\sin(2\omega\tau)}{2\omega}yp\,, \tag{2.1.18}$$

which generate the Newton-Hooke symmetry, [Niederer (1973); Galajinsky (2010); Andrzejewski (2014); Galajinsky (2018)],

$$\{\mathscr{H}_g, \mathscr{D}\} = -(\mathscr{H}_g - 2\omega^2\mathscr{K}), \qquad \{\mathscr{H}_g, \mathscr{K}\} = -2\mathscr{D}, \qquad \{\mathscr{D}, \mathscr{K}\} = -\mathscr{K}, \tag{2.1.19}$$

whose Casimir invariant is $\mathscr{F} = \mathscr{K}\mathscr{H}_g - \mathscr{D}^2 - \omega^2\mathscr{K}^2 = g/4$. Using Eqs. (2.1.17) and (2.1.18), one can find solution to the equation of motion of the system (2.1.15),

$$y^2(\tau) = \frac{2}{\omega^2}(a\sin^2(\omega\tau) + \omega b\sin(2\omega\tau) + \omega^2 c\cos(2\omega\tau)), \tag{2.1.20}$$

where $a > 0$, b and $c > 0$ are constants corresponding to the values of the integrals $\mathscr{H}_g$, $\mathscr{D}$ and $\mathscr{K}$, respectively, and obeying the relation $ac - b^2 - \omega^2 c^2 = g/4$. From the explicit form of the solution we see that it is periodic with the period $T = \pi/\omega$ not depending on the value of the coupling constant[1] g. The finite interval in which the evolution parameter τ varies corresponds to the period of the motion of the system (2.1.15), and one can consider τ as the compact evolution parameter that takes values on the closed interval $[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}]$ with identified ends.

As in the previous case, if one sets $g = 0$, $\mathscr{H}_g$ is formally reduced to the Hamiltonian of the harmonic oscillator, however the object is defined in $\mathbb{R}^+$ (we will call it as *half* harmonic oscillator). If we extend for this case the domain to the entire real line, i.e., we exchange $y \to \tilde{y} \in \mathbb{R}$, the resulting

[1] System given by Hamiltonian (2.1.16) is an isoperiodic deformation of the half-harmonic oscillator of frequency ω [Asorey et al. (2007)].

system $\mathscr{H}_{\rm os}$ has the additional dynamical integrals

$$\chi_\omega = \tilde{y}\cos(\omega\tau) - \frac{p}{\omega}\sin(\omega\tau)\,, \qquad P_\omega = \omega\tilde{y}\sin(\omega\tau) + p\cos(\omega\tau)\,. \tag{2.1.21}$$

They are identified as the initial conditions of the oscillatory motion and in terms of them, the Hamiltonian is read as $\mathscr{H}_{\rm os} = \frac{1}{2}(P_\omega^2 + \omega^2\chi_\omega^2)$. The generators (2.1.21), together with generators $\mathscr{H}_{\rm os}$, $\mathscr{D}_{\rm os} = \frac{\chi_\omega P_\omega}{2}$ and $\mathscr{K}_{\rm os} = \frac{1}{2}\chi_\omega^2$ produce the following Poisson brackets relations

$$\{\mathscr{H}_{\rm os}, \mathscr{D}_{\rm os}\} = -(\mathscr{H}_{\rm os} - 2\omega^2\mathscr{K}_{\rm os})\,, \quad \{\mathscr{H}_{\rm os}, \mathscr{K}_{\rm os}\} = -2\mathscr{D}_{\rm os}\,, \quad \{\mathscr{D}_{\rm os}, \mathscr{K}_{\rm os}\} = -\mathscr{K}_{\rm os}\,, \tag{2.1.22}$$

$$\{\chi_\omega, P_\omega\} = 1\,, \qquad \{\mathscr{H}_{\rm os}, P_\omega\} = \omega^2\chi_\omega\,, \qquad \{\mathscr{H}_{\rm os}, \chi_\omega\} = P_\omega \qquad \{\mathscr{K}_{\rm os}, \chi_\omega\} = 0\,, \tag{2.1.23}$$

$$\{\mathscr{K}_{\rm os}, P_\omega\} = \chi_\omega\,, \qquad \{\mathscr{D}_{\rm os}, \chi_\omega\} = -\tfrac{1}{2}\chi_\omega\,, \qquad \{\mathscr{D}_{\rm os}, P_\omega\} = \tfrac{1}{2}P_\omega\,. \tag{2.1.24}$$

Note that if instead to take $\mathscr{H}_{\rm os}$ we consider $\hat{\mathscr{H}} = \mathscr{H}_{\rm os} - \omega^2\mathscr{K}_{\rm os} = \frac{1}{2}P_\omega^2$ one gets the algebraic relations (2.1.7), (2.1.8) and (2.1.9), which mean that generators $\{\mathscr{H}_{\rm os}\,, \mathscr{D}_{\rm os}\,, \mathscr{K}_{\rm os}\,, \chi_\omega\,, P_\omega\}$ are just another basis for the Schrödinger symmetry. In fact by taking the limit $\omega \to 0$ we recover the free particle generators.

According to [Dirac (1949)], starting from a given symmetry algebra, one can freely designate a particular generator or a linear combination of generators as Hamiltonian, leading to different forms of dynamics. This terminology was introduced in the context of special relativity, however, the two models discussed above are good examples in nonrelativistic mechanics.

At the quantum level, the AFF Hamiltonian takes the form

$$\mathscr{H}_\nu = \frac{1}{2}\left(-\frac{d^2}{dy^2} + \omega^2 y^2 + \frac{g(\nu)}{y^2}\right)\,, \qquad g(\nu) = \nu(\nu+1)\,, \tag{2.1.25}$$

which as well as H_ν in (2.1.10), has a bounded spectrum restricted from below in the domain $\{\psi \in L^2((0,\infty), dy) | \psi(0^+) = 0\}$ for $\nu \geq -1/2$ [Falomir et al. (2002); Falomir and Pisani (2005)]. The normalized eigenstates of the system and its respective energy values are given by

$$\psi_{\nu,n}(y) = \sqrt{\frac{2n!\omega^{\nu+\frac{3}{2}}}{\Gamma(n+\nu+\frac{3}{2})}}\, y^{\nu+1} L_n^{(\nu+\frac{1}{2})}(\omega y^2) e^{-\frac{\omega y^2}{2}}\,, \qquad E_{\nu,n} = \omega(2n+\nu+\frac{3}{2})\,, \tag{2.1.26}$$

where

$$\mathcal{L}_n^{(\alpha)}(\eta) = \sum_{j=0}^{n} \frac{\Gamma(n+\alpha+1)}{\Gamma(j+\alpha+1)} \frac{(-\eta)^j}{j!(n-j)!}\,, \tag{2.1.27}$$

are the generalized Laguerre Polynomials. Note that g in (2.1.25) vanishes for $\nu = 0$ and for $\nu = -1$ (where we have some problems with boundary conditions) and for both cases $\mathscr{H}_\nu$ looks like an harmonic oscillator Hamiltonian. Indeed, the well known relations

$$H_{2n+1}(\eta)=(-1)^n2^{2n+1}L_n^{(1/2)}(\eta^2)\,,\qquad H_{2n}(\eta)=(-1)^n2^{2n}L_n^{(-1/2)}(\eta^2)\,, \tag{2.1.28}$$

where functions $H_n(\eta)$ are the Hermite polynomials, show us that in the first case eigenfunctions (2.1.26) become the odd eigenstates of the harmonic oscillator (vanishing at the origin), and in the second case, they take the form of the even eigenstates of the latter mentioned system (which do not vanish at $x=0$, thereby violating the imposed boundary conditions).

Instead to do a direct quantization of generators $\mathscr{K}$ and $\mathscr{D}$, it is worth it to consider complex combinations of them. In particular, in the Schrödinger picture we construct

$$\mathcal{C}_\nu^\pm=\mathscr{H}_\nu-2\omega^2\mathscr{K}\pm 2i\omega\mathscr{D}=\left(\mathscr{H}_\nu-\omega^2y^2\pm2\omega(y\tfrac{d}{dy}+\tfrac{1}{2})\right)\,, \tag{2.1.29}$$

which, together with $\mathscr{H}_\nu$, produce the commutator relations

$$[\mathscr{H}_\nu,\mathcal{C}_\nu^\pm]=\pm2\omega\mathcal{C}_\nu^\pm\,,\qquad [\mathcal{C}_\nu^-,\mathcal{C}_\nu^+]=4\omega\mathscr{H}_\nu\,, \tag{2.1.30}$$

and by using the identification $\mathscr{H}_\nu=2\omega\mathcal{J}_0$ and $\mathcal{C}_\nu=2\omega\mathcal{J}_\pm$, we recognize the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.0.4). On the Hilbert space of the AFF system, the states (2.1.26) correspond to an infinite-dimensional unitary irreducible representation of the $\mathfrak{sl}(2,\mathbb{R})$ algebra of the discrete type series $\mathcal{D}_\alpha^+$ with $\alpha=\frac{1}{2}\nu+\frac{3}{4}$, and the Casimir operator takes the value $\mathscr{F}_\nu=\mathcal{J}^\mu\mathcal{J}_\mu=-\alpha(\alpha-1)=\frac{3}{16}-\frac{1}{4}\nu(\nu+1)$, [Plyushchay (1993)].

As operators $\mathcal{C}_\nu^\pm$ are not integrals of motion, when we go to the Heisenberg picture, it is necessary to replace operators $\mathcal{C}^\pm$ by the dynamical integrals ${}_H\mathcal{C}^\pm=e^{\mp i2\omega t}\mathcal{C}^\pm$.

Relations (2.1.30) clearly show us that $\mathcal{C}_\nu^\pm$ are ladder operators which change the energy in $\pm2\omega$. Their action can be computed by means of the corresponding recurrence relations of Laguerre polynomials,

$$y\tfrac{d}{dy}L_n^\alpha(y)-yL_n^\alpha(y)+\alpha L_n^\alpha=(n+1)L_{n+1}^{\alpha-1}\,,\qquad \tfrac{d}{dy}L_n^\alpha(y)-L_n^\alpha(y)=-L_n^{\alpha+1}(y)\,,$$
$$\tfrac{d}{dy}L_n^\alpha(y)=-L_{n-1}^{\alpha+1}(y)\,,\qquad y\tfrac{d}{dy}L_n^\alpha(y)+\alpha L_n^\alpha(y)=(n+\alpha)L_n^{\alpha-1}(y)\,. \tag{2.1.31}$$

Using this we get

$$\mathcal{C}_\nu^-\psi_{\nu,n}=2\omega\sqrt{n(n+\nu+\tfrac{1}{2})}\,\psi_{\nu,n-1}\,, \tag{2.1.32}$$
$$\mathcal{C}_\nu^+\psi_{\nu,n}=2\omega\sqrt{(n+1)(n+\nu+\tfrac{3}{2})}\,\psi_{\nu,n+1}\,, \tag{2.1.33}$$

from where we see that the lowering operator $\mathcal{C}_\nu^-$ annihilates the ground state of the system. In next section we will show that these operators have their own origin in supersymmetric quantum mechanics.

2.2 The $\mathfrak{osp}(2|2)$ superconformal symmetry

The aim of this section is to construct an $\mathcal{N}=2$ super-extension of the AFF model (2.1.25). To this end, we apply the method introduced in Chap. 1.

For the construction let us use the ground state $\psi_{\nu,0}\propto y^{\nu+1}e^{-\omega y^2/2}$ as a seed state for the first order Darboux transformation. The associated intertwining operators are

$$A_\nu^- = \frac{1}{\sqrt{2}}\left(\frac{d}{dy}+\omega y-\frac{\nu+1}{y}\right)\,, \qquad A_\nu^+ = (A_\nu^-)^\dagger\,, \tag{2.2.1}$$

which produce

$$A_\nu^+A_\nu^- = \mathscr{H}_\nu-\omega(\nu+\frac{3}{2}) := H_-\,, \qquad A_\nu^-A_\nu^+ = \mathscr{H}_{\nu+1}-\omega(\nu+\frac{1}{2}) := H_+\,, \tag{2.2.2}$$

and intertwining relations take the form (1.1.5). Using the recurrence relations that Laguerre polynomials satisfy (2.1.31), one gets the explicit action of $A_\nu^\pm$ on eigenstates (2.1.26),

$$A_\nu^-\psi_{\nu,n} = -\sqrt{2n\omega}\,\psi_{\nu+1,n-1}\,, \qquad A_\nu^+\psi_{\nu+1,n-1} = -\sqrt{2n\omega}\,\psi_{\nu,n}\,. \tag{2.2.3}$$

With the help of (2.2.1) we can construct the matrix generators

$$\mathcal{H}_\nu^e = \begin{pmatrix} \mathscr{H}_{\nu+1}-\omega(\nu+1/2) & 0 \\ 0 & \mathscr{H}_\nu-\omega(\nu+3/2) \end{pmatrix}\,, \tag{2.2.4}$$

$$\mathcal{Q}_\nu^1 = \begin{pmatrix} 0 & A_\nu^- \\ A_\nu^+ & 0 \end{pmatrix}\,, \quad \mathcal{Q}_\nu^2 = i\Gamma\mathcal{Q}_\nu^1, \tag{2.2.5}$$

where $\Gamma=\sigma_3$ is the $\mathbb{Z}_2$ grading operator. These generators produce the Poincaré supersymmetry (1.1.9). Operator $\mathcal{H}_\nu^e$ has the spectrum $2\omega n$, $n=0,1,\ldots$, and the unique ground state $(0,\ \psi_{\nu,0})^t$ is annihilated by all generators in (2.2.4), therefore supersymmetry is in the exact phase.

On the other hand, the system (2.1.25) possesses the nonphysical solutions $\psi_{\nu,n}^-=\psi_{\nu,n}(iy)$ of the eigenvalues $-E_{n,\nu}$[2]. Then, instead of the ground state we could select the function $\psi_{\nu,0}^-\propto y^{\nu+1}e^{\omega y^2/2}$ as a seed state. The resulting intertwining operators are

$$B_\nu^- = \frac{1}{\sqrt{2}}\left(\frac{d}{dy}-\omega y-\frac{\nu+1}{y}\right)\,, \qquad B_\nu^+ = (B_\nu^-)^\dagger\,. \tag{2.2.6}$$

[2]The stationary Schrödinger equation $\mathscr{H}_\nu\psi_{\nu,n}=E\psi_{\nu,n}$ has a discrete symmetry group, and the transformation defined as $y\to iy$ and $E_{\nu,n}\to -E_{\nu,n}$ is an element of this group, see Chap. 7. The nonphysical eigenstates produced by the action of the mentioned group can be used in the Darboux transformations, resulting in new solvable systems.

Their products give us

$$B_\nu^+ B_\nu^- = \mathscr{H}_\nu + \omega(\nu + \frac{3}{2}) = H_- + \omega(2\nu+3)\,, \qquad B_\nu^- B_\nu^+ = \mathscr{H}_{\nu+1} + \omega(\nu + \frac{1}{2}) = H_+ + \omega(2\nu+1)\,, \tag{2.2.7}$$

and in terms of $H_\pm$ the intertwining relations take the form

$$B_\nu^- H_- = (H_+ - 2\omega) B_\nu^-\,, \qquad B_\nu^+ H_+ = (H_- + 2\omega) B_\nu^+\,. \tag{2.2.8}$$

Coherently with this, the action of operators $B_\nu^\pm$ on the eigenstates is

$$B_\nu^- \psi_{\nu,n} = -\sqrt{(2n+2\nu+3)\omega}\,\psi_{\nu+1,n}\,, \qquad B_\nu^+ \psi_{n,\nu+1} = -\sqrt{(2n+2\nu+3)\omega}\,\psi_{\nu,n}\,. \tag{2.2.9}$$

Just like we did with $A_\nu^\pm$, we can also use $B_\nu^\pm$ to build other matrix operators

$$\mathcal{H}_\nu^b = \begin{pmatrix} \mathscr{H}_{\nu+1} + \omega(\nu+1/2) & 0 \\ 0 & \mathscr{H}_\nu + \omega(\nu+3/2) \end{pmatrix}, \tag{2.2.10}$$

$$\mathcal{S}_\nu^1 = \begin{pmatrix} 0 & B_\nu^- \\ B_\nu^+ & 0 \end{pmatrix}, \qquad \mathcal{S}_\nu^2 = i\Gamma \mathcal{S}_\nu^1\,, \tag{2.2.11}$$

which again will satisfy the $\mathcal{N}=2$ Poincaré supersymmetry, but now, in the spontaneously broken phase[3]; the spectrum of $\mathcal{H}_\nu^b$ is $\omega(2n+2\nu+3)$, $n=0,1,\ldots$, and there is no physical eigenstate which is simultaneously annihilated by both odd operators $\mathcal{S}_\nu^a$. On the other hand one can reinterpret the object $\mathcal{H}_\nu^b$ as a linear combination of $\mathcal{H}_\nu^e$ and the nontrivial integral

$$\mathcal{R}_\nu = \frac{1}{2\omega}(\mathcal{H}_\nu^e - \mathcal{H}_\nu^b) = \frac{1}{2}\sigma_3 - (\nu+1)\,, \tag{2.2.12}$$

that plays the role of what will become an R symmetry generator.

Now, remember that the system (2.1.25) has the two second order ladder operators (2.1.29). Namely, they are constructed from $A_\nu^\pm$ and $B_\nu^\pm$ as follows

$$B_\nu^- A_\nu^+ = \mathcal{C}_{\nu+1}^+\,, \qquad A_\nu^- B_\nu^+ = \mathcal{C}_{\nu+1}^-\,, \tag{2.2.13}$$

$$A_\nu^+ B_\nu^- = \mathcal{C}_\nu^+\,, \qquad B_\nu^+ A_\nu^- = \mathcal{C}_\nu^-\,. \tag{2.2.14}$$

By using this structure together with the Eqs. (2.2.9) and (2.2.3) it is easy to check the relations (2.1.32). Also, by means of the Eqs. (2.2.2) and (2.2.7), in addition with the intertwining relations corresponding to $A_\nu^\pm$ and $B_\nu^\pm$, it is easy to derive the $\mathfrak{sl}(2,\mathbb{R})$ algebra (2.1.30).

Returning to the matrix operators subject, the relations (2.2.13)-(2.2.14) show us that the anti-

[3]The seed state $\psi_{\nu,0}^-$ is nonphysical and $1/\psi_{\nu,0}^-$ does not satisfy the boundary condition at the origin.

commutator between generators $\mathcal{S}_\nu^a$ and $\mathcal{Q}_\nu^a$ produces the even operators

$$\mathcal{G}_\nu^\pm = \begin{pmatrix} \mathcal{C}_{\nu+1}^\pm & 0 \\ 0 & \mathcal{C}_\nu^\pm \end{pmatrix} , \tag{2.2.15}$$

which are the corresponding super-extensions of the ladder operators of systems $\mathcal{H}_\nu^e$ and $\mathcal{H}_\nu^b$. Then, all together the generators $\{\mathcal{H}_\nu^e\,, \mathcal{G}_\nu^\pm\,, \mathcal{R}_\nu\,, \mathcal{Q}_\nu^a\,, \mathcal{S}_\nu^a\}$ satisfy the superalgebraic relations

$$[\mathcal{H}_\nu^e, \mathcal{R}_\nu] = [\mathcal{H}_\nu^e, \mathcal{Q}_\nu^a] = 0\,, \tag{2.2.16}$$

$$[\mathcal{H}_\nu^e, \mathcal{G}_\nu^\pm] = \pm 2\omega \mathcal{G}_\nu^\pm\,, \qquad [\mathcal{G}_\nu^-, \mathcal{G}_\nu^+] = 4\omega\left(\mathcal{H}_\nu^e - \omega^2 \mathcal{R}_\nu\right)\,, \tag{2.2.17}$$

$$[\mathcal{H}_\nu^e, \mathcal{S}_\nu^a] = -2i\omega\epsilon^{ab}\mathcal{S}_\nu^b\,, \qquad [\mathcal{R}_\nu, \mathcal{Q}_\nu^a] = -i\epsilon^{ab}\mathcal{Q}_\nu^b\,, \qquad [\mathcal{R}_\nu, \mathcal{S}_\nu^a] = -i\epsilon^{ab}\mathcal{S}_\nu^b\,, \tag{2.2.18}$$

$$[\mathcal{G}_\nu^-, \mathcal{Q}_\nu^a] = \omega(\mathcal{S}_\nu^a + i\epsilon^{ab}\mathcal{S}_\nu^b), \qquad [\mathcal{G}_\nu^+, \mathcal{Q}_\nu^a] = -\omega(\mathcal{S}_\nu^a - i\epsilon^{ab}\mathcal{S}_\nu^b)\,, \tag{2.2.19}$$

$$[\mathcal{G}_\nu^-, \mathcal{S}_\nu^a] = \omega(\mathcal{Q}_\nu^a - i\epsilon^{ab}\mathcal{Q}_\nu^b)\,, \qquad [\mathcal{G}_\nu^+, \mathcal{S}_\nu^a] = -\omega(\mathcal{Q}_\nu^a + i\epsilon^{ab}\mathcal{Q}_\nu^b)\,, \tag{2.2.20}$$

$$\{\mathcal{Q}_\nu^a, \mathcal{Q}_\nu^b\} = 2\delta^{ab}\mathcal{H}_\nu^e\,, \qquad \{\mathcal{S}_\nu^a, \mathcal{S}_\nu^b\} = 2\delta^{ab}(\mathcal{H}_\nu^e - 2\omega\mathcal{R}_\nu)\,, \tag{2.2.21}$$

$$\{\mathcal{Q}_\nu^a, \mathcal{S}_\nu^b\} = \delta^{ab}(\mathcal{G}_\nu^+ + \mathcal{G}_\nu^-) + i\epsilon^{ab}(\mathcal{G}_\nu^+ - \mathcal{G}_\nu^-)\,. \tag{2.2.22}$$

From here we realize that operators $\mathcal{G}^\pm$ and $\mathcal{S}_\nu^a$ are not integrals of motion, and in the Heisenberg picture we have instead the dynamical integrals ${}_H\mathcal{G}^\pm = e^{\mp 2\omega t}\mathcal{G}^\pm$ and ${}_H\mathcal{S}_\nu^a = e^{-i\sigma_3\omega t}\mathcal{S}_\nu^a$.

Superalgebra (2.2.16)-(2.2.22) is identified with the $\mathfrak{osp}(2|2)$ superconformal symmetry [Inzunza and Plyushchay (2018, 2019a,b)], and has the automorphism $f = f^{-1}$ given by the transformations $\mathcal{H}_\nu^e \to \mathcal{H}_\nu^e - 4\mathcal{R}_\nu = \mathcal{H}_\nu^b$, $\mathcal{R}_\nu \to -\mathcal{R}_\nu$, $\mathcal{G}_\nu^\pm \to \mathcal{G}_\nu^\pm$, $\mathcal{Q}_\nu^1 \to -\mathcal{S}_\nu^1$, $\mathcal{Q}_\nu^2 \to \mathcal{S}_\nu^2$, $\mathcal{S}_\nu^1 \to -\mathcal{Q}_\nu^1$ $\mathcal{S}_\nu^2 \to \mathcal{Q}_\nu^2$. Transformation f shows us what would happen with the superalgebra if we had chosen $\mathcal{H}_\nu^b$ instead of $\mathcal{H}_\nu^e$ as our time translation generator.

For future applications, we present the superalgebraic structure in terms of nilpotent fermionic operators

$$\mathcal{Q}_\nu = \begin{pmatrix} 0 & A_\nu \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{W}_\nu = \begin{pmatrix} 0 & 0 \\ B_\nu^+ & 0 \end{pmatrix}, \tag{2.2.23}$$

and its Hermitian counterpart, as follows,

$$[\mathcal{H}_\nu^e, \mathcal{G}_\nu^\pm] = \pm 2\omega\mathcal{G}_\nu^\pm\,, \qquad [\mathcal{G}_\nu^-, \mathcal{G}_\nu^+] = 4\omega(\mathcal{H}_\nu^e - \omega^2\mathcal{R}_\nu)\,, \tag{2.2.24}$$

$$[\mathcal{H}_\nu^e, \mathcal{W}_\nu] = -2\omega\mathcal{W}_\nu\,, \qquad [\mathcal{R}_\nu, \mathcal{Q}_\nu] = \mathcal{Q}_\nu\,, \qquad [\mathcal{R}_\nu, \mathcal{W}_\nu] = -\mathcal{W}_\nu\,, \tag{2.2.25}$$

$$\{\mathcal{Q}_\nu, \mathcal{Q}_\nu^\dagger\} = \mathcal{H}_\nu^e\,, \qquad \{\mathcal{W}_\nu, \mathcal{W}_\nu^\dagger\} = \mathcal{H}_\nu^e - 2\omega\mathcal{R}_\nu\,, \tag{2.2.26}$$

$$\{\mathcal{Q}_\nu, \mathcal{S}_\nu\} = \mathcal{G}_\nu^-\,, \qquad [\mathcal{G}_\nu^-, \mathcal{Q}_\nu^\dagger] = 2\omega\mathcal{W}_\nu\,, \qquad [\mathcal{G}_\nu^-, \mathcal{W}_\nu^\dagger] = 2\omega\mathcal{Q}_\nu\,, \tag{2.2.27}$$

in addition with corresponding Hermitian conjugate relations. In this base, we have the automorphism $\mathcal{H}^e_\nu \to \mathcal{H}^b_\nu$, $\mathcal{G}^\pm_\nu \to \mathcal{G}^\pm_\nu$, $\mathcal{R}_\nu \to -\mathcal{R}_\nu$, $\mathcal{Q}^a_\nu \leftrightarrow \mathcal{S}^a_\nu$.

As was for the bosinic case, one can use this structure as an approach to the study of the super-harmonic oscillator system, whose corresponding $\mathfrak{osp}(2|2)$ generators are

$$\{\mathcal{H}_{os}\,,\mathcal{G}^\pm\,,\mathcal{R}\,,\mathcal{Q}^a\,,\mathcal{S}^a\}=\{\mathcal{H}^e_\nu\,,\mathcal{G}^\pm_\nu\,,\mathcal{R}_\nu\,,\mathcal{Q}^a_\nu\,,\mathcal{S}^a_\nu\}|_{\nu=-1,y\to\tilde{y}}\,, \tag{2.2.28}$$

where $\tilde{y}\in\mathbb{R}$. The super-Hamiltonian $\mathcal{H}_{os}=\text{diag}(\mathscr{H}_{os}+\omega,\mathscr{H}_{os}-\omega)$ is a composition of two copies of an harmonic oscillator Hamiltonian, displaced from each other. On the other hand, from the perspective of the Darboux transformation, the seed states used to construct the fermionic operators $\mathcal{Q}^a$ and $\mathcal{S}^a$ are $\psi_0(\tilde{y})\propto e^{-\tilde{y}^2/2}$ and $\psi_0(i\tilde{y})\propto e^{\tilde{y}^2/2}$ respectively, see [Inzunza and Plyushchay (2018)], and as a consequence, both resulting systems $\mathcal{H}_{os}$ and $\mathcal{H}_{os}-4\mathcal{R}_0$ have the exact Poincaré supersymmetry, in contrast to the AFF case, since $\psi_0(i\tilde{y})^{-1}\propto\psi_0(\tilde{y})$. Finally, the intertwining operators are reduced to the usual harmonic oscillator ladder operators,

$$A^\pm|_{\nu=-1,y\to\tilde{y}}=a^\pm\,,\qquad B^\pm|_{\nu=-1,y\to\tilde{y}}=-a^\mp\,,\qquad a^\pm=\frac{1}{\sqrt{2}}\left(\omega\tilde{y}\mp\frac{d}{d\tilde{y}}\right)\,. \tag{2.2.29}$$

A radical difference with the super-extended AFF model is that for the super-harmonic oscillator system we can also build the additional operators

$$\mathcal{F}^\pm=\begin{pmatrix} a^\pm & 0\\ 0 & a^\pm\end{pmatrix}\,,\qquad \Sigma_1=\frac{1}{2}\sigma_1\,,\qquad \Sigma_2=-\frac{1}{2}\sigma_2\,, \tag{2.2.30}$$

that supplement the $\mathfrak{osp}(2|2)$ superalgebra with the (anti)-commutation relations

$$[\mathcal{H}_{os},\mathcal{F}^\pm]=\pm\omega\mathcal{F}^\pm\,,\qquad [\mathcal{F}^\mp,\mathcal{G}^\pm]=\mp\omega\mathcal{F}^\pm\,,\qquad [\mathcal{F}^-,\mathcal{F}^+]=\omega\mathbb{I}\,, \tag{2.2.31}$$

$$\{\Sigma_a,\Sigma_b\}=\tfrac{1}{2}\delta_{ab}\mathbb{I}\,,\qquad [\mathcal{H}_{os},\Sigma_a]=-i\omega\epsilon_{ab}\Sigma_b\,,\qquad [\mathcal{R},\Sigma_a]=i\epsilon_{ab}\Sigma_b\,, \tag{2.2.32}$$

$$\{\Sigma_a,\mathcal{Q}_b\}=\tfrac{1}{2}[\delta_{ab}(\mathcal{F}^++\mathcal{F}^-)-i\epsilon_{ab}(\mathcal{F}^+-\mathcal{F}^-)]\,, \tag{2.2.33}$$

$$\{\Sigma_a,\mathcal{S}_b\}=\tfrac{1}{2}[\delta_{ab}(\mathcal{F}^++\mathcal{F}^-)+i\epsilon_{ab}(\mathcal{F}^+-\mathcal{F}^-)]\,, \tag{2.2.34}$$

$$[\mathcal{F}^-,\mathcal{Q}_a]=\omega(\Sigma_a+i\epsilon_{ab}\Sigma_b)\,,\qquad [\mathcal{F}^+,\mathcal{Q}_a]=-\omega(\Sigma_a-i\epsilon_{ab}\Sigma_b)\,, \tag{2.2.35}$$

$$[\mathcal{F}^-,\mathcal{S}_a]=\omega(\Sigma_a-i\epsilon_{ab}\Sigma_b)\,,\qquad [\mathcal{F}^+,\mathcal{S}_a]=-\omega(\Sigma_a+i\epsilon_{ab}\Sigma_b)\,, \tag{2.2.36}$$

$$[\Sigma_a,\mathcal{F}^\pm]=[\Sigma_a,\mathcal{G}^\pm]=0\,. \tag{2.2.37}$$

Again, operators (2.2.30) do not commute with $\mathcal{H}_{\rm os}$ so in the Heisenberg picture we will have the dynamical integrals ${}_H\mathcal{F}^\pm=e^{\mp i\omega t},\mathcal{F}^\pm$ and ${}_H\Sigma^\pm=e^{-i\sigma_3\omega t}\Sigma^\pm$.

Note that generators $\{\mathcal{F}^\pm,\mathbb{I},\mathfrak{S}_a\}$ produce an ideal sub-supergebra, which we identify with the natural super-extension of Heisenberg's symmetry. In fact, the superalgebraic structure generated by (2.2.28), along with the Eqs. (2.2.31)-(2.2.37) is a semi-direct sum of this super-Heisenberg symmetry and the superalgebra $\mathfrak{osp}(2|2)$, corresponding to an $\mathcal{N}=2$ super-extension of the Schrödinger symmetry [Beckers and Hussin (1986); Beckers et al. (1987); Inzunza and Plyushchay (2018)] .

2.3 The zero frequency limit

In this paragraph we take the limit $\omega \to 0$ in supersymmetric generators introduced in last section, getting new $\mathcal{N}=2$ super-extended systems. We start with the supersymmetric AFF model generators, but now we consider the basis

$$\hat{\mathcal{D}}_\nu = \tfrac{i}{4\omega}(\mathcal{G}_\nu^- - \mathcal{G}_\nu^+)\,, \qquad \hat{\mathcal{K}}_\nu = \tfrac{1}{4\omega^2}(\mathcal{H}_\nu^e - \mathcal{G}_\nu^- - \mathcal{G}_\nu^-)\,, \qquad \hat{\mathcal{H}}_\nu = \tfrac{1}{2}(\mathcal{H}_\nu^e + \mathcal{H}_\nu^b) - \omega^2\hat{\mathcal{K}}_\nu\,, \tag{2.3.1}$$

$$\xi_\nu^a = \tfrac{1}{2}\epsilon^{ab}(\mathcal{Q}_\nu^a - \mathcal{S}_\nu^a)\,, \qquad \pi_\nu^a = \tfrac{1}{2\omega}\epsilon^{ab}(\mathcal{Q}_\nu^a + \mathcal{S}_\nu^a)\,, \qquad \mathcal{Z}_\nu = \tfrac{1}{2}\mathcal{R}_\nu\,. \tag{2.3.2}$$

The generators defined in this way satisfy

$$[\hat{\mathcal{D}}_\nu, \hat{\mathcal{H}}_\nu] = i\hat{\mathcal{H}}_\nu\,, \qquad [\hat{\mathcal{D}}_\nu, K_\nu] = -i\hat{\mathcal{K}}_\nu\,, \qquad [\hat{\mathcal{H}}_\nu, \hat{\mathcal{D}}_\nu] = -2\hat{\mathcal{D}}_\nu\,, \tag{2.3.3}$$

$$\{\zeta_\nu^a, \zeta_\nu^b\} = 2\hat{\mathcal{K}}_\nu\delta^{ab}\,, \qquad \{\pi_\nu^a, \pi_\nu^b\} = 2\hat{\mathcal{H}}_\nu\delta^{ab}\,, \qquad \{\zeta_\nu^a, \zeta_\nu^b\} = 2\hat{\mathcal{D}}_\nu\delta^{ab} + 2\epsilon^{ab}\mathcal{Z}_\nu\,, \tag{2.3.4}$$

$$[\hat{\mathcal{D}}_\nu, \pi_\nu^a] = \tfrac{i}{2}\pi_\nu^a\,, \quad [\hat{\mathcal{D}}_\nu, \xi_\nu^a] = -\tfrac{i}{2}\xi_\nu^a\,, \quad [\mathcal{Z}_\nu, \pi_\nu^a] = -\tfrac{i}{2}\epsilon_{ab}\pi_\nu^b\,, \quad [\mathcal{Z}_\nu, \xi_\nu^a] = -\tfrac{i}{2}\epsilon_{ab}\xi_\nu^b\,, \tag{2.3.5}$$

$$[\hat{\mathcal{H}}_\nu, \xi_\nu^a] = -i\pi_\nu^a\,, \qquad [\hat{\mathcal{K}}_\nu, \pi_a] = i\xi_\nu^a\,. \tag{2.3.6}$$

This is the usual way in which the superalgebra $\mathfrak{osp}(2,2)$ is presented for supersymmetric extensions of the conformal model (2.1.1) at the quantum level [Leiva and Plyushchay (2003); Fedoruk et al. (2012)]. So it is not a surprise that at the zero frequency limit we get

$$\hat{\mathcal{H}}_\nu|_{\omega=0} = \tfrac{1}{2}(p^2 + \tfrac{\nu^2}{y^2})\mathbb{I} + \tfrac{\nu}{2y^2}\sigma_3\,, \tag{2.3.7}$$

$$\hat{\mathcal{D}}_\nu|_{\omega=0} = \tfrac{1}{4i}\left(y\tfrac{d}{dy} + \tfrac{d}{dy}y\right)\mathbb{I} := \mathcal{D}\,, \qquad \hat{\mathcal{K}}_\nu|_{\omega=0} = \tfrac{y^2}{2}\mathbb{I} := \mathcal{K}\,, \tag{2.3.8}$$

$$\xi_\nu^a|_{\omega=0} = \tfrac{y}{\sqrt{2}}\sigma_a,\,, \qquad \pi_\nu^a|_{\omega=0} = \tfrac{1}{\sqrt{2}}\left(p\sigma_a - \tfrac{\nu+1}{y}\epsilon_{ab}\sigma_b\right)\,, \tag{2.3.9}$$

where $\mathbb{I} = \text{diag}(1,1)$ and $p = -i\frac{d}{dy}$.

We can repeat this procedure for the super-Schrödinger symmetry, which we have derived for the super-harmonic oscillator system. In this case the generators

$$\{\hat{\mathcal{H}}_\nu, \hat{\mathcal{D}}|_{\omega=0}, \hat{\mathcal{K}}|_{\omega=0}\,, \mathcal{Z}_\nu\,, \xi_\nu^a|_{\omega=0}\,, \pi_\nu^a|_{\omega=0}\}|_{\nu=-1, y\to\tilde{y}} = \{\mathcal{H}_0, \mathcal{D}, \mathcal{K}, \mathcal{Z}\,, \xi_a\,, \pi_a\} \tag{2.3.10}$$

reflect the superconformal symmetry of the super-extended free particle, which in turn includes the additional integrals $\Sigma_1 = \frac{1}{2}\sigma_1$, $\Sigma_2 = -\frac{1}{2}\sigma_2$ and

$$\mathcal{P} = \frac{i}{2}(\mathcal{F}^+ - \mathcal{F}^+)|_{\omega=0, y\rightarrow\tilde{y}} = -\frac{i}{\sqrt{2}}\frac{d}{d\tilde{y}}\mathbb{I}\,, \qquad \mathcal{X} = \frac{1}{2\omega}(\mathcal{F}^+ + \mathcal{F}^+)|_{\omega=0, y\rightarrow\tilde{y}} = \frac{\tilde{y}}{\sqrt{2}}\mathbb{I}\,. \tag{2.3.11}$$

Together, generators $\{\mathcal{H}_0, \mathcal{D}, \mathcal{K}, \mathcal{X}\,, \mathcal{P}\,, \xi_a\,, \pi_a\,, \Sigma_a\}$ produce the super-Schrödinger symmetry, now for the super-free particle system [Aizawa (2011); Inzunza and Plyushchay (2018)],

$$[\mathcal{D}, \mathcal{H}_0] = i\mathcal{H}_0\,, \qquad [\mathcal{D}, \mathcal{K}] = -i\mathcal{K}\,, \qquad [\mathcal{K}, \mathcal{H}_0] = 2i\mathcal{D}\,, \qquad [\mathcal{X}, \mathcal{P}] = \tfrac{1}{2}i\mathbb{I}\,, \tag{2.3.12}$$

$$[\mathcal{H}_0, \mathcal{X}] = -i\mathcal{P}\,, \qquad [\mathcal{K}, \mathcal{P}] = i\mathcal{X}, \qquad [\mathcal{D}, \mathcal{P}] = \tfrac{i}{2}\mathcal{P}\,, \qquad [\mathcal{D}, \mathcal{X}] = -\tfrac{i}{2}\mathcal{X}\,, \tag{2.3.13}$$

$$[\mathcal{D}, \pi_a] = \tfrac{i}{2}\pi_a\,, \quad [\mathcal{D}, \xi_a] = -\tfrac{i}{2}\xi_a\,, \quad [\mathcal{Z}, \pi_a] = -\tfrac{i}{2}\epsilon_{ab}\pi_b\,, \quad [\mathcal{Z}, \xi_a] = -\tfrac{i}{2}\epsilon_{ab}\xi_b\,, \tag{2.3.14}$$

$$[\mathcal{H}_0, \xi_a] = -i\pi_a\,, \qquad [\mathcal{K}, \pi_a] = i\xi_a\,, \tag{2.3.15}$$

$$[\mathcal{Z}, \Sigma_a] = \tfrac{i}{2}\epsilon_{ab}\Sigma_b\,, \qquad [\mathcal{P}, \pi_a] = -i\Sigma_a\,, \qquad [\mathcal{X}, \xi_a] = i\Sigma_a\,, \tag{2.3.16}$$

$$\{\Sigma_a, \pi_b\} = \delta_{ab}\mathcal{P}\,, \qquad \{\Sigma_a, \xi_b\} = \delta_{ab}\mathcal{X}\,, \qquad \{\Sigma_a, \Sigma_b\} = \tfrac{1}{2}\delta_{ab}\mathbb{I}\,, \tag{2.3.17}$$

$$\{\pi_a, \pi_b\} = 2\delta_{ab}\mathcal{H}_0\,, \qquad \{\xi_a, \xi_b\} = 2\delta_{ab}\mathcal{K}\,, \qquad \{\pi_a, \xi_b\} = 2\delta_{ab}\mathcal{D} + 2\epsilon_{ab}\mathcal{Z}\,. \tag{2.3.18}$$

2.4 Remarks

In this chapter we have considered one-dimensional conformal and an $\mathcal{N} = 2$ super-conformal mechanical models. In the bosonic case, there are many models that share the same conformal symmetry and some examples are the charged particle in a Dirac monopole background, Landau problem, rational Calogero models of N particles, geodesic motion in extreme black holes, the free particle and the harmonic oscillator in d dimensions, to name a few. In particular, some systems in various dimensions are especially rich thanks to the presence of conformal symmetry. Such is the case of the rational Calogero model, which is not only integrable, but also super-integrable, see [Correa et al. (2014)] and references therein. On the other hand, higher extensions of superconformal models are also a regular topic in scientific literature [Akulov and Pashnev (1983); Fubini and Rabinovici (1984); Ivanov et al. (1989); Donets et al. (2000); Fedoruk et al. (2012)].

In Sec 2.1 we have emphasized that the models (2.1.1) and (2.1.15) represent two different forms of dynamics associated with conformal algebra. In the next chapter we will show that there is a non-unitary mapping between both models. We call it the conformal bridge transformation, and it might be useful to obtain hidden symmetries for higher dimensional conformal invariant models.

Chapter 3

The conformal bridge

As we highlighted in the previous chapter, the conformal invariant systems with or without a harmonic potential are just two different dynamical phases of the same algebraic structure. However, there seems to be no direct relationship at the eigenstate level because one of the Hamiltonians is a non-compact generator, in contrast to the another Hamiltonian (the harmonically trapped one), which is compact. The objective of this chapter is to show that there is a non-unitary transformation that effectively maps one quantum mechanical system to the other but in an unorthodox way. To do so, let us start with algebra (2.0.1) without specifying a particular form of the generators. Then we construct the operators

$$\mathfrak{S} = e^{-\alpha K} e^{\frac{H}{2\alpha}} e^{i\ln(2)D}\,, \qquad \mathfrak{S}^{-1} = e^{-i\ln(2)D} e^{-\frac{H}{2\alpha}} e^{\alpha K}\,, \tag{3.0.1}$$

which from now on we will call as "conformal bridge", because by means of the Baker-Campbell-Hausdorff formula

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \dots\,, \tag{3.0.2}$$

one can show that

$$\mathfrak{S}(H)\mathfrak{S}^{-1} = \alpha\mathcal{J}_-\,, \qquad \mathfrak{S}(D)\mathfrak{S}^{-1} = -i\mathcal{J}_0\,, \qquad \mathfrak{S}(K)\mathfrak{S}^{-1} = -\tfrac{1}{\alpha}\mathcal{J}_+\,. \tag{3.0.3}$$

Here, $\mathcal{J}_0$ and $\mathcal{J}_\pm$ correspond to the generators of the $\mathfrak{sl}(2,\mathbb{R})$ algebra given in (2.0.4). Note that the transformed generators in (3.0.3) still satisfy the $\mathfrak{so}(2,1)$ symmetry, i.e., the transformation is an automorphism of the algebra.

Anyway, as we showed in the previous chapter, for the one-dimensional case H could represent the Hamiltonian of a free particle or that of the model (2.1.1) and on the other hand, $\mathcal{J}_0$ could be the Hamiltonian of an harmonic oscillator or that of the AFF model. Therefore, the conformal bridge transformation produces a mapping between these two forms of dynamics as follows: the formal eigenstates of $-iD$ are transformed into those of $\mathcal{J}_0$, and on the other hand, eigenstates of H are mapped to eigenstates of the lowering operator $\mathcal{J}_-$, which are in turns coherent states for $\mathcal{J}_0$. Of course, these statements remain true for any other higher-dimensional representation of the generators. In the following sections we explore the scope of this transformation with examples in one and two dimensions.

The content of this chapter is based on [Inzunza et al. (2020b)]. Here we only consider the basic elements and important results related with quantum mechanics examples, even though construction can be extended to the classical level, as we briefly discus in Sec. 3.4.

3.1 Free particle/ harmonic oscillator conformal bridge

Let us identify α with ω and H, K and D with the free particle conformal symmetry generators in the Schrödinger picture ($\hbar = m = 1$),

$$H = -\frac{1}{2}\frac{d^2}{dx^2} := H_0\,, \qquad D = \frac{1}{2i}\left(\frac{d}{dx}+\frac{1}{2}\right)\,, \qquad K = \frac{x^2}{2}\,. \tag{3.1.1}$$

Then, the conformal bridge takes the form

$$\mathfrak{S} = \exp\left(-\omega\frac{x^2}{2}\right)\exp\left(-\frac{1}{4\omega}\frac{d^2}{dx^2}\right)\exp\left(\ln\sqrt{2}\,\left(x\frac{d}{dx}+\frac{1}{2}\right)\right)\,, \tag{3.1.2}$$

besides $\mathcal{J}_0$ and $\mathcal{J}_\pm$ are the symmetry generators of the harmonic oscillator,

$$2\omega\mathcal{J}_0 = \frac{1}{2}\left(-\frac{d^2}{dx^2}+\omega^2x^2\right) := H_{os}\,, \qquad 2\omega\mathcal{J}_\pm = -(a^\pm)^2\,, \qquad a^\pm = \frac{1}{\sqrt{2}}\left(\omega x \mp \frac{d}{dx}\right)\,. \tag{3.1.3}$$

As we saw in the previous chapter, these operators are well defined for $x \in \mathbb{R}$, and there are many more symmetries for these systems. In the case of the free particle we have the momentum operator $p = -i\frac{d}{dx}$ and the Galilean boost, which in the Schrödinger picture at $t = 0$ is just x. These objects are connected with the Heisenberg generators $a^\pm$, appearing in (3.1.3), via the conformal bridge as follow

$$\mathfrak{S}(x)\mathfrak{S}^{-1} = \frac{1}{\omega}a^+\,, \qquad \mathfrak{S}(p)\mathfrak{S}^{-1} = -ia^-\,, \tag{3.1.4}$$

and, therefore, the transformation is also an automorphism of the Schrödinger symmetry.

For the sake of simplicity, we set $\omega = 1$ along the rest of this chapter.

The relation (the inverse Weierstrass transformation of a monomial)

$$e^{-\frac{1}{4}\frac{d^2}{dx^2}}x^n = 2^{-n}H_n(x)\,, \tag{3.1.5}$$

where $H_n(x)$ are the Hermite polynomials, implies that

$$\psi_n(x) = \frac{1}{\sqrt{\pi^{1/2}n!}} H_n(x) e^{-\frac{x^2}{2}} = (2\pi)^{\frac{1}{4}} \sqrt{2^n n!}\, \mathfrak{S} \left(\frac{x}{\sqrt{2}} \right)^n , \tag{3.1.6}$$

which are the eigenstates of H_{os} with eigenvalues $E_n = \omega(n + \frac{1}{2})$.

With the exception of 1 and x, which are annihilated by H_0, the functions x^n are not solutions of the free particle Schrödinger equation. They are in fact the rank n Jordan states of the zero energy, as the relations

$$(H)^j x^{2l} = \left(-\tfrac{1}{2}\right)^{2l} \tfrac{(2l)!(2l-1)!}{(2l-j)!(2l-1-j)!} x^{2(l-j)} , \tag{3.1.7}$$

$$(H)^j x^{2l+1} = \left(-\tfrac{1}{2}\right)^{2l+1} \tfrac{(2l)!(2l+1)!}{(2l-j+1)!(2l-j)!} x^{2(l-j)+1} , \tag{3.1.8}$$

valid for $j = 0, \ldots, l$, show us. If we set $j = l$ in these formulas, a subsequent application of H from the left produces 0 on the right hand side of the equation, and for this reason the nonlocal operator in (3.1.5) produces a polynomial of order n. Also these Jordan states satisfy the equation $2iDx^n = (n + 1/2)x^n$, so it is not really a surprise that after applying $\mathfrak{S}$, one obtains the harmonic oscillator eigenstates.

Now let us put our attention to the plane waves. We know that the functions $e^{i\frac{\kappa}{\sqrt{2}}x}$ are eigenstates of the free particle with energy $E = \kappa^2$, then the application of $\mathfrak{S}$ produces

$$\mathfrak{S} e^{i\frac{\kappa}{\sqrt{2}}x} = 2^{\frac{1}{4}} \exp\left(-\frac{x^2}{2} + \frac{\kappa^2}{4} + i\kappa x \right) = (2\pi)^{\frac{1}{4}} \sum_{n=0}^{\infty} \sqrt{\frac{2^n}{n!}}\, (ik)^n \psi_n(x) := \psi_{CS}(x, \kappa) . \tag{3.1.9}$$

These funtions are eigenstates of a^- and $(a^-)^2$, and up to a normalization factor, they are coherent states of H_{os}, [Schrödinger (1926); Klauder and Skagerstam (1985); Gazeau (2009)]. In fact, by applying the evolution operator $U = e^{iH_{os}t}$ one gets $\psi_{CS}(x, \kappa, t) = e^{\frac{it}{2}} \psi_{CS}(x, \kappa e^{it})$ which is a solution of the harmonic oscillator time-dependent Schrödinger equation. To obtain the overcomplete set of coherent states, an analytical continuation in κ must be done, allowing complex values.

From these results one can formulate a general recipe:

- Under the conformal bridge transformation, the formal states of the operator $2iD$, that are also the rank n Jordan state of zero energy, are mapped to normalizable eigenstates of J_0.
- Eigenstates of the Hamiltonian H are transformed into coherent states of the system J_0, which are eigenstates of J_- and conserve their form with time evolution. To have the overcomplete set, negative energy solutions (complex κ) should be also considered in this map.
- The conformal bridge also serves to map other symmetries from one system to another, as was the case for generators of the Heisenberg algebra (3.1.4).

In [Inzunza et al. (2020b)], it is shown how to obtain the squeezed states by applying the conformal bridge to Gaussian packets, and there is also an interesting discussion about the relation of this transformation and the Stone-von Newman theorem [Takhtadzhian (2008)]. Nevertheless, we prefer to not dwell with these details here.

3.2 Conformal bridge and the AFF model

Let us now set up H as the Hamiltonian operator of the two-body Calogero model with omitted center of mass degree of freedom

$$H=\frac{1}{2}\left(-\frac{d^2}{dx^2}+\frac{\nu(\nu+1)}{x^2}\right):=H_\nu\,, \tag{3.2.1}$$

besides D and K take the same form given in (3.1.1). With this choice, the conformal bridge is also labeled by ν,

$$\mathfrak{S}_\nu=\exp\left(-\frac{x^2}{2}\right)\exp\left(\frac{1}{4}\left(-\frac{d^2}{dx^2}+\frac{\nu(\nu+1)}{x^2}\right)\right)\exp\left(\ln\sqrt{2}\left(x\frac{d}{dx}+\frac{1}{2}\right)\right), \tag{3.2.2}$$

and operators J_0, $J_\pm$ are now

$$2\omega J_0=\frac{1}{2}\left(-\frac{d^2}{dx^2}+\frac{\nu(\nu+1)}{x^2}+x^2\right):=\mathcal{H}_\nu\,,\qquad 2\omega J_\pm=-(a^\pm)^2+\frac{\nu(\nu+1)}{2x^2}:=\mathcal{C}_\nu^\pm\,. \tag{3.2.3}$$

Following the recipe described above, we look for the zero energy solutions and its Jordan states, then consider the set of functions $x^{\nu+1+2n}$, $n=0,1,\ldots$. The function with $n=0$ represents a formal, diverging at infinity, eigenstate of the differential operator H_ν with $\nu\geq-1/2$ of eigenvalue $E=0$. For $n\geq1$ this functions are the Jordan states of rank n corresponding to the same eigenvalue of H_ν. The functions $x^{\nu+1+2n}$ are at the same time eigenstates of the operator $2iD$ with eigenvalues $\nu+2n+3/2$. The Jordan states with $n\geq1$ satisfy the relations

$$(H_\nu)^j x^{\nu+1+2n}=\frac{(-2)^j\Gamma(n+1)}{\Gamma(n+1-j)}\frac{\Gamma(n+\nu+3/2)}{\Gamma(n+\nu+3/2-j)}x^{\nu+1+2(n-j)}\,,\quad j=0,1,\ldots,n\,, \tag{3.2.4}$$

which can be proved by induction. Eq. (3.2.4) extends to the case $j=n+1$ giving $(H_\nu)^{n+1}x^{\nu+1+2n}=0$ due to appearing of a simple pole in the denominator.

Using relation (3.2.4) one can compute the conformal bridge transformation in functions $x^{2n+\nu+1}$, which gives

$$\mathfrak{S}_\nu\left(\frac{x}{\sqrt{2}}\right)^{\nu+1+2n}=2^{\frac{-1}{4}}(-1)^n\sqrt{n!\Gamma(n+\nu+3/2)}\;\psi_{\nu,n}(x)\,, \tag{3.2.5}$$

where eigenstates $\psi_{\nu,n}(x)$ correspond to (2.1.26) (with $\omega=1$ and $y=x$).

On the other hand, application of the operator $\mathfrak{S}_\nu$ to the eigenstates (2.1.12) (with $x = q$) of the system H_ν gives

$$\mathfrak{S}_\nu \psi_{\kappa,\nu}(\tfrac{1}{\sqrt{2}}x) = 2^{\frac{1}{4}} e^{-\frac{1}{2}x^2 + \frac{1}{4}\kappa^2} \sqrt{x} J_{\nu+1/2}(\kappa x) := \phi_\nu(x,\kappa)\,. \tag{3.2.6}$$

These are the coherent states of the AFF model [Perelomov (2012)], which satisfy

$$\mathcal{J}_- \phi_\nu(x,\kappa) = -\frac{1}{4}\kappa^2 \phi_\nu(x,\kappa)\,. \tag{3.2.7}$$

By allowing the $\kappa > 0$ to become a complex parameter z, coherent states can be constructed with complex eigenvalues of the operator $\mathcal{J}_-$. Application of the evolution operator e^{-itH_ν} to these states gives the time-dependent coherent states

$$\phi_\nu(x,z,t) = 2^{1/4}\sqrt{x}\mathcal{J}_{\nu+1/2}(z(t)x)e^{-x^2/2 + z^2(t)/4 - it}\,, \tag{3.2.8}$$

where $z(t) = ze^{-it}$. In the case of $\nu = 0$, these time-dependent coherent states of the AFF model are the odd Schrödinger cat states of the quantum harmonic oscillator [Dodonov et al. (1974)],

$$\phi_0(x,z,t) \propto e^{-\frac{x^2}{2} + \frac{z^2(t)}{4} - \frac{it}{2}} \sin(z(t)x)\,. \tag{3.2.9}$$

3.3 The conformal bridge and Landau problem

The generalization of the conformal bridge between free particle and harmonic oscillator to the $d-$dimensional case is straightforward; since the problem is separable in Cartesian coordinates, the conformal bridge operator is just $\mathfrak{S}(\boldsymbol{r}) = \mathfrak{S}(x_1)\ldots\mathfrak{S}(x_d)$. Each $\mathfrak{S}(x_i)$ touch only the objects constructed in terms of x_i and $p_i = -\frac{d}{dx_i}$, leaving invariant the other coordinates. On the other hand, as both systems posses the $\mathfrak{so}(d)$ symmetry, the generalized angular momentum tensor $M_{ij} = x_i p_j - x_j p_i$ remains intact after the similarity transformation.

On the other hand, a nontrivial relation between two-dimensional free particle, whose conformal symmetry generators are

$$H = \frac{1}{2}(p_x^2 + p_y^2)\,, \qquad D = \frac{1}{2}(xp_x + yp_y + 1)\,, \qquad K = \frac{1}{2}(x^2 + y^2)\,, \tag{3.3.1}$$

and the Landau problem in the symmetric gauge, can be established by means of the two-dimensional conformal bridge operator

$$\mathfrak{S}(x,y) = \mathfrak{S}(x)\mathfrak{S}(y)\,, \tag{3.3.2}$$

with $\mathfrak{S}(x)$ and $\mathfrak{S}(y)$ of the form (3.1.2). This is the subject of this section.

Consider now the Landau problem for a scalar particle on $\mathbb{R}^2$. In the symmetric gauge $\vec{A} = \frac{1}{2}B(-q_2, q_1)$, the Hamiltonian operator (in units $c = m = \hbar = 1$) is given by

$$H_{\rm L} = \frac{1}{2}\vec{\Pi}^2, \qquad \Pi_j = -i\frac{\partial}{\partial q_j} - eA_j\,, \qquad [\Pi_1, \Pi_2] = ieB\,. \tag{3.3.3}$$

Assuming $\omega_c = eB > 0$, this operator can be factorized as

$$H_{\rm L} = \omega_c(\mathcal{A}^+\mathcal{A}^- + \tfrac{1}{2})\,, \tag{3.3.4}$$

$$\mathcal{A}^\pm = \tfrac{1}{\sqrt{2\omega_c}}(\Pi_1 \mp i\Pi_2)\,, \qquad [\mathcal{A}^-, \mathcal{A}^+] = 1\,. \tag{3.3.5}$$

Setting $\omega_c = 2$, we can identify q_i with dimensionless variables $q_1 = x$, $q_2 = y$. Then we present $\mathcal{A}^\pm$ as linear combinations of the usually defined ladder operators $a_x^\pm$ and $a_y^\pm$ (the shape of which corresponds to the third equation in (3.1.3)), in terms of which we also define the operators $\mathcal{B}^\pm$,

$$\mathcal{A}^\pm = \frac{1}{\sqrt{2}}(a_y^\pm \pm ia_x^\pm)\,, \qquad \mathcal{B}^\pm = \frac{1}{\sqrt{2}}(a_y^\pm \mp ia_x^\pm)\,. \tag{3.3.6}$$

The operators $\mathcal{B}^\pm$ satisfy relation $[\mathcal{B}^-, \mathcal{B}^+] = 1$, and commute with $\mathcal{A}^\pm$. They are integrals of motion, and their non-commuting Hermitian linear combinations $\mathcal{B}^+ + \mathcal{B}^-$ and $i(\mathcal{B}^+ - \mathcal{B}^-)$ are identified with the coordinates of the center of the cyclotron motion. In terms of the ladder operators $a_x^\pm$, $a_y^\pm$ the Hamiltonian $H_{\rm L}$ takes the form of a linear combination of the Hamiltonian of the isotropic oscillator $H_{\rm iso}$ and angular momentum operator M,

$$H_{\rm L} = H_{\rm iso} - M\,, \qquad H_{\rm iso} = a_x^+a_x^- + a_y^+a_y^- + 1\,, \qquad M = xp_x - yp_y = -i(a_x^+a_y^- - a_y^+a_y^-)\,. \tag{3.3.7}$$

On the other hand, $H_{\rm iso}$ and M are presented in terms of $\mathcal{A}^\pm$ and $\mathcal{B}^\pm$ as follows,

$$M = \mathcal{B}^+\mathcal{B}^- - \mathcal{A}^+\mathcal{A}^-\,, \qquad H_{\rm iso} = \mathcal{B}^+\mathcal{B}^- + \mathcal{A}^+\mathcal{A}^- + 1\,, \tag{3.3.8}$$

and we have the commutation relations $[M, \mathcal{B}^\pm] = \pm\mathcal{B}^\pm$, $[M, \mathcal{A}^\pm] = \mp\mathcal{A}^\pm$. By taking into account the invariance of the angular momentum under similarity transformation, we find that its linear combination with the dilatation operator is transformed into the Hamiltonian of the Landau problem,

$$\mathfrak{S}(x,y)(2iD - M)\mathfrak{S}^{-1}(x,y) = H_{\rm L}\,. \tag{3.3.9}$$

Let us now introduce complex coordinate in the plane,

$$w = \frac{1}{\sqrt{2}}(y + ix)\,, \quad \text{and} \qquad \bar{w} = \frac{1}{\sqrt{2}}(y - ix)\,. \tag{3.3.10}$$

The elements of conformal algebra and angular momentum operator take then the form

$$H = -\frac{\partial^2}{\partial w \partial \bar{w}}\,, \quad D = \frac{1}{2i}\left(w\frac{\partial}{\partial w} + \bar{w}\frac{\partial}{\partial \bar{w}} + 1\right)\,, \quad K = w\bar{w}\,, \quad M = \bar{w}\frac{\partial}{\partial \bar{w}} - w\frac{\partial}{\partial w}\,, \tag{3.3.11}$$

and we find that the operator (3.3.2) generates the similarity transformations

$$\mathfrak{S}(x,y) w \mathfrak{S}^{-1}(x,y) = \mathcal{A}^+\,, \qquad \mathfrak{S}(x,y)\left(\tfrac{\partial}{\partial w}\right)\mathfrak{S}^{-1}(x,y) = \mathcal{A}^-\,, \tag{3.3.12}$$

$$\mathfrak{S}(x,y) \bar{w} \mathfrak{S}^{-1}(x,y) = \mathcal{B}^+\,, \qquad \mathfrak{S}(x,y)\left(\tfrac{\partial}{\partial \bar{w}}\right)\mathfrak{S}^{-1}(x,y) = \mathcal{B}^-\,, \tag{3.3.13}$$

$$\mathfrak{S}(x,y)\left(w\tfrac{\partial}{\partial w}\right)\mathfrak{S}^{-1}(x,y) = \mathcal{A}^+\mathcal{A}^-\,, \qquad \mathfrak{S}(x,y)\left(\bar{w}\tfrac{\partial}{\partial \bar{w}}\right)\mathfrak{S}^{-1}(x,y) = \mathcal{B}^+\mathcal{B}^-\,. \tag{3.3.14}$$

Observe that each pair of relations in (3.3.12) and (3.3.13) has a form similar as the one-dimensional transformation (3.1.4), where, however, the coordinate and momentum are Hermitian operators.

Simultaneous eigenstates of the operators $w\frac{\partial}{\partial w}$ and $\bar{w}\frac{\partial}{\partial \bar{w}}$, which satisfy the relations $w\frac{\partial}{\partial w}\phi_{n,m} = n\phi_{n,m}$ and $\bar{w}\frac{\partial}{\partial \bar{w}}\phi_{n,m} = m\phi_{n,m}$ with $n, m = 0, 1, \ldots$, are

$$\phi_{n,m}(x,y) = w^n(\bar{w})^m = 2^{-(n+m)/2}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}(i)^{n-m+l-k}y^{k+l}x^{n+m-k-l}\,, \tag{3.3.15}$$

where the binomial theorem has been used. Employing Eq. (3.3.11) we find that

$$M\phi_{n,m} = (m-n)\phi_{n,m}\,, \qquad 2iD\phi_{n,m} = (n+m+1)\phi_{n,m}\,, \tag{3.3.16}$$

$$K\phi_{n,m} = \phi_{n+1,m+1}\,, \qquad H\phi_{n,m} = -nm\phi_{n-1,m-1}\,. \tag{3.3.17}$$

The last equality shows that $\phi_{0,m}$ and $\phi_{n,0}$ are the zero energy eigenstates of the two-dimensional free particle, while the $\phi_{n,m}$ with $n, m > 0$ are the Jordan states corresponding to the same zero energy value. Application of the operator $\mathfrak{S}(x,y)$ to these functions yields

$$\mathfrak{S}(x,y)\phi_{n,m}(x,y) = 2^{2(n+m)+\frac{1}{2}}e^{-\frac{(x^2+y^2)}{2}}H_{n,m}(y,x) = \psi_{n,m}(x,y)\,, \tag{3.3.18}$$

where

$$H_{n,m}(y,x) = 2^{-(n+m)}\sum_{k=0}^{n}\sum_{l=0}^{m}\binom{n}{k}\binom{m}{l}(i)^{n-m+l-k}H_{k+l}(y)H_{n+m-k-l}(x)\,, \tag{3.3.19}$$

are the complex Hermite polynomials, see [Ghanmi (2012)]. These functions are eigenstates of the operators $H_{\rm L}$, M and $H_{\rm iso}$,

$$H_{\rm L}\psi_{n,m} = (n+\tfrac{1}{2})\psi_{n,m}\,, \qquad M\psi_{n,m} = (m-n)\psi_{n,m}\,, \tag{3.3.20}$$

$$H_{\rm iso}\psi_{n,m} = (n+m+1)\psi_{n,m}\,, \tag{3.3.21}$$

and we note that $\psi_{n,n}$ is rotational invariant.

Eqs. (3.3.12), (3.3.13), and (3.3.17) show that the operators $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ act as the ladder operators for the indexes n and m, respectively, while the operators $\hat{\mathcal{J}}_{\pm} = -\frac{1}{2}((a_x^{\pm})^2 + (a_y^{\pm})^2)$, increase or decrease simultaneously n and m by one.

Application of the operator $\mathfrak{S}(x,y)$ to exponential functions of the most general form $e^{\alpha w+\beta\bar{w}}$ with $\alpha, \beta \in \mathbb{C}$ gives here, similarly to the one-dimensional case, the coherent states of the Landau problem as well of the isotropic harmonic oscillator,

$$\begin{aligned} \psi_{\mathrm{L}}(x,y,\alpha,\beta) &= \mathfrak{S}(x,y)e^{\frac{1}{\sqrt{2}}((\alpha+\beta)y+i(\alpha-\beta)x)} = \sqrt{2}e^{-\frac{(x^2+y^2)}{2}+(\alpha+\beta)y+i(\alpha-\beta)x-\alpha\beta} \\ &= \textstyle\sum_{n=0}^{\infty}\sum_{l=0}^{n}\frac{1}{n!}\binom{n}{l}\alpha^l\beta^{n-l}\psi_{l,n-l}(x,y)\,. \end{aligned} \tag{3.3.22}$$

Applying to them, in particular, the evolution operator $e^{-itH_{\mathrm{L}}}$, we obtain the time dependent solution to the Landau problem,

$$\psi_{\mathrm{L}}(x,y,\alpha,\beta,t) = e^{-\frac{it}{2}}\psi_{\mathrm{L}}(x,y,\alpha e^{-it},\beta)\,, \tag{3.3.23}$$

whereas under rotations these states transform as

$$e^{i\varphi M}\psi_{\mathrm{L}}(x,y,\alpha,\beta) = \psi_{\mathrm{L}}(x,y,\alpha e^{-i\varphi},\beta e^{i\varphi})\,. \tag{3.3.24}$$

As the function $e^{\alpha w+\beta\bar{w}}$ is a common eigenstate of the differential operators $\frac{\partial}{\partial w}$ and $\frac{\partial}{\partial \bar{w}}$ with eigenvalues α and β, respectively, then our transformation yields

$$\mathcal{A}^{-}\psi_{\mathrm{L}}(x,y,\alpha,\beta) = \alpha\psi_{\mathrm{L}}(x,y,\alpha,\beta)\,, \qquad \mathcal{B}^{-}\psi_{\mathrm{L}}(x,y,\alpha,\beta) = \beta\psi_{\mathrm{L}}(x,y,\alpha,\beta)\,, \tag{3.3.25}$$

that provides another explanation why the wave functions (3.3.22) are the coherent states for the planar harmonic oscillator as well as for the Landau problem.

3.4 Remarks

Note that if we apply $\mathfrak{S}$ from the right to the equations in (3.0.3), we get intertwining relations of the form

$$\mathfrak{S}H = \alpha\mathcal{J}_{-}\mathfrak{S}\,, \qquad \mathfrak{S}D = -i\mathcal{J}_0\mathfrak{S}\,, \qquad \mathfrak{S}K = -\tfrac{1}{\alpha}\mathcal{J}_{+}\mathfrak{S}\,, \tag{3.4.1}$$

which are very similar to the usual intertwining relations of the supersymmetric quantum mechanics, however, here the "intertwining operators" are nonlocal and non-unitary operators described by an infinite series of powers of second derivatives. This is the reason why non-normalizable functions are mapped to bound states and vice-versa.

One can go further and try to obtain a classical version of the conformal bridge by using "Hamiltonian flows" of the form

$$f(\alpha)=\exp(\alpha F)\star f:=f+\sum_{n=1}^{\infty}\frac{\alpha^n}{n!}\{F,\{\ldots,\{F,f\underbrace{\}\ldots\}\}}_{n}=:T_F(\alpha)(f)\,. \tag{3.4.2}$$

where F represents a symmetry generator, α is a transformation parameter and $f=f(q,p)$ corresponds to a function on phase space. The composed transformation

$$T_{\beta\alpha\gamma}:=T_K(\beta)\circ T_{H_0}(\alpha)\circ T_D(\gamma)=T_K(\beta)\circ T_D(\gamma)\circ T_{H_0}(2\alpha)\,, \tag{3.4.3}$$

with the election

$$\alpha=\frac{i}{2}\,,\qquad \beta=-i\,,\qquad \gamma=-\ln 2\,, \tag{3.4.4}$$

is the classical analog, the operator (3.0.1) (generators should be fixed at $t=0$). This is a complex canonical transformation, so one should expect that there is some relation with $\mathcal{PT}$ symmetry [Dorey et al. (2001); Bender (2007); El-Ganainy et al. (2018)]. Actually, in the case of the classical bridge between free particle and harmonic oscillator, the function $T_{iD}(\tau)(x)$, i.e, the "imaginary" flux of x due to D, is the one that is mapped to a complex combination of position and momentum of the harmonic oscillator. Besides, the transformation of the free particle trajectory does not have a clear interpretation.

Finally it is worth emphasizing that Hamiltonians of the form xp have found application in mathematics, namely, in the study of Riemann hypothesis, see [Connes (1999); Berry and Keating (1999); Regniers and Van der Jeugt (2010); Sierra and Rodriguez-Laguna (2011); Bender et al. (2017)].

Chapter 4

Hidden bosonized superconformal symmetry

It is well known that the one-dimensional quantum harmonic oscillator system is characterized by a bosonized superconformal symmetry [de Crombrugghe and Rittenberg (1983); Balantekin et al. (1988); Cariñena and Plyushchay (2016a); Bonezzi et al. (2017)], however, the origin of this symmetry had not been clarified, until the article [Inzunza and Plyushchay (2018)] appeared, and this chapters summarize the main results of that work. We show that this supersymmetry can be derived by applying a nonlocal transformation (of the nature of a Foldy-Wouthuysen transformation) to a particular super-extended system. The latter system itself can not be obtained directly from a given superpotential, i.e., is outside of the Darboux transformation scheme, however its corresponding generators are, in fact, linear combinations of the $\mathfrak{osp}(2|2)$ symmetry generators that the super-harmonic oscillator system possesses. They were introduced in Chap 2, Sec. 2.2. The mentioned system can also be obtained by taking a certain limit in an isospectral deformation of the harmonic oscillator, produced with a confluent Darboux transformation.

4.1 Dimensionless generators

So far we have turned our attention to Hamiltonian operators of the form

$$H = \frac{1}{2}\left(-\frac{d^2}{dy^2} + V(y)\right), \qquad [y] = \sqrt{t}\,, \tag{4.1.1}$$

where $V(y)$ is the potential of the harmonic oscillator or that of the AFF model. However, when we are working with the DCKA transformation, it is worth using dimensionless operators. For this reason, we consider the change of variables $x = \sqrt{\omega}y$, in term of which the Hamiltonian (4.1.1) takes the form $H = \frac{\omega}{2}L$, where depending on the situation we are looking at, the operator L as well as its eigenstates and its spectrum could be

$$L_{\rm os}=-\frac{d^2}{dx^2}+x^2\,,\qquad \psi_n(x)=\frac{H_n(x)e^{-\frac{x^2}{2}}}{\sqrt{\pi^{1/2}n!}}\,,\qquad E_n=2n+1\,,\tag{4.1.2}$$

or

$$L_\nu=-\tfrac{d^2}{dx^2}+x^2+\tfrac{\nu(\nu+1)}{x^2}\,,\tag{4.1.3}$$

$$\psi_{\nu,n}(x)=\sqrt{\tfrac{2n!}{\Gamma(n+\nu+\frac{3}{2})}}\;x^{\nu+1}L_n^{(\nu+\frac{1}{2})}(x^2)e^{-\frac{x^2}{2}}\,,\qquad E_{\nu,n}=4n+2\nu+3\,.\tag{4.1.4}$$

It is also convenient to redefine the first order ladder operators of the harmonic oscillator as

$$a^\pm=\mp\frac{d}{dx}+x,\qquad [a^+,a^-]=2\,,\qquad [L_{\rm os},a^\pm]=\pm 2a^\pm\,.\tag{4.1.5}$$

and the same for the second order ladder operators of the AFF system which are now given by

$$\mathcal{C}_\nu^\pm=-(a^\pm)^2+\tfrac{\nu(\nu+1)}{x^2}\,,\tag{4.1.6}$$

$$[L_\nu,\mathcal{C}_\nu^\pm]=\pm\Delta E\mathcal{C}_\nu^\pm\,,\qquad [\mathcal{C}_\nu^-,\mathcal{C}_\nu^+]=8L_\nu\,,\qquad \Delta E=4\,.\tag{4.1.7}$$

In the Heisenberg picture operators $a^\pm$ and $\mathcal{C}_\nu^\pm$ are respectively replaced by ${}_Ha^\pm=e^{\mp 2it}a^\pm$ and ${}_H\mathcal{C}^\pm=e^{\mp 4it}\mathcal{C}^\pm$, which will be dynamical integrals of motion for the corresponding systems.

4.2 Hidden superconformal symmetry of the quantum harmonic oscillator

In this paragraph we show how the aforementioned superconformal symmetry appears for the one-dimensional bosonic harmonic oscillator system, the Hamiltonian of which is given by (4.1.2).

As the ladder operators (4.1.5) anticommute with reflection operator $\mathcal{R}$ defined by $\mathcal{R}^2=1$, $\mathcal{R}x=-x\mathcal{R}$, and their anti-commutator produces $\{a^+,a^-\}=2L_{\rm os}$, it is clear that if one set $\mathcal{R}$ as the $\mathbb{Z}_2$-grading operator, then:

- $a^\pm$ are identified as odd, fermionic generators,
- $L_{\rm os}$ and quadratic operators $(a^\pm)^2$ are identified as even, bosonic generators since $[\mathcal{R},L_{\rm os}]=[\mathcal{R},(a^\pm)^2]=0$.

Consider now the dynamical integrals of motion

$$J_0=\frac{1}{4}L_{\rm os},\qquad J_\pm=\frac{1}{4}e^{\mp 4it}(a^\pm)^2\,,\qquad \alpha_\pm=\frac{1}{4}e^{\mp i2t}a^\pm\,.\tag{4.2.1}$$

They produce the (anti)commutator relations

$$[J_0, J_\pm] = \pm J_\pm\,, \qquad [J_-, J_+] = 2J_0\,, \tag{4.2.2}$$

$$\{\alpha_+, \alpha_-\} = \tfrac{1}{2}J_0\,, \qquad \{\alpha_\pm, \alpha_\pm\} = \tfrac{1}{2}J_\pm\,, \tag{4.2.3}$$

$$[J_0, \alpha_\pm] = \pm\tfrac{1}{2}\alpha_\pm, \qquad [J_\pm, \alpha_\mp] = \mp\alpha_\pm\,. \tag{4.2.4}$$

The superalgebra (4.2.2), (4.2.3), (4.2.4) describes the hidden superconformal $\mathfrak{osp}(1|2)$ symmetry of the quantum harmonic oscillator [de Crombrugghe and Rittenberg (1983); Balantekin et al. (1988)]. The set of even integrals J_0, $J_\pm$ generates the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra (4.2.2), and relations (4.2.4) mean that fermionic generators $\alpha_\pm$ form a spin-1/2 representation of this Lie subalgebra. One can extend this superalgebra by introducing the fermionic operators

$$\beta_\pm = i\mathcal{R}\alpha_\pm\,. \tag{4.2.5}$$

which give rise to the additional super-algebraic relations

$$[J_0, \beta_\pm] = \pm\tfrac{1}{2}\beta_\pm\,, \qquad [J_\pm, \beta_\mp] = \mp\beta_\pm\,, \tag{4.2.6}$$

$$\{\beta_\pm, \beta_\pm\} = \tfrac{1}{2}J_\pm\,, \qquad \{\beta_+, \beta_-\} = \tfrac{1}{2}J_0\,, \qquad \{\alpha_\pm, \beta_\mp\} = \mp\tfrac{i}{2}Z\,, \tag{4.2.7}$$

$$[Z, \alpha_\pm] = \tfrac{i}{2}\beta_\pm\,, \qquad [Z, \beta_\pm] = -\tfrac{i}{2}\alpha_\pm\,, \tag{4.2.8}$$

where

$$Z = -\frac{1}{4}\mathcal{R}\,. \tag{4.2.9}$$

However this extension is nonlocal since $\mathcal{R}$ can be presented as $\mathcal{R} = \sin\big(\frac{\pi}{2}L_{\rm os}\big)$.

We will show soon that superalgebra given by Eqs. (4.2.2)-(4.2.4) and (4.2.6)-(4.2.8) is just another basis for the $\mathfrak{osp}(2|2)$ superconformal algebra presented in Chap. 2, Sec 2.2.

4.3 Extended system with super-Schrödinger symmetry and nonlocal Foldy-Wouthuysen transformation

The approach with nonlocal Foldy-Wouthuysen transformation and a subsequent reduction was used to clarify the origin of the hidden bosonized supersymmetry (that is outside the conformal symmetry) in [Gamboa et al. (1999); Jakubský et al. (2010)], and in this section we demonstrate that the bosonized superconformal symmetry introduced above can be "extracted" from the symmetry generators of the extended quantum harmonic oscillator system described by the matrix

Hamiltonian

$$\mathcal{H}=\begin{pmatrix} L_{\rm os} & 0 \\ 0 & L_{\rm os}\end{pmatrix}. \tag{4.3.1}$$

It is natural to identify the diagonal matrix $\Gamma=\sigma_3$ as a $\mathbb{Z}_2$-grading operator, implying that Hamiltonian (4.3.1) is an even generator, besides the anti-diagonal integrals σ_a, $a=1,2$, can be considered as odd supercharges. The peculiarity of the system (4.3.1) is that these supercharges anticommute not for Hamiltonian but for central element, $\{\sigma_a,\sigma_b\}=2\delta_{ab}\mathbb{I}$, $\mathbb{I}=\text{diag}\,(1,1)$. On the other hand, all the energy levels of the extended system $\mathcal{H}$ (including the lowest $E_0=1>0$) are doubly degenerate. Furthermore, neither the supercharges nor the Hamiltonian can annihilate any eigenstate or linear combination of them, so the system is in the spontaneously broken supersymmetric phase. Additionally, one can also construct the dynamical integrals

$$\mathcal{J}_\pm=\tfrac{1}{4}e^{\mp i4t}\begin{pmatrix} (a^\pm)^2 & 0 \\ 0 & (a^\pm)^2\end{pmatrix}=\begin{pmatrix} J_\pm & 0 \\ 0 & J_\pm\end{pmatrix}, \tag{4.3.2}$$

$$\mathcal{F}_\pm=\tfrac{1}{4}e^{\mp i2t}\begin{pmatrix} a^\pm & 0 \\ 0 & a^\pm\end{pmatrix}=\begin{pmatrix} \alpha_\pm & 0 \\ 0 & \alpha_\pm\end{pmatrix}, \tag{4.3.3}$$

$$\mathcal{Q}_\pm=\tfrac{1}{4}e^{\mp i2t}\begin{pmatrix} 0 & a^\pm \\ a^\pm & 0\end{pmatrix}=\begin{pmatrix} 0 & \alpha_\pm \\ \alpha_\pm & 0\end{pmatrix}, \qquad \mathcal{S}_\pm=i\sigma_3\mathcal{Q}_\pm. \tag{4.3.4}$$

Diagonal operators $\mathcal{J}_\pm$ and $\mathcal{F}_\pm$ are identified here as even generators, and antidiagonal dynamical integrals $\mathcal{Q}_\pm$ and $\mathcal{S}_\pm$ are odd. All these generators produce the superalgebra:

$$[\mathcal{J}_0,\mathcal{J}_\pm]=\pm\mathcal{J}_\pm\,,\qquad [\mathcal{J}_-,\mathcal{J}_+]=2\mathcal{J}_0\,, \tag{4.3.5}$$

$$[\mathcal{J}_0,\mathcal{F}_\pm]=\pm\tfrac{1}{2}\mathcal{F}_\pm\,,\qquad [\mathcal{J}_\pm,\mathcal{F}_\mp]=\mp\mathcal{F}_\pm\,,\qquad [\mathcal{F}_-,\mathcal{F}_+]=\tfrac{1}{2}\mathcal{I}\,, \tag{4.3.6}$$

$$[\mathcal{J}_0,\mathcal{Q}_\pm]=\pm\tfrac{1}{2}\mathcal{Q}_\pm\,,\quad [\mathcal{J}_0,\mathcal{S}_\pm]=\pm\tfrac{1}{2}\mathcal{S}_\pm\,,\quad [\mathcal{J}_\pm,\mathcal{Q}_\mp]=\mp\mathcal{Q}_\pm\,,\quad [\mathcal{J}_\pm,\mathcal{S}_\mp]=\mp\mathcal{S}_\pm\,, \tag{4.3.7}$$

$$\{\Sigma_a,\Sigma_b\}=2\delta_{ab}\,\mathcal{I}\,,\qquad \{\Sigma_1,\mathcal{Q}_\pm\}=\mathcal{F}_\pm\,,\qquad \{\Sigma_2,\mathcal{S}_\pm\}=\mathcal{F}_\pm\,, \tag{4.3.8}$$

$$\{\mathcal{Q}_\pm,\mathcal{Q}_\pm\}=\tfrac{1}{2}\mathcal{J}_\pm\,,\quad \{\mathcal{Q}_+,\mathcal{Q}_-\}=\tfrac{1}{2}\mathcal{J}_0\,,\quad \{\mathcal{S}_\pm,\mathcal{S}_\pm\}=\tfrac{1}{2}\mathcal{J}_\pm\,,\quad \{\mathcal{S}_+,\mathcal{S}_-\}=\tfrac{1}{2}\mathcal{J}_0\,, \tag{4.3.9}$$

$$\{\mathcal{Q}_+,\mathcal{S}_-\}=-\tfrac{i}{2}\mathcal{Z}\,,\qquad \{\mathcal{Q}_-,\mathcal{S}_+\}=\tfrac{i}{2}\mathcal{Z}\,, \tag{4.3.10}$$

$$[\mathcal{Z},\Sigma_a]=\tfrac{i}{2}\epsilon_{ab}\Sigma_b\,,\qquad [\mathcal{Z},\mathcal{Q}_\pm]=\tfrac{i}{2}\mathcal{S}_\pm\,,\qquad [\mathcal{Z},\mathcal{S}_\pm]=-\tfrac{i}{2}\mathcal{Q}_\pm\,, \tag{4.3.11}$$

$$[\mathcal{F}_\pm,\mathcal{Q}_\mp]=\mp\tfrac{1}{4}\Sigma_1,\qquad [\mathcal{F}_\pm,\mathcal{S}_\mp]=\mp\tfrac{1}{4}\Sigma_2\,, \tag{4.3.12}$$

where

$$\mathcal{J}_0=\tfrac{1}{4}\mathcal{H}=\begin{pmatrix} J_0 & 0 \\ 0 & J_0\end{pmatrix}, \tag{4.3.13}$$

$$\Sigma_1=\tfrac{1}{2}\sigma_1\,,\qquad \Sigma_2=-\tfrac{1}{2}\sigma_2\,,\qquad \mathcal{Z}=-\tfrac{1}{4}\sigma_3\,,\qquad \mathcal{I}=\tfrac{1}{4}\mathbb{I}\,. \tag{4.3.14}$$

The not shown (anti)commutators between generators are equal to zero. The system presented here cannot be obtained by the usual Darboux transformation procedure, since it is not possible to find a superpotential, so that the potentials relative to the superpartners are exactly x^2, see [Inzunza and Plyushchay (2018)]. However, when considering the base change

$$\mathcal{H}_{os} = 2(\mathcal{J}_0 - \mathcal{Z})\,, \qquad \mathcal{G}^{\pm} = -2\mathcal{J}_{\pm}\,, \tag{4.3.15}$$

$$\mathcal{Q}_1 = 2\sqrt{2}\left(Re(\mathcal{Q}^-) + Im(\mathcal{S}^-)\right)\,, \qquad \mathcal{Q}_2 = 2\sqrt{2}\left(Re(\mathcal{S}^-) - Im(\mathcal{Q}^-)\right)\,, \tag{4.3.16}$$

$$\mathcal{S}_1 = 2\sqrt{2}\left(Re(\mathcal{Q}^-) - Im(\mathcal{S}^-)\right)\,, \qquad \mathcal{S}_2 = 2\sqrt{2}\left(Re(\mathcal{S}^-) + Im(\mathcal{Q}^-)\right)\,, \tag{4.3.17}$$

and identifying $2\mathcal{Z}$ with the generator of the R symmetry, one realizes that the generators defined in this way satisfy the $\mathfrak{osp}(2|2)$ superconformal algebra (2.2.16)-(2.2.22). Actually, the generators (4.3.16) - (4.3.17) match with $\mathcal{Q}_a$ and $\mathcal{S}_a$ in (2.2.28) when $t = 0$[1], and in addition, the operators Σ_a and $\mathcal{F}_{\pm}$ are, up to a proportionality factor, the generators of Heisenberg's superextended symmetry. With this information at hand we identify (4.3.5)-(4.3.12) as another expression for super-Schrödinger symmetry.

By comparing with what we have in the previous section, it is obvious that the matrix integrals $\mathcal{J}_0$, $\mathcal{J}_{\pm}$, $\mathcal{Z}$, $\mathcal{Q}_{\pm}$, $\mathcal{S}_{\pm}$ of the extended system (4.3.1) are analogous to the corresponding integrals J_0, $J_{\pm}$, Z, $\alpha_{\pm}$, $\beta_{\pm}$ of the quantum harmonic oscillator. Because of the extension, the nonlocal integrals Z and $\beta_{\pm}$ of the system (4.1.2) are changed here for the corresponding local matrix integrals $\mathcal{Z}$ and $\mathcal{S}_{\pm}$. The anti-commutator of additional fermionic integrals Σ_a with Σ_b generates a central charge $\mathcal{I}$, and via the anti-commutators with odd dynamical integrals $\mathcal{Q}_{\pm}$ and $\mathcal{S}_{\pm}$ they produce additional bosonic integrals $\mathcal{F}_{\pm}$, see Eq. (4.3.8).

The comparison of the symmetries and generators of the systems (4.3.1) and (4.1.2) indicates that the local $\mathfrak{osp}(1|2)$ and nonlocal $\mathfrak{osp}(2|2)$ hidden superconformal symmetries of the quantum harmonic oscillator can be obtained by a certain projection (reduction) of the local symmetries of the matrix system (4.3.1). To find the exact relation between these two systems and their symmetries, we apply to the extended system a unitary transformation $\mathcal{O} \mapsto \widetilde{\mathcal{O}} = U\mathcal{O}U^{\dagger}$ generated by the nonlocal matrix operator

$$U = U^{\dagger} = U^{-1} = \frac{1}{2}\begin{pmatrix} 1+\mathcal{R} & 1-\mathcal{R} \\ 1-\mathcal{R} & 1+\mathcal{R} \end{pmatrix}. \tag{4.3.18}$$

Under this transformation, the central element $\mathcal{I}$ and generators of the $\mathfrak{sl}(2,\mathbb{R})$ subalgebra, $\mathcal{J}_0$ and

[1] putting $\omega = 1$ and therefore $y = x$.

$\mathcal{J}_\pm$, do not change, while other generators take the following form:

$$\widetilde{Z} = \frac{1}{4}\begin{pmatrix} -\mathcal{R} & 0 \\ 0 & \mathcal{R} \end{pmatrix}, \tag{4.3.19}$$

$$\widetilde{\mathcal{Q}_\pm} = \begin{pmatrix} \alpha_\pm & 0 \\ 0 & \alpha_\pm \end{pmatrix}, \qquad \widetilde{\mathcal{S}_\pm} = \begin{pmatrix} i\mathcal{R}\alpha_\pm & 0 \\ 0 & -i\mathcal{R}\alpha_\pm \end{pmatrix} = \begin{pmatrix} \beta_\pm & 0 \\ 0 & -\beta_\pm \end{pmatrix}, \tag{4.3.20}$$

$$\widetilde{\Sigma_1} = \frac{1}{2}\sigma_1\,, \qquad \widetilde{\Sigma_2} = -\frac{1}{2}\sigma_2\mathcal{R}\,, \qquad \widetilde{\mathcal{F}_\pm} = \sigma_1\alpha_\pm\,. \tag{4.3.21}$$

Note that the transformation diagonalizes the dynamical odd integrals $\mathcal{Q}_\pm$ and $\mathcal{S}_\pm$ which initially have had the anti-diagonal form. Therefore, the transformation is of the same nature as a Foldy-Wouthuysen transformation for a Dirac particle in external electromagnetic field [Foldy and Wouthuysen (1950)]. On the other hand, the transformed even, $\widetilde{\mathcal{Z}}$, and odd, $\widetilde{\mathcal{S}_\pm}$, generators of the super-extended Schrödinger symmetry of the system (4.3.1) take a nonlocal form. We can reduce (or, in other words, project) the transformed system and its symmetries to the proper subspace of eigenvalue $+1$ of the matrix σ_3 which corresponds, according to Eq. (4.3.13), to the single (non-extended) quantum harmonic oscillator system. In this procedure (which can be done using projector $\Pi_+ = \frac{1}{2}(1+\sigma_3)$) we looses operators $\widetilde{\mathcal{F}_\pm}$ and $\widetilde{\Sigma_b}$ because they are anti-diagonal, but on the other hand, we retrieve all the generators of the bosonized superconformal symmetry given in the previous section.

4.4 Two-step isospectral Darboux chain

As we have indicated previously, the extended system (4.3.1) cannot be produced by means of the usual supersymmetric algorithm based on some superpotential $W(x)$. In this section we will show that an option to generate this system is through a two-step confluent Darboux transformation: The extended system obtained in this way will have a set of true and dynamical integrals of motion, and after the application of a certain limit, these integrals will give us the generators of the super-extended Schrödinger symmetry related to (4.3.1).

Consider the functions $\psi_0(x)$, which is the normalized ground state of (4.1.2), and $\chi_0(x)$, given by

$$\chi_0(x;\mu) = \mu\widetilde{\psi_0(x)} + \Omega_0\,, \tag{4.4.1}$$

where Ω_0 is a Jordan state of energy $E=1$, whose form corresponds to (1.3.3), and μ is a real constant. By construction χ_0 satisfy $H_-^2\chi_0 = 0$ with $H_- = a^+a^- = L_{\text{os}} - 1$, and the application of

a^- on it gives us

$$\varphi_{-0}(x;\mu) = a^-\chi_0(x;\mu) = \frac{\mu + I_0(x)}{\psi_0(x)} = \mu\psi_{-0}(x) + \widetilde{\psi_{-0}(x)}\,, \qquad I_0(x) = \int_{-\infty}^{x} (\psi_0(t))^2 dt\,, \quad (4.4.2)$$

where $\psi_{-0}(x) = e^{x^2/2}$ is a nonphysical eigenstate of L_{os} with negative energy $E = -1$ and $\widetilde{\psi_{-0}(x)}$ is its corresponding linear independent partner constructed according to (1.1.7).

If we choose the value of parameter μ in one of the infinite intervals $(-\infty, -1)$ or $(0, \infty)$ for which $\varphi_{-0}(x;\mu)$ is a nodeless on a real line function being a nonphysical eigenstate of $H_+ = a^-a^+$ of zero eigenvalue, $H_+\varphi_{-0}(x;\mu) = 0$, then we can use it as a seed state for a new Darboux transformation which produces the first order differential operators

$$A_\mu^- = \varphi_{-0}(x;\mu)\frac{d}{dx}\frac{1}{\varphi_{-0}(x;\mu)} = \frac{d}{dx} + W(x;\mu)\,, \qquad A_\mu^+ = (A_\mu^-)^\dagger\,, \qquad (4.4.3)$$

where

$$W(x;\mu) = -(\ln\varphi_{-0}(x;\mu))' = -x - \frac{\psi_0(x)}{\varphi_{-0}(x;\mu)}\,. \qquad (4.4.4)$$

These operators factorize H_+ and

$$H_\mu = H_+ + 2W' = H_- - 2\left(\ln(I_0(x) + \mu)\right)''\,, \qquad (4.4.5)$$

$A_\mu^+A_\mu^- = H_+$, $A_\mu^-A_\mu^+ = H_\mu$, and intertwine them, $A_\mu^-H_+ = H_\mu A_\mu^-$, $A_\mu^+H_\mu = H_+A_\mu^+$. Considering the second order differential operators given by a composition of the first order Darboux generators,

$$\mathbb{A}_\mu^- = A_\mu^-a^-\,, \qquad \mathbb{A}_\mu^+ = a^+A_\mu^+\,, \qquad (4.4.6)$$

we find that they intertwine the Hamiltonian operators H_- and H_μ,

$$\mathbb{A}_\mu^-H_- = H_\mu\mathbb{A}_\mu^-\,, \qquad \mathbb{A}_\mu^+H_\mu = H_-\mathbb{A}_\mu^+\,, \qquad (4.4.7)$$

and also satisfy relations $\mathbb{A}_\mu^+\mathbb{A}_\mu^- = (H_-)^2$, $\mathbb{A}_\mu^-\mathbb{A}_\mu^+ = (H_\mu)^2$. By construction,

$$\ker(\mathbb{A}_\mu^-) = \text{span}\{\psi_0(x), \chi_0(x;\mu)\}\,. \qquad (4.4.8)$$

The Darboux-deformed oscillator system described by the Hamiltonian operator H_μ is *completely isospectral* to the system H_-. Its eigenstates with eigenvalues $E = 2n$, $n = 1, 2\ldots$, are obtained by the mapping $\mathbb{A}_\mu^-:\ \psi_n(x) \mapsto \psi_n(x;\mu) = \mathbb{A}_\mu^-\psi_n(x)$, $H_\mu\psi_n(x;\mu) = 2n\psi_n(x;\mu)$. The (not normalized) ground state of zero energy of the system H_μ is described by wave function $\psi_0(x;\mu) = \frac{1}{\varphi_{-0}(x;\mu)}$, where $\varphi_{-0}(x;\mu)$ corresponds to (4.4.2).

Thus, we obtained the completely isospectral pair H_- and H_μ, from which we compose the extended system described by the matrix Hamiltonian operator

$$\mathcal{H}_\mu = \begin{pmatrix} H_\mu & 0 \\ 0 & H_- \end{pmatrix} . \tag{4.4.9}$$

On the other hand, A_μ^- and A_μ^+ intertwine $H_+ = H_- + 2$ and H_μ, which implies

$$A_\mu^- H_- = (H_\mu - 2)A_\mu^- \,, \qquad A_\mu^+(H_\mu - 2) = H_- A_\mu^+ \,. \tag{4.4.10}$$

For this system we have in fact three Darboux schemes:

- Scheme $(\psi_0(x), \chi_0(x;\mu))$ which provides us with the intertwining operator $\mathbb{A}_\mu^\pm$.
- Scheme $(\varphi_{-0}(x;\mu))$, the intertwining operators of which are $A_\mu^\pm$.
- Scheme $(\psi_0(x), \psi_1(x), a^+\chi_0(x;\mu))$, which gives us the third order intertwining operators $\mathcal{A}_\mu^- = A_\mu^-(a^-)^2 = \mathbb{A}_\mu^- a^-$ and $\mathcal{A}_\mu^+ = (\mathcal{A}_\mu^-)^\dagger$, that satisfy $\mathcal{A}_\mu^- H_- = (H_\mu{+}2)\mathcal{A}_\mu^-$, $\mathcal{A}_\mu^+(H_\mu{+}2) = H_-\mathcal{A}_\mu^+$.

Using the intertwining operators of these three Darboux schemes, we construct the odd integrals

$$\mathcal{Q}_{\mu 1} = \begin{pmatrix} 0 & \mathbb{A}_\mu^- \\ \mathbb{A}_\mu^+ & 0 \end{pmatrix}, \quad \mathcal{Q}_{\mu 2} = i\sigma_3 \mathcal{Q}_{\mu 1} \,, \quad \mathcal{S}_{\mu 1} = \begin{pmatrix} 0 & A_\mu^- \\ A_\mu^+ & 0 \end{pmatrix}, \quad \mathcal{S}_{\mu 2} = i\sigma_3 \mathcal{S}_{\mu 1} \,, \tag{4.4.11}$$

$$\mathcal{L}_{\mu 1} = \begin{pmatrix} 0 & \mathcal{A}_\mu^- \\ \mathcal{A}_\mu^+ & 0 \end{pmatrix}, \qquad \mathcal{L}_{\mu 2} = i\sigma_3 \mathcal{L}_{\mu 1} \,. \tag{4.4.12}$$

and by means of the relations

$$\mathbb{A}_\mu^- A_\mu^+ = A_\mu^- a^- A_\mu^+ \,, \qquad A_\mu^+ \mathbb{A}_\mu^- = (H_- + 2)a^- \,, \tag{4.4.13}$$

$$\mathcal{A}_\mu^- A_\mu^+ = A_\mu^-(a^-)A_\mu^+ \,, \qquad A_\mu^+ \mathcal{A}_\mu^- = (H_- + 2)(a^-)^2 \,, \tag{4.4.14}$$

we also construct diagonal (even) operators

$$\mathcal{F}_{\mu -} = \begin{pmatrix} A_\mu^- a^- A_\mu^+ & 0 \\ 0 & (H_- + 2)a^- \end{pmatrix}, \qquad \mathcal{J}_{\mu -} = \begin{pmatrix} A_\mu^-(a^-)^2 A_\mu^+ & 0 \\ 0 & (H_- + 2)(a^-)^2 \end{pmatrix}, \tag{4.4.15}$$

and Hermitian conjugate operators $\mathcal{F}_{\mu +}$ and $\mathcal{J}_{\mu +}$. With respect to the Hamiltonian $\mathcal{H}_\mu$, the only pair of time-independent integrals are the supercharges $\mathcal{Q}_{\mu a}$, $a = 1, 2$. To obtain dynamical integrals one should unitary transform other operators with $U(t) = \exp{(i\mathcal{H}_\mu t)}$.

The generators considered here produce a kind of a nonlinear deformation of the super-Schrödinger symmetry. We are not interested here in explicit form of such a nonlinear superalgebra, but just note that when $\mu \to \pm\infty$, we have $(\ln(I(x)+\mu))' \to 0$. As a result, in any of the two limits the Hamiltonian H_μ transforms into H_-, and the matrix Hamiltonian transforms into extended Hamiltonian (4.3.1) shifted for the minus unit matrix: $\mathcal{H}_\mu \to \mathcal{H} - \mathbb{I}$. In this limit we also have $A_\mu^\pm \to -a^\mp$, and find that the constructed operators transform as follows:

$$\mathcal{Q}_{\mu 1} \to -(\mathcal{H}-1)\sigma_1\,, \qquad \mathcal{Q}_{\mu 2} \to (\mathcal{H}-1)\sigma_2\,, \tag{4.4.16}$$

$$\mathcal{S}_{\mu a} \to -\check{\mathcal{S}}_a\,, \qquad \mathcal{L}_{\mu a} \to -(\mathcal{H}-2+\sigma_3)\widehat{\mathcal{Q}}_a\,, \tag{4.4.17}$$

$$\mathcal{F}_{\mu -} \to (\mathcal{H}-\sigma_3)\mathcal{F}_-\,, \qquad \mathcal{F}_{\mu +} \to \mathcal{F}_+(\mathcal{H}-\sigma_3)\,, \tag{4.4.18}$$

$$\mathcal{J}_{\mu -} \to (\mathcal{H}-\sigma_3)\mathcal{J}_-\,, \qquad \mathcal{J}_{\mu +} \to \mathcal{J}_+(\mathcal{H}-\sigma_3)\,. \tag{4.4.19}$$

In such a way we reproduce all the corresponding integrals of the system (4.3.1) that generate the super-extended Schrödinger symmetry lying behind the hidden superconformal symmetries $\mathfrak{osp}(1|2)$ and $\mathfrak{osp}(2|2)$ of a single quantum harmonic oscillator.

The isospectral deformation $V_\mu(x)$ of the harmonic oscillator potential is illustrated by Figure 4.1, while Figure 4.2 illustrates the action of the intertwining operators $\mathbb{A}_\mu^\pm$ and $\mathcal{A}_\mu^\pm$.

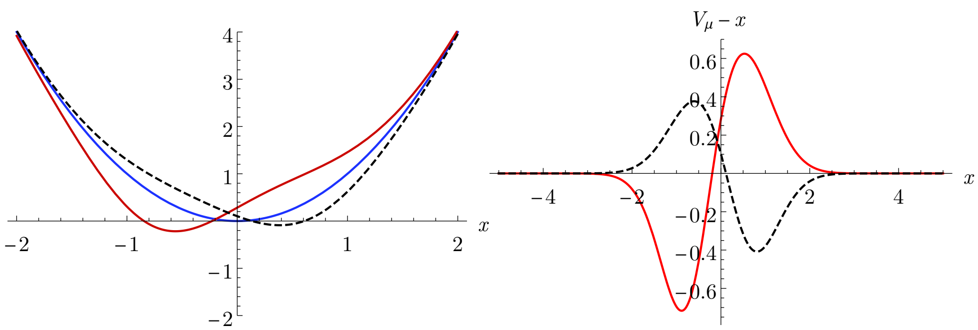

Figure 4.1: On the left: Isospectrally deformed potential V_μ at $\mu = 1$ and $\mu = -3$ is shown by continuous red and dashed black lines, respectively. On the right: The difference $V_\mu(x) - x^2$ given by the last term in Eq. (4.4.5) is shown for the same values $\mu = 1$ and $\mu = -3$. With increasing value of modulus of the deformation parameter μ the amplitudes of minimum and maximum of the difference $V_\mu(x) - x^2$ decrease, and in both limits $\mu \to \pm\infty$ the deformed potential $V_\mu(x)$ transforms into harmonic potential $V = x^2$ shown on the left by continuous blue line.

In conclusion of this section we note that the Hamiltonian (4.4.9) and the second order intertwining operators $\mathbb{A}_\mu^\pm$ can be presented in alternative form which corresponds to the anomaly-free scheme of quantization of classical systems with second-order supersymmetry [Plyushchay (2017)]. For this we introduce the quasi-amplitude [Brezhnev (2008)]

$$\Xi(x) = \sqrt{\psi_{-0}(x)\varphi_{-0}(x;\mu)}\,. \tag{4.4.20}$$

It is a square root of the product of two nonphysical eigenstates of eigenvalue -1 of the quantum harmonic oscillator $L_{\rm os}$. The rescaled function $\Xi(x)/\sqrt{\mu}$ transforms in the limit $\mu \to \pm\infty$ into

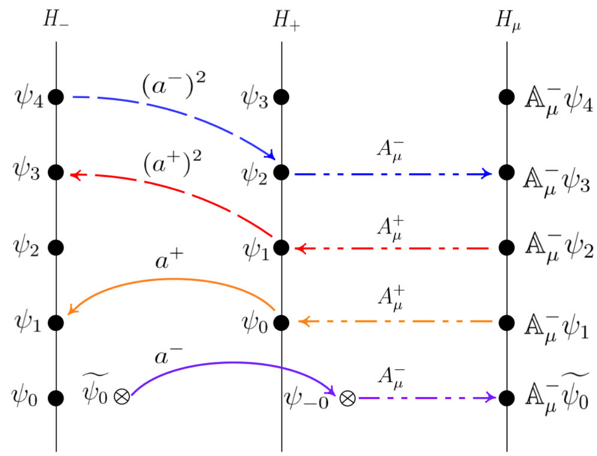

Figure 4.2: Mapping of eigenstates of the systems H_- and H_μ by intertwining operators $\mathbb{A}_\mu^\pm$ and $\mathcal{A}_\mu^\pm$ via eigenstates of intermediate system H_+. The ground state $\mathbb{A}_\mu^-\widetilde{\psi_0}$ of H_μ is obtained by applying $\mathbb{A}_\mu^-$ to nonphysical eigenstate $\widetilde{\psi_0}$ of H_-. It also can be generated by a not shown here action of $\mathcal{A}_\mu^-$ on nonphysical eigenstate $\widetilde{\psi_1}$ of H_- via nonphysical eigenstate ψ_{-0} of H_+.

the nonphysical eigenstate ψ_{-0}. This function satisfies Ermakov-Pinney equation [Ermakov (1880); Milne (1930); Pinney (1950); Cariñena and De Lucas (2011)]

$$-\Xi'' + (x^2+1)\Xi = \frac{1}{4\Xi^3}\,. \tag{4.4.21}$$

In terms of quasi-amplitude, the first order differential operators

$$A_\Xi^- = \Xi(x)\frac{d}{dx}\frac{1}{\Xi(x)} = \frac{d}{dx} - x - \mathcal{W}(x)\,, \qquad A_\Xi^+ = (A_\Xi)^\dagger\,, \tag{4.4.22}$$

can be defined, where

$$\mathcal{W}(x) = \frac{1}{2\Xi^2(x)} = \frac{1}{2}\left(\ln(I_0(x)+\mu)\right)'\,. \tag{4.4.23}$$

Then the Hamiltonian H_μ and the intertwining operator $\mathbb{A}_\mu^-$ can be presented in the form

$$\mathcal{H}_\mu = A_\Xi^- A_\Xi^+ + \mathcal{W}^2 - 2\mathcal{W}'\sigma_3\,, \qquad \mathbb{A}_\mu^- = -(A_\Xi^- - \mathcal{W})(A_\Xi^+ + \mathcal{W})\,. \tag{4.4.24}$$

Function $\mathcal{W}(x)$ in the anomaly-free scheme of quantization plays a role of superpotential for corresponding classical system with second order supersymmetry, [Plyushchay (2000a); Klishevich and Plyushchay (2001); Plyushchay (2017)].

4.5 Remarks

Along with the harmonic oscillator, there are many bosonic systems that have hidden bosonized supersymmetry and the idea of the Foldy-Wouthuysen transformation is not new, see [Plyushchay (1996, 2000a); Gamboa et al. (1999); Correa and Plyushchay (2007); Correa et al. (2008); Jakubský et al. (2010)]. In fact, one can use the transformation (4.3.18) in the generators of the super-extended free particle given in Chap. 2, Sec. 2.3, to obtain the hidden superconformal symmetry of the bosonic free particle.

The exotic feature here is the supersymmetric system from where we get the bosonic superalgebra which, in principle, does not correspond to the Darboux transformation scheme. However, it is possible to obtain such a system starting from the classical level: Consider a classical system described by a Hamiltonian

$$H = p^2 + W^2 + W'[\theta^+, \theta^-] \tag{4.5.1}$$

with superpotential $W(x) = \sqrt{x^2+c^2}$, where $c > 0$ is a constant, besides θ^+ and $\theta^- = (\theta^+)^*$ are Grassmann variables with a nonzero Poisson bracket $\{\theta^+, \theta^-\}_{_{PB}} = -i$, that after quantization are realized as the creation-annihilation fermionic operators $\theta^\pm \to \sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. A direct quantum analog of this system is a composition of two isospecral systems and is in the phase of spontaneously broken supersymmetry, with nonsingular superpartner potentials $V_\pm = x^2 + c^2 \pm x/\sqrt{x^2+c^2}$. The spectrum of subsystems is different from that of the quantum harmonic oscillator. On the other hand, if before the quantization we realize a canonical transformation $x \to X = x + N\partial G(x,p)/\partial p$, $p \to P = p - N\partial G(x,p)/\partial x$, $\theta^\pm \to \Theta^\pm = e^{\pm iG(x,p)}\theta^\pm$, where $N = \theta^+\theta^-$ and $G = \frac{1}{2}\arcsin\big((p^2 - x^2 - c^2)/(p^2 + x^2 + c^2)\big)$ [Klishevich and Plyushchay (2001); Inzunza and Plyushchay (2018)], we obtain the canonically equivalent form of the Hamiltonian $H = P^2 + X^2 + c^2$. In the canonically transformed system, the new classical Grassmann variables $\Theta^\pm$ completely decouple and are the odd integrals of motion with Poisson bracket $\{\Theta^+, \Theta^-\}_{_{PB}} = -i$. The quantization of the canonically transformed system gives us exactly the extended quantum system (4.3.1) shifted just for the additive constant c^2.

Another possibility is a "naive" application of the comformal bridge. To do so, let us start by setting the super-Schrödinger symmetry generators for the super-free particle system,

$$\mathcal{H} = -\tfrac{1}{2}p^2\mathbb{I}\,, \qquad \mathcal{D} = \tfrac{1}{4}\{x,p\}\mathbb{I}\,, \qquad \mathcal{K} = \tfrac{x^2}{2}\mathbb{I}\,, \qquad \mathcal{Z} = -\tfrac{\sigma_3}{4}\,, \tag{4.5.2}$$

$$\mathcal{P} = p\mathbb{I}\,, \qquad \mathcal{X} = x\mathbb{I}\,, \qquad \Sigma_1 = \sigma_1\,, \qquad \Sigma_2 = -\sigma_2\,, \qquad \pi_a = p\Sigma_a \qquad \xi_a = x\Sigma_a\,, \tag{4.5.3}$$

where $p = i\frac{d}{dx}$. The conformal bridge transformation produces

$$\mathfrak{S}\mathcal{H}\mathfrak{S}^{-1} = -\tfrac{(a^-)^2}{2}\mathbb{I}\,, \qquad \mathfrak{S}\mathcal{D}\mathfrak{S}^{-1} = -\tfrac{i}{4}L_{\mathrm{os}}\mathbb{I}\,, \qquad \mathfrak{S}\mathcal{K}\mathfrak{S}^{-1} = \tfrac{(a^+)^2}{2}\mathbb{I} \tag{4.5.4}$$

$$\mathfrak{S}\mathcal{Z}\mathfrak{S}^{-1} = \mathcal{Z}\,, \qquad \mathfrak{S}\mathcal{X}\mathfrak{S}^{-1} = \tfrac{a^+}{\sqrt{2}}\mathbb{I}\,, \qquad \mathfrak{S}\mathcal{P}\mathfrak{S}^{-1} = -i\tfrac{a^-}{\sqrt{2}}\mathbb{I}\,, \tag{4.5.5}$$

$$\mathfrak{S}\Sigma_a\mathfrak{S}^{-1} = \Sigma_a\,, \qquad \mathfrak{S}\xi_a\mathfrak{S}^{-1} = \tfrac{a^+}{\sqrt{2}}\Sigma_a\,, \qquad \mathfrak{S}\pi_a\mathfrak{S}^{-1} = -i\tfrac{a^-}{\sqrt{2}}\Sigma_a\,, \tag{4.5.6}$$

that up to a complex proportionality constant, they match with the generators presented in Sec. 4.3 at $t = 0$. The conformal bridge works fine in this case because $\mathcal{D}$ and its transformed version are matrix generators containing two copies of the same differential operator. In the general case, both superpartners are different from each other and the transformation fails.

Chapter 5

Rationally extended conformal mechanics

As we have shown in Chap. 1, Sec. 1.2, DCKA transformation allows us to construct new quantum systems starting from a well known original one. In this context, the systems that appear due to these transformations applied to the harmonic oscillator are the rationally extended harmonic oscillators, that is, a harmonic potential plus a regular rational function of x, and to obtain a well-defined system, we have to follow some rules for selecting the set of seed states for transformation. The selection rule that gives us a regular potential is known as the Krein-Adler theorem, [Krein (1957); Adler (1994); Dubov et al. (1994); Quesne (2012); Gómez-Ullate et al. (2013)]. In the research carried out in the article [Cariñena et al. (2018)], we found new selection rules to construct completely isospectral rational extensions for the AFF model with integer coupling constant $m(m+1)$, where $m = 1, 2, \ldots$, as well as deformations with gaps in their spectrum. We also learned how to construct the spectrum-generating ladder operators of these deformed systems by using what we call Darboux dualities. The content presented in this chapter is a summary of the results obtained in [Cariñena et al. (2018)], an article that in turn was inspired by previous research on rational deformations of the harmonic oscillator [Cariñena and Plyushchay (2017)].

Before to start, let us explain what a Darboux duality is with a simple example: consider the half-harmonic oscillator Hamiltonian L_0[1]. When the first m physical states are considered as seed states for the DCKA transformation, it is not difficult to show that the resulting system is the AFF model L_m, defined in (4.1.3), shifted by the constant $-2m$. Now, by performing the transformation $x \rightarrow ix$ in the physical eigenstates, we produce new nonphysical solutions, and when the first m functions obtained in this way are taken as seed states for the DCKA transformation, the resulting system is again L_m but now shifted by the positive constant $2m$. So both Darboux transformation

[1] This Hamiltonian is formally (4.1.2), but defined in the domain $\{\psi \in L^2((0,\infty), dy)|\psi(0^+) = 0\}$. The physical states are the odd eigenstates of the harmonic oscillator system.

schemes generate essentially the same quantum system, and in this sense we call them as dual Darboux schemes. The intertwining operators of both dual schemes are independent of each other and it can be shown that operators constructed by means of products of these intertwiners are equivalent to powers of $\mathfrak{sl}(2,\mathbb{R})$ generators [Cariñena et al. (2018)].

Here we study rational extended systems built on the basis of the half-harmonic oscillator, and for simplicity, we present the following notation to refer to the physical and nonphysical eigenstates of the quantum harmonic oscillator system (from now on QHO),

$$n \equiv \psi_n(x), \qquad -n \equiv \psi_n^- = \psi_n(ix)\,, \qquad \widetilde{n} \equiv \widetilde{\psi_n}\,, \qquad \widetilde{-n} \equiv \widetilde{\psi_n^-}\,. \tag{5.0.1}$$

5.1 Generation of rationally extended systems

Rational deformations (extensions) of the QHO system are constructed following the Krein-Adler theorem [Krein (1957); Adler (1994)], which ensures that the Wronskian of the seed states (or henceforth Darboux scheme) $(n_1, n_1+1, \ldots, n_\ell, n_\ell+1)$, where the numbers $n_j \in \mathbb{N}$, $j = 1, \ldots, \ell$, indicate the chosen seed states, see notations (5.0.1), does not have zeros on the real axis. The corresponding DCKA transformation produces

$$L_{(n_1,n_1+1,\ldots,n_\ell,n_\ell+1)} = L + 4\ell + \frac{F(x)}{Q(x)}, \tag{5.1.1}$$

where $F(x)$ and $Q(x)$ are even polynomials, with $Q(x)$ taking positive values on real line and having degree higher by two of degree of $F(x)$. According with Chap. 1, the spectrum of the system (5.1.1) is almost isospectral to the QHO spectrum: there are missing energy levels or gaps, related to the energy levels corresponding to seed states.

On the other hand, deformations of the AFF model L_m can be obtained from the half-harmonic oscillator by considering the scheme $(n_1, n_1+1, \ldots, n_\ell, n_\ell+1, 2k_1+1, \ldots, 2k_m+1)$, where even indexes inside the set $n_1, n_1+1, \ldots, n_\ell, n_\ell+1$ represent nonphysical eigenstates of L_0 and k_i, $i = 1, \ldots, m$, are identified as m odd states which were not considered in the first set of $2n_\ell$ states. The Hamiltonian operator

$$L_{(n_1,n_1+1,\ldots,n_\ell,n_\ell+1,2k_1+1,\ldots,2k_m+1)} = L_m + 2m + 4\ell + \frac{\widetilde{F(x)}}{\widetilde{Q(x)}}, \tag{5.1.2}$$

appears as a final result of the DCKA transformation, where polynomials $\widetilde{F(x)}$ and $\widetilde{Q(x)}$ have the properties similar to those of $F(x)$ and $Q(x)$ in (5.1.1). Note that in this way we can only construct deformations of L_m. Rational deformations of L_ν, with arbitrary values for parameter ν, cannot be connected with the harmonic oscillator as we did here, and the issue about their construction is discussed properly in Chap. 7. In general such a system has gaps in its spectrum. If, however, the set $n_1, n_1+1, \ldots, n_\ell, n_\ell+1, 2k_1+1, \ldots, 2k_m+1$ contains all the $\ell+m$ odd indexes from 1 to $2k_m+1$, the generated deformed AFF system will have no gaps in its spectrum and we obtain a system completely isospectral to $L_0+4\ell+2m$. Such completely isospectral (gapless) deformations in the QHO case are only possible if we include Jordan states in the construction.

The mirror diagram method developed and used in [Cariñena et al. (2018)] is a technique such that a dual scheme with nonphysical "negative" eigenstates (5.0.1), is derived from a "positive" scheme with physical states of L_{os} and, vice-versa. This can be done by using the algorithmic procedure described in Appendix B.1 and the final picture is the following:

- For a given positive scheme $\Delta_+ \equiv (l_1^+, \ldots, l_{n_+}^+)$, where l_i^+ with $i=1,\ldots,n_+$, one gets the negative scheme $\Delta_- = (-\check{0}, \ldots, -\check{n}_i^- = l_i^+ - l_{n_+}^+, \ldots, -l_{n_+}^+)$, where $-\check{n}_i^-$, means that the corresponding number $-n_i^-$ is omitted from the set Δ_-.

- If we have instead the negative scheme $\Delta_- \equiv (-l_1^-, \ldots, -l_{n_-}^-)$, where $-l_j^-$ with $j=1,\ldots,n_-$, one obtains the positive scheme $\Delta_+ = (\check{0}, \ldots, \check{n}_j^+ = l_{n_-}^- - l_j^-, \ldots, l_{n_-}^-)$, where symbols $\check{n}_j^+$ represent again the states missing from the list of the chosen seed states.

Obviously, Darboux scheme must be constructed in such a way that the generated Hamiltonian is a non-singular operator, that is, by means of the rules discussed above. Then, having two dual schemes on hand, the relation

$$e^{-n_+x^2/2}W(\Delta_-) = e^{n_-x^2/2}W(\Delta_+)\,, \tag{5.1.3}$$

is valid modulo a multiplicative constant. From here one can see that the Hamiltonians of dual schemes satisfy

$$L_{(+)} - L_{(-)} = 2N\,, \qquad N \equiv n_+ + n_- = l_{n^+}^+ + 1 = l_{n^-}^- + 1\,, \tag{5.1.4}$$

where $L_{(+)}$ and $L_{(-)}$ correspond to

$$L_{(\pm)} = -\frac{d^2}{dx^2} + V(x) - 2\frac{d^2}{dx^2}\ln W(\Delta_\pm)\,. \tag{5.1.5}$$

On the other hand, the intertwining operators $\mathbb{A}_{n_+}^-$ and $\mathbb{A}_{n_-}^-$ that correspond to each scheme are constructed following (1.2.5), however, we prefer to use the more generic notation

$$\mathbb{A}_{n_+}^- = A_{(+)}^-\,, \qquad \mathbb{A}_{n_-}^- = A_{(-)}^-\,, \qquad A_{(+)}^+ = (A_{(+)}^-)^\dagger\,, \qquad A_{(-)}^+ = (A_{(-)}^-)^\dagger\,. \tag{5.1.6}$$

By means of the negative scheme we do not eliminate any energy level from the spectrum, but instead energy levels can be introduced, but not obligatorily, in its lower part. In the particular special case of completely isospectral deformations of the (shifted) L_m systems, all m seed states composing negative scheme are nonphysical odd eigenstates of L_0, and the transformation does not introduce any additional energy level.

The construction of the mirror diagram can be better understood with the following example: Consider the illustration in figure 5.1. In the upper line we have represented the first eleven physical eigenstates of the harmonic oscillator by circles, where the black ones are the seed states of the positive scheme $(1, 4, 5, 10, 11)$, which produces a system of the type (5.1.2). In a similar way, the first eleven nonphysical states with negative energy are indicated by the circles in the bottom line, and the marked ones are the seed states of the corresponding dual negative scheme. In general, when considering a scheme of the form $(\ldots, N)$, in the upper line we ordered from left to right all the physical state from ψ_0 to ψ_N, besides in the bottom line we set from right to left all the states between ψ_0^- to ψ_N^-. After marking the states of the positive scheme, the construction of the negative scheme is by means of a sort of an "anti-reflection" transformation with respect to an imaginary line in the center, that is parallel to the other two lines. The construction of the positive scheme from the negative one is analogous.

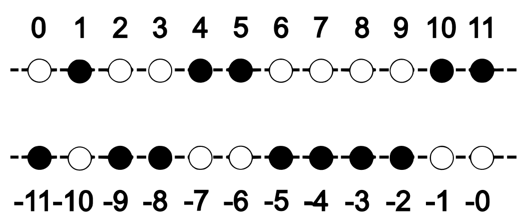

Figure 5.1: A mirror diagram example.

This construction seems to be related with the Maya diagram formalism, for a review see Gómez-Ullate and Milson (2019). However, our technique is completely based on the existence of the first-order ladder operators and their relationship with the Darboux transformation (this is the key to its generalization for the AFF model in Chap. 7), besides for the Maya diagrams it is important to study the proprieties of an additional structure called the pseudo-Wronskian, which we do not introduce in our work.

5.2 Spectrum-generating ladder operators: completely isospectral case

In this section we explore the possibilities of constructing spectrum-generating ladder operators for rationally extended isospectral systems. We start with the simplest example and then expand on the ideas for the general case.

Consider the simplest deformed AFF system generated via the Darboux transformation based on the nonphysical eigenstate $\psi_3^- = (2x^3+3x)e^{x^2/2}$ of the half-harmonic oscillator L_0. The resulting Hamiltonian takes the form

$$L_{(-)} := L_1 - 2 + 8\frac{2x^2-3}{(2x^2+3)^2}\,. \tag{5.2.1}$$

By the method of the mirror diagram, we find that up to a constant shift, the system can be generated alternatively by the DCKA transformation based on the set $(1,2,3)$[2] ,

$$L_{(+)} := L_{(-)} + 8\,. \tag{5.2.2}$$

The intertwiners of the negative scheme are

$$A^-_{(-)} = \psi_3^- \frac{d}{dx}\frac{1}{\psi_3^-} = \frac{d}{dx} - x - \frac{1}{x} - \frac{4x}{2x^2+3}\,, \qquad A^+_{(-)} = (A^-_{(-)})^\dagger\,. \tag{5.2.3}$$

They provide us the factorization relations $A^+_{(-)}A^-_{(-)} = L_0 + 7\,,\ A^-_{(-)}A^+_{(-)} = L_{(-)} + 7 = L_{(+)} - 1$. In correspondence with them, $A^-_{(-)}$ intertwines the Hamiltonian operators L_0 and $L_{(-)}$,

$$A^-_{(-)}L_0 = L_{(-)}A^-_{(-)} = (L_{(+)} - 2\Delta E)A^-_{(-)}\,, \qquad \Delta E = 4\,, \tag{5.2.4}$$

and the intertwining relation for $A^+_{(-)}$ is obtained by Hermitian conjugation.

The systems L_0 and $L_{(+)}$ are also intertwined by the third order operators $A^\pm_{(+)}$, where the operator $A^-_{(+)}$ is uniquely specified by its kernel: $\ker A^-_{(+)} = \mathrm{span}\,\{\psi_1,\psi_2,\psi_3\}$. We have the intertwining relation $A^-_{(+)}L_0 = L_{(+)}A^-_{(+)} = (L_{(-)}+8)A^-_{(+)}$, and the conjugate relation for $A^+_{(+)}$.

To construct ladder operators for this deformed system we can "Darboux dress" the ladder operators of the half-harmonic oscillator which are nothing else than $(a^\pm)^2$. The first pairs of operators produced in this way are

$$\mathcal{A}^\pm = A^-_{(-)}(a^\pm)^2 A^+_{(-)}\,. \tag{5.2.5}$$

These operators together with the Hamiltonian $L_{(-)}$ generate a nonlinear deformation of the conformal symmetry given by the commutation relations

$$[L_{(-)},\mathcal{A}^\pm] = \pm 4\mathcal{A}^\pm, \qquad [\mathcal{A}^-,\mathcal{A}^+] = 16\left(L_{(-)}+3\right)\left(L_{(-)}+7\right)\left(L_{(-)}+1/2\right). \tag{5.2.6}$$

The roots of the fourth order polynomial in the relation

$$\mathcal{A}^+\mathcal{A}^- = (L_{(-)}+7)(L_{(-)}+3)(L_{(-)}-1)(L_{(-)}-3)\,, \tag{5.2.7}$$

[2] The state ψ_2 is not a physical state of the half-harmonic oscillator L_0.

correspond to eigenstates of $L_{(-)}$, which belong to the kernel of the lowering operator,

$$\ker \mathcal{A}^- = \text{span}\,\{A^-_{(-)}\widetilde{\psi^-_3}\,,\ A^-_{(-)}\psi^-_1\,, A^-_{(-)}\psi_0,\ A^-_{(-)}\psi_1\}. \tag{5.2.8}$$

The last state $A^-_{(-)}\psi_1 = A^-_{(+)}\psi_5$ describes here the ground state of $L_{(-)}$ of eigenvalue $E = 3$, and other states are nonphysical.

On the other hand, the roots in the product

$$\mathcal{A}^-\mathcal{A}^+ = (L_{(-)} + 11)(L_{(-)} + 7)(L_{(-)} + 3)(L_{(-)} + 1)\,, \tag{5.2.9}$$

correspond to eigenvalues of the eigenstates of $L_{(-)}$ which appear in the kernel of the raising ladder operator,

$$\ker \mathcal{A}^+ = \text{span}\{A^-_{(-)}\psi^-_5\,,\ A^-_{(-)}\widetilde{\psi^-_3}\,,\ A^-_{(-)}\psi^-_1\,,\ A^-_{(-)}\psi^-_0\}\,. \tag{5.2.10}$$

All the states in this kernel are nonphysical. In correspondence with the described properties of the ladder operators (5.2.5) they are the spectrum-generating ladder operators for the system $L_{(-)}$: acting by them on any physical eigenstate of $L_{(-)}$, we can generate any other physical eigenstate. The kernels of the ladder operators contain here the same nonphysical eigenstate $A^-_{(-)}\widetilde{\psi^-_3} = A^-_{(-)}\psi^-_1$. Below we shall see that in the case of **non-isospectral** rational deformations of the AFF system the kernels of analogs of such lowering and raising ladder operators contain some common physical eigenstates, see for example Figure 5.2 in next section.

In a similar way, one can construct the ladder operators for $L_{(-)}$ via Darboux-dressing of $(a^\pm)^2$ by the third order intertwining operators,

$$\mathcal{B}^\pm = A^-_{(+)}(a^\pm)^2 A^+_{(+)}\,, \qquad [L_{(-3)}, \mathcal{B}^\pm] = \pm 4\mathcal{B}^\pm\,. \tag{5.2.11}$$

However, these differential operators of order 8 are not independent and reduce to the fourth order ladder operators (5.2.5) multiplied by the second order polynomials in the Hamiltonian,

$$\mathcal{B}^- = \mathcal{A}^-(L_{(-)} + 1)(L_{(-)} + 5) \qquad \text{and} \qquad \mathcal{B}^+ = (\mathcal{B}^-)^\dagger\,. \tag{5.2.12}$$

As the first and third order operators $A^\pm_{(-)}$ and $A^\pm_{(+)}$ intertwine the half-harmonic oscillator with the system $L_{(-)}$ with a nonzero relative shift, we can construct yet another pair of the ladder operators for the quantum system $L_{(-)}$,

$$\mathcal{C}^- = A^-_{(+)}A^+_{(-)}\,, \qquad \mathcal{C}^+ = A^-_{(-)}A^+_{(+)}\,, \tag{5.2.13}$$

$$[L_{(-)}, \mathcal{C}^\pm] = \pm 8\,\mathcal{C}^\pm\,, \qquad [\mathcal{C}^-, \mathcal{C}^+] = 32\left(L^3_{(-)} + 6L^2_{(-)} - L_{(-)} + 30\right)\,. \tag{5.2.14}$$

The kernel of the lowering ladder operator is

$$\ker \mathcal{C}^- = \operatorname{span}\{(\psi^-_{(-)})^{-1},\, A^-_{(-)}\psi_1,\, A^-_{(-)}\psi_2,\, A^-_{(-)}\psi_3\}\,. \tag{5.2.15}$$

Here $A^-_{(-)}\psi_1 = A^-_{(+)}\psi_5$ and $A^-_{(-)}\psi_3 = A^-_{(+)}\psi_7$ are the ground and the first exited states of $L_{(-)}$. On the other hand, all the states in the kernel of the raising ladder operator are nonphysical:

$$\ker \mathcal{C}^+ = \operatorname{span}\{A^-_{(-)}\psi^-_7\,,\, A^-_{(-)}\psi^-_2\,,\, A^-_{(-)}\psi^-_1\,,\, A^-_{(-)}\psi^-_0\}\,. \tag{5.2.16}$$

As a result, the space of states of $L_{(-)}$ is separated into two subspaces, on each of which the ladder operators $\mathcal{C}^+$ and $\mathcal{C}^-$ act irreducibly. One subspace is spanned by the even eigenstates and the another subspace corresponds to the odd eigenstates. The ladder operators $\mathcal{C}^\pm$, unlike $\mathcal{A}^\pm$, are therefore not spectrum-generating operators for the system $L_{(-)}$. Notice that from the point of view of the basic properties of the ladder operators $\mathcal{C}^\pm$, they are similar to the operators $(a^\pm)^4$ in the case of the half-harmonic oscillator L_0. The essential difference here, however, is that the ladder operators $\mathcal{C}^\pm$ are independent from the spectrum-generating ladder operators $\mathcal{A}^\pm$ and have the same differential order equal to four. We shall see that for **non-isospectral** rational extensions of the AFF systems the direct analogs of the operators $\mathcal{C}^\pm$ will constitute an inseparable part of the set of the spectrum-generating operators.

The described properties of this particular example are extended for the general case of isospectral deformations and can be summarized as follows. No matter what set of the m odd nonphysical eigenstates of the quantum harmonic oscillator we select, the lower order ladder operators $\mathcal{A}^\pm$ obtained by Darboux-dressing of the ladder operators of the half-harmonic oscillator are spectrum-generating operators for the rationally deformed AFF system. They commute for a polynomial of order $2m{+}1$ in the corresponding Hamiltonian with which they produce a deformation of the conformal $\mathfrak{sl}(2,\mathbb{R})$ symmetry of the type of W-algebra [de Boer et al. (1996)]. Other spectrum-generating ladder operators, which can be constructed on the basis of other DCKA schemes via the Darbox-dressing procedure, act on physical states in the same way as the operators $\mathcal{A}^\pm$ of order $2(m{+}1)$, and are equal to them modulo the multiplicative factor in the form of the polynomial in the Hamiltonian operator of the system. The ladder operators $\mathcal{C}^\pm$ constructed by "gluing" intertwining operators of the two dual schemes are not spectrum-generating. Particularly, for the isospectral deformation of the system L_{l_m+1} based on the set of the seed states $(-(2l_1+1), -(2l_2+1), \ldots, -(2l_m+1))$ with $0 \le l_1 < l_2 < \ldots < l_m$, $l_m \ge 1$, the operator $\mathcal{C}^-$ annihilates the lowest l_m+1 states in the spectrum of the system.

5.3 Spectrum-generating ladder operators: non-isospectral case

As in the previous section, here we explore the construction of spectrum-generating ladder operators for non-isospectral deformations of the AFF system through a particular example, and then generalize the ideas.

Let us start with Darboux's positive scheme $(1, 4, 5, 10, 11)$ that we have already used as example to explain the mirror diagram technique in Sec. 5.1. There we had already obtained the negative scheme which is $(-2, -3, -4, -5, -8, -9, -11)$.

After performing the DCKA transformation using the positive scheme, we obtain the Hamiltonian operator

$$L_{(+)} := -\frac{d^2}{dx^2} + x^2 - 2(\ln W(1, 4, 5, 10, 11))'' \,, \tag{5.3.1}$$

where

$$\begin{aligned} W(1, 4, 5, 10, 11) \propto \quad & xe^{-\frac{5}{2}x^2}(467775 + 4x^2(155925 - 93555x^2 + 8x^4(62370 - 21945x^2 + \\ & +4x^4(735 + 1145x^2 - 504x^4 + 358x^6 - 88x^8 + 8x^{10})))) \end{aligned} \,. \tag{5.3.2}$$

The graph of the resulting potential and the quantum spectrum of the system (5.3.1) are shown on Figure 5.2.

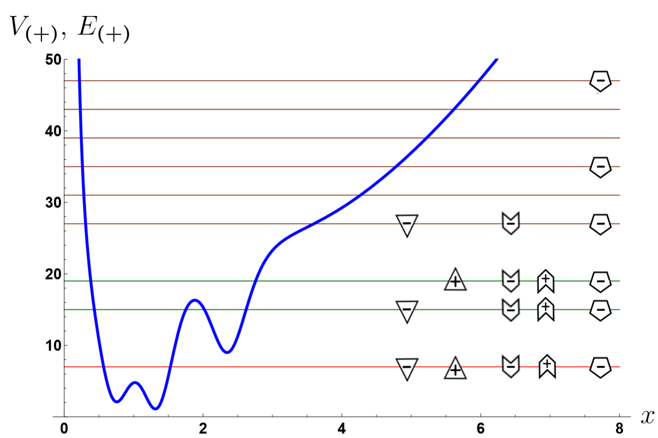

Figure 5.2: Potential of the system (5.3.1). The energy levels of the corresponding physical states annihilated by ladder operators $\mathcal{B}^-$, $\mathcal{B}^+$, $\mathcal{A}^-$, $\mathcal{A}^+$, and $\mathcal{C}^-$ are indicated from left to right.

The potential has three local minima and the system supports three separated states in its spectrum which are organized in two "valence bands" of one and two states. On the other hand, the dual scheme produces the same Hamiltonian operator but shifted by a constant, $L_{(+)} - L_{(-)} = 6\Delta E = 24$. The fact that the mutual shift of both Hamiltonians is proportional to the difference of two consecutive energy levels in the spectrum of the AFF model allows us to use below exactly the same rule for the construction of the ladder operators of the type $\mathcal{C}^{\pm}$ as in the previous section. As we shall see, the number of physical states annihilated by the lowering operator $\mathcal{C}^-$ in this case is equal exactly to six. Later, we also shall see that in some cases of the rational gapped deformations of the AFF systems, the mutual shift of the corresponding Hamiltonian operators can be equal to the half-integer multiple of ΔE, and then the procedure for the construction of the ladder operators of the type $\mathcal{C}^{\pm}$ will require some modification.

In the DCKA construction of the Hamiltonian operator $L_{(+)}$, the energy levels corresponding to the physical seed eigenstates of the half-harmonic oscillator L_0 were removed from the spectrum producing two gaps. In the (up to a shifted constant) equivalent system $L_{(-)}$ based on nonphysical seed eigenstates of L_0, the energy levels were added under the lowest energy of the ground state of L_0. The intertwining operators associated with the positive scheme $A^{\pm}_{(+)}$ have differential order five, while the operators $A^{\pm}_{(-)}$, obtained from the negative scheme, have differential order eleven.

The three lowest physical states of the system (5.3.1) which correspond to the three separated energy levels can be presented in two equivalent forms

$$\phi_0 = A^-_{(-)}\widetilde{\psi^-_8} = A^-_{(+)}\psi_3\,, \qquad \phi_1 = A^-_{(-)}\widetilde{\psi^-_4} = A^-_{(+)}\psi_7\,, \qquad \phi_2 = A^-_{(-)}\widetilde{\psi^-_2} = A^-_{(+)}\psi_9\,, \tag{5.3.3}$$

where equalities are modulo a nonzero constant multiplier. We have here the intertwining relations

$$A^-_{(+)}L_0 = L_{(+)}A^-_{(+)} = (L_{(-)}+24)A^-_{(+)}\,, \qquad A^-_{(-)}L_0 = L_{(-)}A^-_{(-)} = (L_{(+)}-24)A^-_{(-)}, \tag{5.3.4}$$

and the conjugate relations for $A^+_{(+)}$ and $A^+_{(-)}$.

Let us turn now to the construction of the ladder operators for the system under consideration. Like in the isospectral case, here we have two ways to realize Darboux-dressing of the ladder operators $-\mathcal{C}^{\pm}_0 = (a^{\pm})^2$. Using $A^{\pm}_{(+)}$ for this purpose , we obtain the operators of order twelve:

$$\mathcal{B}^{\pm} = A^-_{(+)}(a^{\pm})^2A^+_{(+)}\,, \qquad [L_{(-)},\mathcal{B}^{\pm}] = \pm\Delta E\mathcal{B}^{\pm}\,. \tag{5.3.5}$$

The kernel of $\mathcal{B}^-$ contains three physical states ϕ_0, ϕ_1 and $\phi_3 = A^-_{(-)}\psi_1 = A^-_{(+)}\psi_{13}$ among other 9 nonphysical solutions with negative energy. They correspond to the ground state, the lowest states in the isolated "valence band", and the first state in the equidistant part of the spectrum, see Figure 5.2. On the other hand $\mathcal{B}^+$ annihilates ϕ_0, the upper state in the valance band ϕ_2 and other 10 nonphysical states. Then, due to the incapacity of these operators to connect the isolates states with the equidistant part of the spectrum, it is obvious that $\mathcal{B}^{\pm}$ are not spectrum-generating.

We also can construct ladder operators by using $A^{\pm}_{(-)}$ instead,

$$\mathcal{A}^{\pm} = A^-_{(-)}(a^{\pm})^2A^+_{(-)}\,, \qquad [L_{(+)},\mathcal{A}^{\pm}] = \pm\Delta E\mathcal{A}^{\pm}\,. \tag{5.3.6}$$

These are also not spectrum-generating operartors because the leap they make does not allow to overcome the gaps. Operator $\mathcal{A}^+$ detects all the states in both separated valence bands by annihilating them. In addition to the indicated physical states, the lowering operator $\mathcal{A}^-$ also annihilates the lowest state in the half-infinite equidistant part of the spectrum.

Therefore, the essential difference of the non-isospectral rational deformations of the AFF model from their isospectral rational extensions is that there is no pair of spectrum-generating ladder operators constructed via the Darboux-dressing procedure. This situation is similar to that in the rationally extended QHO systems [Cariñena and Plyushchay (2017)].

We now construct the ladder operators $\mathcal{C}^\pm$ by "gluing" the intertwining operators of different types. As in the case of the isospectral deformations, they also will not be the spectrum-generating operators, but together with any pair of the ladder operators $\mathcal{B}^\pm$, or $\mathcal{A}^\pm$ they will form a spectrum-generating set. So, let us consider

$$\mathcal{C}^- = A^-_{(-)}A^+_{(+)}\,, \qquad \mathcal{C}^+ = A^-_{(+)}A^+_{(-)}\,, \qquad [L_{(-)},\mathcal{C}^\pm] = \pm 6\Delta E\mathcal{C}^\pm\,. \tag{5.3.7}$$

They are independent from the ladder operators constructed via the Darboux-dressing procedure, and their commutator $[\mathcal{C}^-,\mathcal{C}^+]$ is a certain polynomial of order 11 in the Hamiltonian $L_{(-)}$. The operators $\mathcal{C}^\pm$ divide the Hilbert space of the system into six infinite subsets on which they act irreducibly: The $\mathcal{C}^-$ transforms a physical eigenstate into another physical eigenstate by making it skip six levels below and annihilates the first six eigenstates of the spectrum. The operator $\mathcal{C}^+$ does not annihilate any physical state here and skip the energy of an arbitrary state in to six levels above. Therefore they connect the separated states with the equidistant part of the spectrum.

As a result, the pair $\mathcal{C}^\pm$ together with any pair of the ladder operators, $\mathcal{B}^\pm$ or $\mathcal{A}^\pm$ are the spectrum-generating set. Figure 5.3 illustrates the action of the ladder operators and show how we can use them to obtain a particular state, starting from an arbitrary one.

All the described picture is generalized directly in the case when the index of the last seed state used in the corresponding DCKA transformation is odd. Then the corresponding scheme based on physical eigenstates of L_0 is of the form $(\ldots,2l_m,2l_m+1)$, and the dual scheme is $(\ldots,-(2l_m+1))$. Following the same notation as we used in the particular examples, the Hamiltonian operators generated in these two dual schemes are shifted by the distance equal to the separation $\Delta E=4$ of energy levels in the equidistant part of the spectrum times integer number l_m+1: $L_{(+)}-L_{(-)}=4l_m+4$, see (5.1.4), and the picture is the following:

- Operators $\mathcal{A}^\pm=\mathbb{A}^-_{(-)}(a^\pm)^2\mathbb{A}^+_{(-)}$ are of differential order $2n_-+2$. Rising and lowering operators of this kind annihilate all the states in the isolated valence bands, in the sense of a group of energy levels separated by a gap from the equidistant part of the spectrum. They act as regular ladder operators in the equidistant part of the spectrum.

- Operators $\mathcal{B}^\pm=\mathbb{A}^-_{(+)}(a^\pm)^2\mathbb{A}^+_{(+)}$ are of differential order $2n_++2$. $\mathcal{B}^-$ annihilates all the lowest states in each valence band and the lowest state in the equidistant part of the spectrum. The

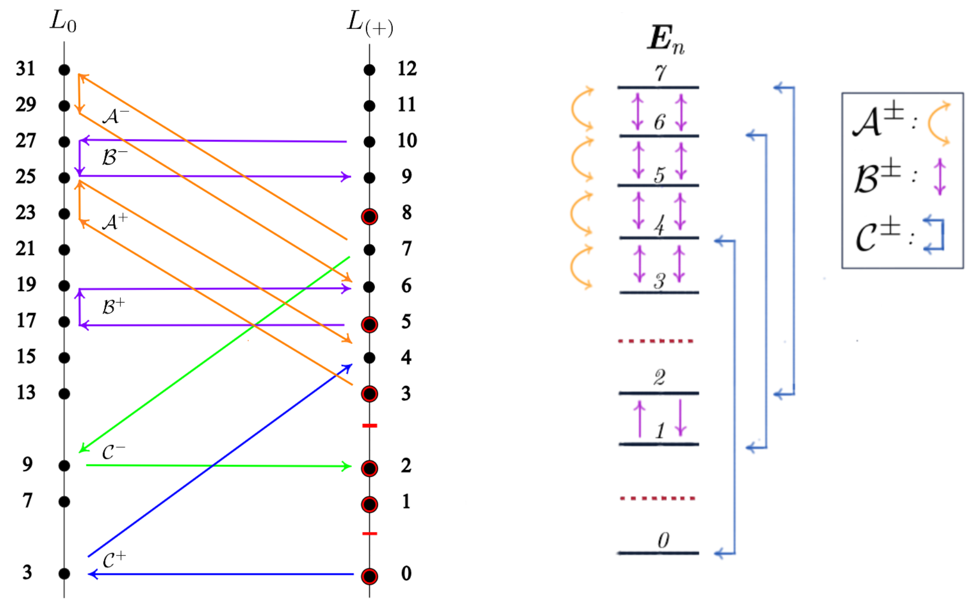

Figure 5.3: On the left: The numbers on the left correspond to the indices of the physical eigenstates ψ_{2l+1} of the half-harmonic oscillator that are mapped "horizontally" by operator $\mathbb{A}^-_{(+)}$ into eigenstates Ψ_n of the system (5.3.1). Lines show the action of the ladder operators coherently with their structure (5.3.6), (5.3.5) and (5.3.7). The marked set of the states 0, 1, 2, 3, 5, 8 on the right corresponds to six eigenstates of $L_{(+)}$ annihilated by $\mathcal{C}^-$. On the right: Horizontal lines correspond to the energy levels of $L_{(+)}$. Upward and downward arrows represent the action of the rising and lowering ladder operators, respectively. As it is shown in the figure on the right, following the appropriate paths, any eigenstate can be transformed into any other eigenstate by applying subsequently the corresponding ladder operators.

raising operator $\mathcal{B}^+$ annihilates all the highest states in each valence band. They act in the same way as $\mathcal{A}^\pm$ in the equidistant part of the spectrum.

- Operators $\mathcal{C}^\pm$ of the form (5.3.7) have a differential order $n_- + n_+ = 2l_m + 2$, and their commutation with Hamiltonian produces:

$$[L_{(-)}, \mathcal{C}^\pm] = \pm(l_m + 1)\Delta E \mathcal{C}^\pm \,. \tag{5.3.8}$$

  Lowering operator $\mathcal{C}^-$ annihilates $l_m + 1$ physical states, where we find all of the isolated states and some exited states of the equidistant part. Rising operator $\mathcal{C}^+$ does not annihilate any physical state.

When we have the schemes $(\ldots, 2l_m - 1, 2l_m) \sim (\ldots, -2l_m)$ generating a gapped rational extension of some AFF system, the corresponding Hamiltonian operators associated with them are shifted mutually for the distance $L_{(+)} - L_{(-)} = 4l_m + 2 = (l_m + \frac{1}{2})\Delta E$, that is equal to the half-integer multiple of the energy spacing in the equidistant part of the spectrum and in the valence bands with more than one state. In this case the procedure related to the construction of the ladder operators $\mathcal{A}^\pm$ and $\mathcal{B}^\pm$ and their properties are similar to those in the systems generated by the schemes $(\dots, 2l_m, 2l_m+1) \sim (\dots, -(2l_m+1))$. However, the situation with the construction of the ladder operators of the type $\mathcal{C}^{\pm}$ in this case is essentially different. We still can construct the operators $\mathcal{C}^{\pm}$ of the form (5.3.7). Such operators will be of odd differential order $2l_m+1$, and their commutation relations with any of the Hamiltonian operators $L_{(+)}$ and $L_{(-)}$ will be of the form $[L, \mathcal{C}^{\pm}] = \pm(4l_m+2)\mathcal{C}^{\pm}$. This means that these operators acting on physical eigenstates of L will produce nonphysical eigenstates excepting the case when the lowering operator $\mathcal{C}^-$ acts on the states from its kernel. The square of these operators will not have the indicated deficiency and will form together with the ladder operators $\mathcal{A}^{\pm}$ or $\mathcal{B}^{\pm}$ the set of the spectrum-generating operators. This picture can be compared with the case of the half-harmonic oscillator L_0, where the first order differential operators $a^{\pm}$ will have the properties similar to those of the described operators $\mathcal{C}^{\pm}$. In this case we can however modify slightly the construction of the ladder operators of the $\mathcal{C}^{\pm}$ type by taking

$$\widetilde{\mathcal{C}}^{-} = A^{-}_{(-)}(a^-)A^{+}_{(+)}\,, \qquad \widetilde{\mathcal{C}}^{+} = A^{-}_{(+)}(a^+)A^{+}_{(-)}\,. \tag{5.3.9}$$

These ladder operators satisfy the commutation relations $[L_{(\pm)}, \widetilde{\mathcal{C}}^{\pm}] = 4(l_m+1)\widetilde{\mathcal{C}}^{\pm}$, and transform a particular physical states into other physical states with different energy.

To conclude this section, let us summarize the structure of the nonlinearly deformed conformal symmetry algebras generated by different pairs of the corresponding ladder operators and Hamiltonians of the rationally deformed conformal mechanics systems. The commutators of the ladder operators $\mathcal{A}^{\pm}$, $\mathcal{B}^{\pm}$ and $\mathcal{C}^{\pm}$ with Hamiltonian operators are given, respectively, by Eqs. (5.3.6), (5.3.5) and (5.3.7) with $\Delta E = 4$. The commutation relations of the form (5.3.6) also are valid for the case of the isospectral deformations discussed in the previous section. To write down the commutation relations between raising and lowering operators of the same type in general case, let us introduce the polynomial functions

$$P_{n_+}(x) = \Pi_{k=1}^{n_+}(x - 2n_k - 1)\,, \qquad R_{n_-}(x) = \Pi_{l=1}^{n_-}(x + 2n_l + 1), \tag{5.3.10}$$

where $n_k > 0$ are the indices of the corresponding seed states in the positive scheme and $-n_l < 0$ are the indices of the seed states in the negative scheme. With this notation, we have the relations $A^{+}_{(+)}A^{-}_{(+)} = P_{n_+}(L_0)$, $A^{-}_{(+)}A^{+}_{(+)} = P_{n_+}(L_{(+)}) = P_{n_+}(L_{(-)}+2(n_-+n_+))$, and $A^{+}_{(-)}A^{-}_{(-)} = R_{n_-}(L_0)$, $A^{-}_{(-)}A^{+}_{(-)} = R_{n_-}(L_{(-)})$. Then we obtain

$$[\mathcal{A}^-, \mathcal{A}^+] = (x+1)(x+3)R_{n_-}(x)R_{n_-}(x+4)\big|^{L_{(-)}-4}_{x=L_{(-)}}\,, \tag{5.3.11}$$

$$[\mathcal{B}^-, \mathcal{B}^+] = (x+1)(x+3)P_{n_+}(x+4)P_{n_+}(x)\big|^{x=L_{(-)}+2N-4}_{x=L_{(-)}+2N}\,, \tag{5.3.12}$$

$$[\mathcal{C}^-, \mathcal{C}^+] = R_{n_-}(x)P_{n_+}(x)\big|^{x=L_{(-)}+2N}_{x=L_{(-)}}\,, \tag{5.3.13}$$

where $N = n_-+n_+$, and relation (5.3.11) also is valid in the case of isospectral deformations. In the case of the non-isospectral deformations given by the dual schemes $(\ldots,2l_m-1,2l_m)\sim(\ldots,-2l_m)$, the corresponding modified operators (5.3.9) satisfy the commutation relation

$$[\widetilde{\mathcal{C}}^-,\widetilde{\mathcal{C}}^+]=(x+1)R_{n_-}(x)P_{n_+}(x+2)\big|_{x=L_{(-)}}^{x=L_{(-)}+2N-2}\,. \tag{5.3.14}$$

Thus, in any rational deformation of the conformal mechanics model we considered, each pair of the conjugate ladder operators of the types $\mathcal{A}^\pm$, $\mathcal{B}^\pm$ or $\mathcal{C}^\pm$ generates a nonlinear deformation of the conformal $\mathfrak{sl}(2,\mathbb{R})$ symmetry. The commutation relations between ladder operators of different types of the form $[\mathcal{A}^\pm,\mathcal{C}^\pm]$, etc. is considered in next chapter, and their taking into account gives rise naturally to different nonlinearly extended versions of the superconformal $\mathfrak{osp}(2|2)$ symmetry [Inzunza and Plyushchay (2019a)].

5.4 Remarks

The construction of the spectrum-generating ladder operators can also be explored by using intertwining operators between the final rational extended model and some intermediate system in the Darboux chain. This possibility was explored in [Cariñena et al. (2018)]. Anyway, the final conclusion of this is that one always has a triad of pairs of ladder operators $\mathcal{A}^\pm$, $\mathcal{B}^\pm$ and $\mathcal{C}^\pm$ which behaves as described above. The only difference here is the number of nonphysical states that appear in the corresponding kernels.

An unresolved question for us is if there is any relationship between rationally extended systems and other systems of quantum mechanics, such as the conformal model (2.1.1) or a $\mathcal{PT}$ deformation of it [Mateos Guilarte and Plyushchay (2017, 2019); Plyushchay (2020)], we are thinking of something like the conformal bridge. It can be speculated that if such a relationship exists, it would be useful in applications related to integrable systems of infinite degrees of freedom, since $\mathcal{PT}$ symmetric systems have opened new branches in the search for solitonic solutions for the KdV equation and other integrable models [Correa and Fring (2016); Mateos Guilarte and Plyushchay (2019); Cen et al. (2020)].

In the next chapter we continue with rationally extended AFF models characterized by integer coupling constants as well as extended QHO systems, but now from the perspective of supersymmetric quantum mechanics.

Chapter 6

Nonlinear supersymmetries in rationally extended systems

We now turn to the study of the extensions and deformations of the superconformal and super-Schrödinger symmetries that appear in the $\mathcal{N}=2$ super-extended systems described by the super-partners ($L_{\rm os}$, L_{def}) and (L_0, $L_{m,def}$). Here L_{def} and $L_{m,def}$ correspond to rational deformations of the QHO system and the AFF model with integer values of the parameter $\nu = m$, $m \in \mathbb{N}$, respectively. As we have seen in the last chapter, the rational deformations of the QHO system and the AFF model are characterized, in the general case, by a finite number of missing energy levels, or gaps, in their spectra, and the description of such systems requires more than a couple of spectrum-generating operators. It is because of this expansion of the sets of ladder operators, whose differential order exceeds two, that nonlinearly deformed superconformal and super-Schrödinger structures appear. This chapter, based on the article [Inzunza and Plyushchay (2019a)], is devoted to the description of the complete sets of generators of the indicated symmetries. At this point, we will again take advantage of the Darboux duality property of the QHO system.

6.1 Basic intertwining operators

According to [Cariñena and Plyushchay (2017); Cariñena et al. (2018)], with each of the dual schemes it is necessary first to associate two basic pairs of the intertwining operators. Here, we discuss general properties of such operators without taking care of the concrete nature of the system built by the DCKA transformation. On the way, however, some important distinctions between rational deformations of the AFF model and harmonic oscillator have to be taken into account, and for this reason, it is convenient to speak of two classes of the systems. We distinguish between them by introducing the class index c, where $c = 1$ and $c = 2$ will correspond to deformed harmonic oscillator and deformed AFF conformal mechanics model, respectively.

As already established in the previous chapter, we will denote the Hamiltonian produced by the positive scheme Δ_+ (negative scheme Δ_-) by $L_{(+)}$ ($L_{(-)}$), and the corresponding intertwining operators by $A^-_{(+)}$ and $(A^-_{(+)})^\dagger \equiv A^+_{(+)}$ ($A^-_{(-)}$ and $(A^-_{(-)})^\dagger \equiv A^+_{(-)}$), see Sec. (5.1). These operators satisfy the relations

$$L_{(+)} - L_{(-)} = 2N\,, \qquad N = n_+ + n_-\,, \tag{6.1.1}$$

$$A^-_{(+)}L = L_{(+)}A^-_{(+)}\,, \qquad A^-_{(-)}L = L_{(-)}A^-_{(-)}\,, \tag{6.1.2}$$

and the corresponding Hermitian conjugate relations for $A^+_{(+)}$ and $A^+_{(-)}$. Here L could be L_{os} or L_0, depending on the class index c of the rationally deformed system $L_{(\pm)}$ that we want to study. Applying operator identities (6.1.2) to an arbitrary physical or nonphysical (formal) eigenstate φ_n of L different from any seed state of the positive scheme and using Eq. (5.1.4), one can derive the equality

$$A^-_{(-)}\varphi_n = A^-_{(+)}\varphi_{n+N}\,, \tag{6.1.3}$$

to be valid modulo a multiplicative constant. As a result, both operators acting on the same state of the harmonic oscillator produce different states of the new system. We have seen this behavior before in last chapter, Sec. 5.3. The Hermitian conjugate operators $A^+_{(-)}$ and $A^+_{(+)}$ do a similar job but in the opposite direction. Eq. (6.1.3) suggests that some peculiarities should be taken into account for class 2 systems: the infinite potential barrier at $x = 0$ assumes that physical states of L_0 and $L_{(\pm)}$ systems are described by odd wave functions. Then, in order for $A^-_{(+)}$to transform physical states of L_0 into physical states of $L_{(\pm)}$, we must take $n+N$ to be odd for odd n in (6.1.3). This means that $A^-_{(-)}$ transforms physical states into physical only if N is even. In the case of odd N, it is necessary to take $A^-_{(-)}a^-$ or $A^-_{(-)}a^+$ as a physical intertwining operator. It is convenient to take into account this peculiarity by denoting the remainder of the division N/c by $r(N,c)$: it takes value 1 in the class $c = 2$ of the systems with odd N and equals zero in all other cases.

The products of the described intertwining operators are of the form (1.2.7), and for further analysis it is useful to write down them explicitly:

$$A^+_{(\pm)}A^-_{(\pm)} = P_{n_\pm}(L)\,, \qquad A^-_{(\pm)}A^+_{(\pm)} = P_{n_\pm}(L_{(\pm)})\,, \tag{6.1.4}$$

$$P_{n_+}(\eta) \equiv \textstyle\prod_{k=1}^{n_+}(\eta - 2l^+_k - 1)\,, \qquad P_{n_-}(\eta) \equiv \textstyle\prod_{k=1}^{n_-}(\eta + 2l^-_k + 1)\,. \tag{6.1.5}$$

Here l^+_k are indexes of physical states with eigenvalues $2l^+_k + 1$ in the set Δ_+, and $-l^-_k$ correspond to nonphysical states with eigenvalues $-2l^-_k - 1$ in the negative scheme Δ_-. In the same vein, it is useful to write

$$(a^+)^k(a^-)^k = T_k(L_0), \qquad (a^-)^k(a^+)^k = T_k(L_0+2k)\,, \tag{6.1.6}$$

$$T_k(\eta) \equiv \prod_{s=1}^{k}(\eta - 2s + 1)\,, \qquad T_k(\eta+2k) \equiv \prod_{s=1}^{k}(\eta + 2s - 1)\,. \tag{6.1.7}$$

We also have the operator identities

$$(a^-)^N = (-1)^{n_-}A^+_{(-)}A^-_{(+)}\,, \qquad f(L_{(-)})A^-_{(+)}(a^+)^{n_-} = (-1)^{n_-}h(L_{(-)})A^-_{(-)}(a^-)^{n_+}\,, \tag{6.1.8}$$

and their Hermitian conjugate versions, where $f(\eta)$ and $h(\eta)$ are polynomials whose explicit structure is given in Appendix C.1. In one-gap deformations of the harmonic oscillator and gapless deformations of L_1 these polynomials reduce to 1.

6.2 Extended sets of ladder and intertwining operators

Actually, instead of three types of ladder operators, we have a total of three families of operators

$$\mathfrak{A}^\pm_k \equiv A^-_{(-)}(a^\pm)^k A^+_{(-)}\,, \qquad \mathfrak{B}^\pm_k \equiv A^-_{(+)}(a^\pm)^k A^+_{(+)}\,, \tag{6.2.1}$$

$$\mathfrak{C}^-_{N\pm k'} \equiv A^-_{(+)}(a^\mp)^{k'} A^+_{(-)}\,, \qquad \mathfrak{C}^+_{N\pm k'} \equiv (\mathfrak{C}^-_{N\pm k'})^\dagger\,, \tag{6.2.2}$$

where, formally, k can take any nonnegative integer value and k' is such that $N-k' \geq 0$, otherwise operators (6.2.2) reduce to $\mathfrak{A}^\pm_k$, [Inzunza and Plyushchay (2019a)]. Due to relations (6.1.4)-(6.1.8) one concludes that at $k=0$ and $N-k'=0$ all these operators are reduced to certain polynomials in $L_{(\pm)}$. These objects are generated by taking the commutator relations between two arbitrary representatives of the spectrum generator set described in the previous chapter, and behave like powers of the ladder operator in the QHO system. Calculations with these operators are discussed in detail in Appendix C.2, so this chapter contains only the main results.

Independently of the class of the system, or on whether the operators are physical or not, the three families $\mathfrak{D}^\pm_{\rho,j} = (\mathfrak{A}^\pm_j, \mathfrak{B}^\pm_j, \mathfrak{C}^\pm_j)$, $\rho = 1,2,3$, $j = 1,2,\ldots$, satisfy the commutation relations of the form

$$[L_{(\pm)}, \mathfrak{D}^\pm_{\rho,j}] = \pm 2j\mathfrak{D}^\pm_{\rho,j}\,, \qquad [\mathfrak{D}^-_{\rho,j}, \mathfrak{D}^+_{\rho,j}] = \mathcal{P}_{\rho,j}(L_{(-)})\,, \tag{6.2.3}$$

where $\mathcal{P}_{\rho,j}(L_{(-)})$ is a certain polynomial of the corresponding Hamiltonian operator of the system, whose order of polynomial is equal to differential order of $\mathfrak{D}^\pm_{\rho,j}$ minus one, see Appendix C.2. Algebra (6.2.3) can be considered as a deformation of $\mathfrak{sl}(2,\mathbb{R})$, [Mateos Guilarte and Plyushchay (2019)].

Of all the operators that can be built, our objective is to discriminate against those that are physical and cannot be written as products of lower order elements, belonging to others or to the same family. Having this in mind, we have the following assertion related to the three families:

- From (6.2.3) one concludes that $2j \propto \Delta E = 2c$. Then, for $\mathfrak{A}$ and $\mathfrak{B}$ families, the physical operators are those whose index is $j = lc$ with $l \in \mathbb{N}$, while for $\mathfrak{C}$ family index should be $j = N + r(N,c) + cs$, where s is integer such that $j > 0$.

- For isospectral deformations of the AFF model, the spectrum-generating set is given by any pair of the conjugate operators $\mathfrak{A}_2^{\pm}$, $\mathfrak{B}_2^{\pm}$, or $\mathfrak{C}_2^{\pm}$.

- Due to Eq. (6.1.8) one realizes that the basic operators in the general case are

$$\begin{cases} \mathfrak{A}_k^{\pm}\,, & 0 < k < N\,, \\ \mathfrak{B}_k^{\pm}\,, & 0 < k < N\,, \\ \mathfrak{C}_k^{\pm}\,, & 0 < k < 2N + r(N,c)\,, \end{cases} \tag{6.2.4}$$

- For one-gap deformations of the harmonic oscillator, the set of basic ladder operators can be reduced further to the set

$$\begin{cases} \mathfrak{A}_k^{\pm}\,, & 0 < k < n_+\,, \\ \mathfrak{B}_k^{\pm}\,, & 0 < k < n_-\,, \\ \mathfrak{C}_k^{\pm}\,, & M < k < n_+\,, \end{cases} \qquad M = \begin{cases} \max{(n_-,n_+)} & \text{if} \quad n_- \neq n_+\,, \\ N/2 & \text{if} \quad n_- = n_+\,, \end{cases} \tag{6.2.5}$$

  where the relations $\mathfrak{A}_{n_+}^{\pm} = (-1)^{n_-}\mathfrak{C}_{n_+}^{\pm}$ and $\mathfrak{B}_{n_-}^{\pm} = (-1)^{n_-}\mathfrak{C}_{n_-}^{\pm}$ were taken into account.

As is obvious from their explicit form, any of the basic elements belonging to one of the three families of ladder operators can be constructed by "gluing" two different intertwining operators associated with an alternative DCKA transformation, which are of the form $A_{(\pm)}a^{\pm}$ and $A_{(\pm)}a^{\mp}$, so their number should also be reduced. Indeed, for general deformations only the operators

$$\begin{cases} A_{(\pm)}^{-}(a^{\pm})^n\,, & 0 \leq n < N\,, \\ A_{(\pm)}^{-}(a^{\mp})^n\,, & 0 < n < N + r(N,c)\,, \end{cases} \tag{6.2.6}$$

and their Hermitian conjugate counterparts can be considered as basic, see Appendix C.2. One can note that the total number of the basic intertwining operators $\#_f = 2[(4N - 2 + r(N,c))/c]$ is greater than the number of the basic ladder operators $\#_{lad} = 2[(4N - 3 + r(N,c))/c]$ which can be constructed with their help. In particular case of gapless deformations of the AFF model, the indicated set of Darboux generators can be reduced to those which produce, by 'gluing' procedure, one conjugate pair of the spectrum-generating ladder operators of the form $\mathfrak{D}_{2,\rho}^{\pm}$.

For $c = 1$ one-gap systems, identity (6.1.8) allows us to reduce further the set of the basic intertwining operators, which, together with corresponding Hermitian conjugate ones, is given by any of the two options,

$$
\mathfrak{S}_z \equiv \begin{cases} A^-_{(-)}(a^+)^{|z|}\,, & -N < z \leq 0\,, \\ A^-_{(-)}(a^-)^{z}\,, & 0 < z \leq n_+\,, \\ A^-_{(+)}(a^+)^{N-z}\,, & n_+ < z \leq N\,, \\ A^-_{(+)}(a^-)^{N-z}\,, & N < z < 2N\,, \end{cases} \qquad \text{or} \qquad \mathfrak{S}'_z \equiv \mathfrak{S}_{N-z}\,, \tag{6.2.7}
$$

see Appendix C.2. Here we have reserved $z = 0$ and $z = N$ values for index z to the dual schemes intertwining operators: in the first choice, $\mathfrak{S}_0 = A^-_{(-)}$ and $\mathfrak{S}_N = A^-_{(+)}$, and for the second choice we have $\mathfrak{S}'_0 = A^-_{(+)}$ and $\mathfrak{S}'_N = A^-_{(-)}$. Written in this way, these operators satisfy the intertwining relations $\mathfrak{S}_z L = (L_{(-)} + 2z)\mathfrak{S}_z$ or $\mathfrak{S}'_z L = (L_{(+)} - 2z)\mathfrak{S}'_z$, and their Hermitian conjugate versions. Then, to study supersymmetry, we have to choose either positive or negative scheme to define the $\mathcal{N} = 2$ super-extended Hamiltonian. We take $\mathfrak{S}_z$ if we work with a negative scheme, and $\mathfrak{S}'_z$ if positive scheme is chosen for the construction of super-extension.

6.3 Supersymmetric extensions

For each of the two dual schemes, one can construct an $\mathcal{N} = 2$ super-extended Hamiltonian operator following the recipe given in Chap. 1, equation (1.2.8). The task is to choose appropriately $H_1 = \breve{L} - \lambda^*$ and $H_0 = L - \lambda^*$. We put $\breve{L} = L_{(+)}$ and $\lambda^* = \lambda_+ = 2l_1^+ + 1$ for positive scheme, and choose $\breve{L} = L_{(-)}$ and $\lambda^* = \lambda_- = -2l_1^- - 1$ for negative scheme. For both options, we set $L = L_{\text{os}}$ if we are dealing with a rational extension of harmonic oscillator, and $L = L_0$ if we work with a deformation of the AFF model. We name the matrix Hamiltonian associated with negative scheme as $\mathcal{H}$, and denote by $\mathcal{H}'$ the Hamiltonian of positive scheme. The spectrum of these systems can be found using the properties of the corresponding intertwining operators described in Sec. 1.2.2, see also refs. [Cariñena and Plyushchay (2017); Cariñena et al. (2018)]. The two Hamiltonians are connected by relation $\mathcal{H} - \mathcal{H}' = -N(1 + \sigma_3) - \lambda_- + \lambda_+$, and σ_3 plays a role of the $\mathcal{R}$ symmetry generator for both super-extended systems. In this subsection we finally construct the corresponding spectrum-generating superalgebra for $\mathcal{H}$ and $\mathcal{H}'$. The resulting structures are based on the physical operators $\mathfrak{D}^{\pm}_{\rho,j}$. As we shall see, the supersymmetric versions of the $c = 1$ systems are described by a nonlinearly extended super-Schrödinger symmetry with bosonic generators to be differential operators of even and odd orders, while in the case of the $c = 2$ systems we obtain nonlinearly extended superconformal symmetry in which bosonic generators are of even order only.

We construct a pair of fermionic operators on the basis of each intertwining operator from the set (6.2.6) and their Hermitian conjugate counterparts. Let us consider first the extended nonlinear super-Schrödinger symmetry of a one-gap deformed harmonic oscillator, and then we generalize the picture. If we choose the negative scheme, then we use $\mathfrak{S}_z$ defined in (6.2.7) to construct the set of operators

$$\mathcal{Q}_1^z = \begin{pmatrix} 0 & \mathfrak{S}_z \\ \mathfrak{S}_z^\dagger & 0 \end{pmatrix}, \qquad \mathcal{Q}_2^z = i\sigma_3 \mathcal{Q}_1^z, \qquad -N < z < 2N. \tag{6.3.1}$$

They satisfy the (anti)-commutation relations

$$[\mathcal{H}, \mathcal{Q}_a^z] = 2iz\epsilon_{ab}\mathcal{Q}_b^z, \qquad \{\mathcal{Q}_a^z, \mathcal{Q}_b^z\} = 2\delta_{ab}\mathbb{P}_z(\mathcal{H}, \sigma_3), \qquad [\Sigma, \mathcal{Q}_a^z] = -i\epsilon_{ab}\mathcal{Q}_b^z, \tag{6.3.2}$$

where $\Sigma = \frac{1}{2}\sigma_3$ and $\mathbb{P}_z$ are some polynomials whose structure is described in Appendix C.3. For the choice of the positive scheme to fix extended Hamiltonian, according to (6.2.7), the corresponding fermionic operators are given by $\mathcal{Q}_1^{\prime z} \equiv \mathcal{Q}_1^{N-z}$. They satisfy relations of the same form (6.3.2) but with replacement $\mathcal{H} \to \mathcal{H}'$, $\Sigma = \frac{1}{2}\sigma_3 \to \Sigma' = -\frac{1}{2}\sigma_3$, $\mathbb{P}_z(\mathcal{H}, \sigma_3) \to \mathbb{P}'_z(\mathcal{H}', \sigma_3) = \mathbb{P}_{N-z}(\mathcal{H}' - N(1+\sigma_3) - \lambda_- + \lambda_+, \sigma_3)$, $\mathcal{Q}_1^z \to \mathcal{Q}_2^{\prime z}$ and $\mathcal{Q}_2^z \to \mathcal{Q}_1^{\prime z}$. The fermionic operators $\mathcal{Q}_a^0$ (or $\mathcal{Q}_a^{\prime 0}$) are the supercharges of the (nonlinear in general case) $\mathcal{N} = 2$ Poincaré supersymmetry, which are integrals of motion of the system $\mathcal{H}$ (or $\mathcal{H}'$), and $\mathbb{P}_0 = P_{n_-}(\mathcal{H}+\lambda_-)$ (or $\mathbb{P}_0 = P_{n_+}(\mathcal{H}'+\lambda_+)$) with polynomials $P_{n_\pm}$ defined in (6.1.5). The operators $\mathcal{Q}_a^{\prime 0}$ are analogous here to supercharges in Q_ν^a in the linear case, see Chap. 2. On the other hand, we have here the fermionic operators $\mathcal{Q}_a^{\prime N}$ as analogs of dynamical integrals $\mathcal{S}_\nu^a$ there. We recall that in the simple linear case considered in section 2.2, the interchange between positive and negative schemes corresponds to the automorphism of superconformal algebra, and this observation will be helpful for us for the analysis of the nonlinearly extended super-Schrödinger structures. Here, actually, each of the $(\#_f - 2)/2$ pairs of fermionic operators distinct from supercharges provides a possible dynamical extension of the super-Poincaré symmetry. As we will see, all of them are necessary to obtain a closed nonlinear spectrum-generating superalgebra of the super-extended system.

To construct any extension of the deformed Poincaré supersymmetry, we calculate $\{\mathcal{Q}_a^0, \mathcal{Q}_a^z\}$, in the negative scheme, or $\{\mathcal{Q}_a^{\prime 0}, \mathcal{Q}_a^{\prime z}\}$ in the positive one. In the first case we have

$$\{\mathcal{Q}_a^0, \mathcal{Q}_b^z\} = \delta_{ab}(\mathcal{G}_{-z}^{(2\theta(z)-1)} + \mathcal{G}_{+z}^{(2\theta(z)-1)}) + i\epsilon_{ab}(\mathcal{G}_{-z}^{(2\theta(z)-1)} - \mathcal{G}_{+z}^{(2\theta(z)-1)}), \tag{6.3.3}$$

where $z \in (-N, 0) \cup (0, 2N)$, $\theta(z) = 1\,(0)$ for $z > 0\,(z < 0)$, and $\mathcal{G}_{\pm z}^{(2\theta(z)-1)}$ are given by

$$\mathcal{G}_{+z}^{(2\theta(z)-1)} = \begin{pmatrix} \mathfrak{S}_0(\mathfrak{S}_z)^\dagger & 0 \\ 0 & (\mathfrak{S}_z)^\dagger\mathfrak{S}_0 \end{pmatrix}, \qquad \mathcal{G}_{-z}^{(2\theta(z)-1)} = (\mathcal{G}_{+z}^{(2\theta(z)-1)})^\dagger. \tag{6.3.4}$$

Following definition (6.2.7), one finds directly that $\mathfrak{S}_0(\mathfrak{S}_z)^\dagger$ is equal to $\mathfrak{A}_{|z|}^-$ when $-N < z < 0$, while for $0 < z \leq n_+$, this operator is equal to $\mathfrak{A}_z^+$, and takes the form of $\mathfrak{C}_z^+$ for $n_+ < z < 2N$.

The operators $(\mathfrak{S}_z)^\dagger\mathfrak{S}_0$ reduce to

$$(\mathfrak{S}_z)^\dagger\mathfrak{S}_0 = \begin{cases} P_{n_-}(L-2k)(a^-)^{|z|}\,, & -N<z<0\,,\\ (a^+)^z P_{n_-}(L)\,, & 0<z\leq n_+\,,\\ (-1)^{n_-}(a^+)^z T_{N-z}(L+2N)\,, & n_+<z<N\,,\\ (-1)^{n_-}(a^+)^z\,, & N\leq z<2N\,. \end{cases} \tag{6.3.5}$$

Note that $\mathcal{G}^{(-1)}_{\pm k}$ and $\mathcal{G}^{(+1)}_{\pm k}$ with $k=|z|\leq n_-$ are two different matrix extensions of the same operator $\mathfrak{A}^{\pm}_k$.

For a super-extended system based on the positive scheme, we obtain

$$\{\mathcal{Q}'^{0}_a,\mathcal{Q}'^{z}_b\} = \delta_{ab}(\mathcal{G}'^{(2\theta(z)-1)}_{-z}+\mathcal{G}'^{(2\theta(z)-1)}_{+z}) - i\epsilon_{ab}(\mathcal{G}'^{(2\theta(z)-1)}_{-z}-\mathcal{G}'^{(2\theta(z)-1)}_{+z})\,, \tag{6.3.6}$$

where, again, $z\in(-N,0)\cup(0,2N)$, and $\mathcal{G}'^{(2\theta(z)-1)}_{\pm z}$ are given by

$$\mathcal{G}'^{(2\theta(z)-1)}_{-z} = \begin{pmatrix} \mathfrak{S}'_0(\mathfrak{S}'_z)^\dagger & 0\\ 0 & (\mathfrak{S}'_z)^\dagger\mathfrak{S}'_0 \end{pmatrix}\,, \qquad \mathcal{G}'^{(2\theta(z)-1)}_{+z} = (\mathcal{G}'^{(2\theta(z)-1)}_{-z})^\dagger\,. \tag{6.3.7}$$

Now, $\mathfrak{S}'_0(\mathfrak{S}'_z)^\dagger = \mathfrak{B}^+_{|z|}$ when $-N<z<0$, while for positive index z this operator reduces to $\mathfrak{B}^-_z$ when $0<z\leq n_-$, and to $\mathfrak{C}^-_z$ when $n_-<z<2N$. For the other matrix element we have

$$(\mathfrak{S}'_z)^\dagger\mathfrak{S}'_0 = \begin{cases} (a^+)^{|z|}P_{n_+}(L)\,, & -N<z<0\,,\\ (a^-)^z P_{n_+}(L)\,, & 0<z\leq n_-\,,\\ (-1)^{n_-}T_{N-k}(L)(a^-)^z\,, & n_-<z<N\,,\\ (-1)^{n_-}(a^-)^z\,, & N<z<2N\,. \end{cases} \tag{6.3.8}$$

Here, again, there are two different matrix extensions of the operators of the $\mathfrak{B}$-family given by $\mathcal{G}'^{(+1)}_{\pm k}$ and $\mathcal{G}'^{(-1)}_{\pm k}$ when $k\leq n_-$.

By comparing both schemes one can note two other special features. It turns out that $\mathcal{G}^{(1)}_{\pm k} = \mathcal{G}'^{(1)}_{\pm k}$ when $k\geq N$, and this corresponds to the automorphism discussed in section 2.2. In the same way, for $\max(n_-,n_+)<k<N$, operators $\mathcal{G}^{(1)}_{\pm k}$ and $\mathcal{G}'^{(1)}_{\pm k}$ are different matrix extensions of $\mathfrak{C}^{\pm}_k$.

From here and in what follows we do not specify whether we have the super-extended system corresponding to the negative or the positive scheme, and will just use, respectively, the unprimed or primed notations for operators of the alternative dual schemes. In particular, we have

$$[\mathcal{H},\mathcal{G}^{(2\theta(z)-1)}_{\pm k}] = \pm 2k\mathcal{G}^{(2\theta(z)-1)}_{\pm k}\,, \qquad k\equiv|z|\,, \qquad z\in(-N,0)\cup(0,2N)\,, \tag{6.3.9}$$

that shows explicitly that our new bosonic operators have the nature of ladder operators of the super-extended system $\mathcal{H}$. Commutators $[\mathcal{G}^{(1)}_{-k},\mathcal{G}^{(1)}_{+k}]$ and $[\mathcal{G}^{(-1)}_{-k},\mathcal{G}^{(-1)}_{+k}]$ produce polynomials in $\mathcal{H}$ and σ_3, which can be calculated by using the polynomials $\mathcal{P}_{\rho,j}$ defined in (6.2.3). The algebra generated by $\mathcal{H}$, $\mathcal{G}^{(2\theta(z)-1)}_{\pm k}$ and σ_3 is identified as a deformation of $\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{u}(1)$, where a concrete form of deformation depends on the system, $\mathcal{H}$, and on z. Each of these nonlinear bosonic algebras expands further up to a certain closed nonlinear deformation of superconformal $\mathfrak{osp}(2|2)$ algebra generated by the subset of operators

$$\mathcal{U}^{(2\theta(z)-1)}_{0,z}\equiv\{\mathcal{H},\sigma_3,\mathbb{I},\mathcal{G}^{(2\theta(z)-1)}_{\pm|z|},\mathcal{Q}^0_a,\mathcal{Q}^z_a\}\,,\qquad z\in(-N,0)\cup(0,2N)\,,\tag{6.3.10}$$

see Appendix C.3.

The deficiency of any of these nonlinear superalgebras is that none of them is a spectrum-generating algebra for the super-extended system: application of operators from the set (6.3.10) and of their products does not allow one to connect two arbitrary eigenstates in the spectrum of $\mathcal{H}$. To find the spectrum-generating superalgebra for this kind of the super-extended systems, one can try to include into the superalgebra simultaneously the operators $\mathcal{G}^{(1)}_{\pm N}$ and, say, $\mathcal{G}^{(1)}_{\pm 1}$ or $\mathcal{G}^{(-1)}_{\pm 1}$. The operators $\mathcal{G}^{(1)}_{\pm N}$ provide us with matrix extension of the operators $\mathfrak{C}^{\pm}_N$ being ladder operators for deformed subsystems $L_{(-)}$ or $L_{(+)}$. Analogously, operators $\mathcal{G}^{(1)}_{\pm 1}$ or $\mathcal{G}^{(-1)}_{\pm 1}$ supply us with matrix extensions of the ladder operators $\mathfrak{A}^{\pm}_1$ or $\mathfrak{B}^{\pm}_1$ ($\mathfrak{A}^{\pm}_2$ or $\mathfrak{B}^{\pm}_2$) when systems $L_{(\pm)}$ are of the class $c=1$ or $c=2$ with even (odd) N. Therefore, it is enough to unify the sets of generators $\mathcal{U}^{(1)}_{0,1}$ and $\mathcal{U}^{(1)}_{0,N}$. Having in mind the commutation relations between operators of the three families $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$, one can find, however, that the commutators of the operators $\mathcal{G}^{(1)}_{\pm N}$ with $\mathcal{G}^{(1)}_{\pm 1}$ generate other bosonic matrix operators $\mathcal{G}^{(1)}_{\pm k}$. The commutation of these operators with supercharges $\mathcal{Q}^0_a$ generates the rest of the fermionic operators we considered, see Appendix C.3 for details. The set of higher order generators is completed by considering all non-reducible bosonic and fermionic generators, which do not decompose into the products of other generators. In correspondence with that was noted above, we arrive finally at two different extensions of the sets of operators with index less than N. By this reason it is convenient also to introduce the operators

$$\mathcal{G}^{(0)}_{\pm k}\equiv\Pi_-(a^{\pm})^k,\qquad k=1,\ldots,N-1,\qquad \Pi_-=\tfrac{1}{2}(1-\sigma_3)\,,\tag{6.3.11}$$

which help us to fix in a unique way the bosonic set of generators. For our purposes we choose to write all the operators $\mathcal{G}^{(-1)}_{\pm k}$ in terms of $\mathcal{G}^{(1)}_{\pm k}$ and $\mathcal{G}^{(0)}_{\pm k}$ when $k\leq n_+$ in the negative scheme, and when $k\leq n_-$ in the extended system associated with the positive scheme. For indexes outside the indicated scheme-dependent range, we neglect operators $\mathcal{G}^{(-1)}_{\pm k}$ because they are not basic in correspondence with the discussion on reduction of ladder operators in the previous Sec. 6.2. As a result, we have to drop from (6.3.10) all the operators $\mathcal{G}^{(2\theta(z)-1)}_{\pm|z|}$ with $z\in(-N,0)$.

By taking anti-commutators of fermionic operators $\mathcal{Q}_a^N$ with $\mathcal{Q}_a^z$, $z \neq 0$, we produce bosonic dynamical integrals $\mathcal{J}_{\pm|z-N|}^{(1-2\theta(z-N))}$, which have exactly the same structure of the even generators $\mathcal{G}'^{(2\theta(z)-1)}_{\pm|z|}$ in the extension associated with the dual scheme. In this way we obtain the subsets of operators

$$\mathcal{I}_{N,z}^{(1-2\theta(z-N))} \equiv \{\mathcal{H}, \sigma_3, \mathbb{I}, \mathcal{J}_{\pm|z-N|}^{(1-2\theta(z-N))}, \mathcal{Q}_a^N, \mathcal{Q}_a^z\} \qquad z \in (-N,0)\cup(0,2N)\,, \tag{6.3.12}$$

which also generate closed nonlinear super-algerabraic structures. With the help of (6.3.11), we find similarly to the subsets (6.3.10), that a part of the sets (8.1.16) also can be reduced.

Having in mind the ordering relation between n_- and n_+, the super-extended systems associated with the negative schemes can be characterized finally by the following irreducible, in the sense of subection 6.2, subsets of symmetry generators:

$n_- \leq n_+$	$n_+ < n_-$
$\mathcal{U}_{0,k}^{(1)}\,, \qquad 0 < k < 2N$	$\mathcal{U}_{0,k}^{(1)}\,, \qquad k \in (0,n_+)\cup(n_-,2N)$
$\mathcal{I}_{N,z}^{(1-2\theta(N-z))}\,, \qquad z \in (-N,0)\cup(n_+,N)$	$\mathcal{I}_{N,z}^{(1-2\theta(N-z))}\,, \qquad z \in (-N,0)\cup[n_+,N)$

Table 6.1: Symmetry generators subset.

For more details, see Appendix C.2. A similar result can be obtained for super-extended systems associated with positive schemes, where the roles played by families $\mathfrak{A}$ and $\mathfrak{B}$, and of numbers n_- and n_+ are interchanged.

Finally, we arrive at the following picture. Any operator that can be generated via (anti)-commutation relations and which does not belong to the sub-sets appearing in Table 6.1, can be written as a product of the basic generators. For super-extensions of rationally deformed one-gap harmonic oscillator systems we have considered, the spectrum-generating algebra is composed from the sets $\mathcal{U}_{0,k}^{(1)}$ and $\mathcal{I}_{N,z}^{(1-2\theta(N-z))}$ and from those operators generated by them via (anti)-commutation relations which cannot be written as a product of the basic generators. It is worth to stress that in this set of generators the unique true integrals of motion, in addition to $\mathcal{H}$ and σ_3, are the supercharges $\mathcal{Q}_a^0$, while the rest has to be promoted to the dynamical integrals by unitary transforming them with the evolution operator.

For gapless rational extensions of the systems of class $c = 2$, only the subset $\mathcal{U}_{0,2}^{(1)}$ has to be considered instead of the family of sets $\mathcal{U}_{0,k}^{(1)}$. For super-extensions of rationally deformed systems of arbitrary form in the sense of the class c and arbitrary number of gaps and their dimensions, the identification of their generalized super-Schrödinger or superconformal structures is realized in a similar way. The procedure is based on the sets of operators (8.2.6) and (6.2.6), which include the operators (6.2.5) and (6.2.7) of the discussed one-gap case as subsets. As a result, for every irreducible pair of ladder operators (8.2.6) with index less than N we have two super-extensions which are related by operators of the form (6.3.11). When we put together the subsets containing the spectrum-generating set of operators, we obtain all the other structures.

We would like to end this section highlighting some of the peculiarities of the simplest systems that can be treated with this machinery and these are

Peculiarities of one-gap deformations of the QHO : The super-extended Hamiltonian constructed on the base of the negative scheme with $n_- = 1$ is characterized by unbroken $\mathcal{N} = 2$ Poincaré supersymmetry, whose supercharges, being the first order differential operators, generate a Lie superalgebra. The $\mathfrak{B}$ family of ladder operators in the sense of (6.2.5) does not play any role in this scheme. On the other hand, the super-Hamiltonian provided by the positive scheme possesses n_+ singlet states while the ground state is a doublet. The $\mathcal{N} = 2$ super-Poincaré algebra of such a system is nonlinear as its supercharges are of differential order $n_+ = 2\ell \geq 2$.

Peculiarities of gapless deformations of L_1 : The negative scheme produces a super-Hamiltonian with spontaneously broken supersymmetry, whose all energy levels are doubly degenerate; its $\mathcal{N} = 2$ super-Poincaré algebra has linear nature. To construct the spectrum-generating algebra we only need a matrix extension of the operators $\mathfrak{A}_2^{\pm}$. In a super-extended system produced by the positive scheme, $n_+ > 1$ physical and nonphysical states of L_0 of positive energy (the latter being even eigenstates of harmonic oscillator) are used as seed states for DCKA transformation. Its supersymmetry is spontaneously broken, and the $\mathcal{N} = 2$ super-Poincaré algebra is nonlinear. The nonlinearly deformed super-Poincaré symmetry cannot be expanded to spectrum-generating superalgebra by combining it with matrix extension of the $\mathfrak{A}_2^{\pm}$, but this can be done by using matrix extensions of the $\mathfrak{B}_2^{\pm}$ or $\mathfrak{C}_2^{\pm}$ ladder operators, see (6.3.7). The resulting spectrum-generating superalgebra is a certain nonlinear deformation of the $\mathfrak{osp}(2|2)$ superconformal symmetry.

6.4 Example 1: Gapless deformation of AFF model

The example considered here corresponds to the same system analyzed in the previous chapter, in Sec. 5.2. By construction, the super-Hamiltonian and its spectrum correspond to

$$\mathcal{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix}, \qquad \mathcal{E}_n = 4n + 10\,, \qquad n = 0, 1, \ldots\,, \tag{6.4.1}$$

where $H_1 = L_{(-)} + 7$, with $L_{(-)}$ given in (5.2.1), and $H_0 = L_0 + 7$. Due to complete isospectrality of H_1 and H_0, all the energy levels of the system (6.4.1) including the lowest one $\mathcal{E}_0 = 10 > 0$ are doubly degenerate and we have here the case of spontaneously broken $\mathcal{N} = 2$ super-Poincaré symmetry generated by Hamiltonian $\mathcal{H}$, the supercharges $\mathcal{Q}_a^0$ constructed in terms of $A_{(-)}^{\pm}$, and by $\Sigma = \frac{1}{2}\sigma_3$.

The generators that should be considered for the super-extension correspond to

$$\mathcal{U}_{0,2}^{(1)} = \{\mathcal{H}, \mathbb{I}, \mathcal{G}_{\pm 2}^{(1)}, \sigma_3, \mathcal{Q}_a^0, \mathcal{Q}_a^2\}\,, \tag{6.4.2}$$

where

$$\mathcal{Q}_1^z = \begin{pmatrix} 0 & A_{(-)}^-(a^-)^z \\ (a^+)^z A_{(-)}^+ & 0 \end{pmatrix}, \; z = 0, 2\,, \tag{6.4.3}$$

$$\mathcal{G}_{-2}^{(1)} = \begin{pmatrix} A_{(-)}^-(a^-)^2 A_{(-)}^+ & 0 \\ 0 & H_0(a^-)^2 \end{pmatrix}, \tag{6.4.4}$$

$$\mathcal{Q}_2^z = i\sigma_3 \mathcal{Q}_1^z\,, \qquad \mathcal{G}_{+2}^{(1)} = (\mathcal{G}_{-2}^{(1)})^\dagger\,, \tag{6.4.5}$$

and the explisit form of $A_{(-)}^{\pm}$ is given in (5.2.3). The complete set of superalgebraic relations they satisfy is

$$[\mathcal{H}, \mathcal{Q}_a^0] = 0\,, \qquad [\mathcal{H}, \mathcal{Q}_a^2] = 4i\epsilon_{ab}\mathcal{Q}_b^2\,, \qquad [\sigma_3, \mathcal{Q}_a^z] = -2i\epsilon_{ab}\mathcal{Q}_b^z\,, \quad z = 0, 2\,, \tag{6.4.6}$$

$$\{\mathcal{Q}_a^0, \mathcal{Q}_a^0\} = 2\delta_{ab}\mathcal{H}\,, \qquad \{\mathcal{Q}_a^0, \mathcal{Q}_b^2\} = \delta_{ab}(\mathcal{G}_{-2}^{(1)} + \mathcal{G}_{+2}^{(1)}) + i\epsilon_{ab}(\mathcal{G}_{-2}^{(1)} - \mathcal{G}_{+2}^{(1)})\,, \tag{6.4.7}$$

$$[\mathcal{H}, \mathcal{G}_{\pm 2}^{(1)}] = \pm 4\mathcal{G}_{\pm 2}^{(1)}\,, \qquad [\mathcal{G}_{\mp 2}^{(1)}, \mathcal{Q}_a^0] = \pm 2(\mathcal{Q}_a^2 \mp i\epsilon_{ab}\mathcal{Q}_b^2)\,, \tag{6.4.8}$$

$$[\mathcal{G}_{-2}^{(1)}, \mathcal{G}_{+2}^{(1)}] = 8(\mathcal{H} - 4)(\mathcal{H}(2\mathcal{H} - 9) + \Pi_-(\mathcal{H}^2 - 4\mathcal{H} + 24))\,, \tag{6.4.9}$$

$$[\mathcal{G}_{\mp 2}^{(1)}, \mathcal{Q}_a^2] = \pm 2(-80 + 4\mathcal{H} + \mathcal{H}^2)(\mathcal{Q}_a^0 \pm i\epsilon_{ab}\mathcal{Q}_b^0)\,, \tag{6.4.10}$$

$$\{\mathcal{Q}_a^2, \mathcal{Q}_b^2\} = 2\delta_{ab}(\eta + 1)(\eta + 3)(\eta + 7)|_{\eta = \mathcal{H} + 2\sigma_3 - 9}\,, \tag{6.4.11}$$

where $\Pi_- = \frac{1}{2}(1 - \sigma_3)$. The common eigenstates of $\mathcal{H}$ and $\mathcal{Q}_1^0$ are

$$\Psi_n^+ = \begin{pmatrix} (\mathcal{E}_n)^{-1/2} A_{(-)}^- \psi_{2n+1} \\ \psi_{2n+1} \end{pmatrix}, \qquad \Psi_n^- = \sigma_3 \Psi_n^+, \tag{6.4.12}$$

where $\mathcal{Q}_1^0 \Psi_n^{\pm} = \pm\sqrt{\mathcal{E}_n}\Psi_n^{\pm}$, and we have here the relations $\Psi_n^{\pm} = (\mathcal{G}_{+2}^{(1)})^n \Psi_0^{\pm}$ and $\mathcal{G}_{-2}^{(1)}\Psi_0^{\pm} = 0$. As a result one can generate all the complete set of eigenstates of the system by applying the generators of superalgebra to any of the two ground states Ψ_0^+ or Ψ_0^-, and therefore the restricted set of generators we have chosen is the complete spectrum-generating set for the super-extended system (6.4.1).

The complete set of (anti)-commutation relations (6.4.8)-(6.4.11) corresponds to a nonlinear deformation of superconformal algebra $\mathfrak{osp}(2|2)$. The first relation from (6.4.8) and equation (6.4.9) represent a nonlinear deformation of $\mathfrak{sl}(2,\mathbb{R})$ with commutator $[\mathcal{G}_{-2}^{(1)}, \mathcal{G}_{+2}^{(1)}]$ to be a cubic polynomial in $\mathcal{H}$. From the superalgebraic relations it follows that like in the linear case of superconformal $\mathfrak{osp}(2|2)$ symmetry discussed in Chap 2, Sec. 2.2, here the extension of the set of generators $\mathcal{H}$, $\mathcal{Q}_a^0$

and Σ of the $\mathcal{N}=2$ Poincaré super-symmetry by any one of the dynamical integrals $\mathcal{Q}^2_a$, $a=1,2$, $\mathcal{G}^{(1)}_{+2}$ or $\mathcal{G}^{(1)}_{-2}$ recovers all the complete set of generators of the nonlinearly deformed superconformal $\mathfrak{osp}(2|2)$ symmetry.

Due to a gapless deformation of the AFF model, here similarly to the case of the non-deformed superconformal $\mathfrak{osp}(2|2)$ symmetry, the super-extension based on the positive scheme is characterized by essentially different physical properties. The positive scheme of the system corresponds to the states$(1,2,3)$ and in this case we identify $\mathcal{H}'=\text{diag}\,(L_{(+)}-3, L_0-3)$ as the extended Hamiltonian. This $\mathcal{H}'$ is related to $\mathcal{H}$ defined by Eq. (6.4.1) by the equality $\mathcal{H}'=\mathcal{H}-6+4\sigma_3$. For extended system $\mathcal{H}'$, supercharges $\mathcal{Q}'^0_a$ have the form similar to $\mathcal{Q}^0_a$ in (6.4.3) but with $A^{\pm}_{(-)}$ changed for the third order intertwining operators $A^{\pm}_{(+)}$, constructed with the formula (1.2.5). Being differential operators of the third order, they satisfy relations $[\mathcal{H}', \mathcal{Q}'^0_a]=0$ and $\{\mathcal{Q}'^0_a, \mathcal{Q}'^0_b\}=2\delta_{ab}P_{n_+}(\mathcal{H}'+3)$ with $P_{n_+}(\mathcal{H}'+3)=\mathcal{H}'(\mathcal{H}'-2)(\mathcal{H}'-4)$. The linear $\mathcal{N}=2$ super-Poincaré algebra of the system (6.4.1) is changed here for the nonlinearly deformed superalgebra with anti-commutator to be polynomial of the third order in Hamiltonian. This system has two nondegenerate states $(0,\psi_1)^t$ and $(0,\psi_3)^t$ of energies, respectively, 0 and 4, and both them are annihilated by both supercharges $\mathcal{Q}'^0_a$. All higher energy levels $\mathcal{E}'_n=4n$ with $n=2,3,\ldots$ are doubly degenerate. Thus, the nonlinearly deformed $\mathcal{N}=2$ super-Poincaré symmetry of this system can be identified as partially unbroken [Klishevich and Plyushchay (2001)] since the supercharges have differential order three but annihilate only two nondegenerate physical states. Here instead of the spectrum-generating set $\mathcal{U}^{(1)}_{0,2}$, formed by true and dynamical integrals, the same role is played by the set of integrals $\mathcal{U}'^{(1)}_{0,2}=\{\mathcal{H}', \mathcal{G}'^{(1)}_{\pm 2}, \mathbb{I}, \sigma_3, \mathcal{Q}'^0_a, \mathcal{Q}'^2_a\}$, where fermionic generators are $\mathcal{Q}'^z_a=\mathcal{Q}^{4-z}_a$ with $z=0,2$ according with (6.2.7) and (6.3.1). Bosonic dynamical integrals $\mathcal{G}'^{(1)}_{\pm 2}$ are given here by

$$\mathcal{G}'^{(1)}_{-2}=\begin{pmatrix} A^-_{(+)}(a^+)A^+_{(-)} & 0 \\ 0 & (L_0-1)(a^-)^2 \end{pmatrix}, \qquad \mathcal{G}'^{(1)}_{+2}=(\mathcal{G}'^{(1)}_{-2})^\dagger\,, \tag{6.4.13}$$

where equations in (6.3.7) have been used for the case of the present positive scheme. They are generated via anticommutation of $\mathcal{Q}'^0_a$ with $\mathcal{Q}'^2_b$. The set of operators $\mathcal{U}'^{(1)}_{0,2}$ generates the nonlinearly deformed superconformal $\mathfrak{osp}(2|2)$ symmetry given by superalgebra of the form (6.4.6)–(6.4.11), but with coefficients to be polynomials of higher order in Hamiltonian $\mathcal{H}'$ in comparison with the case of the system (6.4.1).

6.5 Example 2: Rationally extended harmonic oscillator

The example we discuss in this subsection corresponds to the rational extension of QHO based on the dual schemes $(1,2)\sim(-2)$, for which $N=3$. Different aspects of this system were extensively studied in literature [Cariñena and Plyushchay (2017); Cariñena et al. (2018)]. Here, we investigate it in the light of the nonlinearly extended super-Schrödingerr symmetry.

The Hamiltonian produced via Darboux transformation based on the negative scheme is

$$L_{(-)} = -\frac{d^2}{dx^2} + x^2 + 8\frac{2x^2-1}{(1+2x^2)^2} - 2\,, \tag{6.5.1}$$

whose spectrum is $E_0 = -5$, $E_{n+1} = 2n+1$, $n = 0, 1, \ldots$. In this system a gap of size 6 separates the ground state energy from the equidistant part of the spectrum, where levels are separated from each other by a distance $\Delta E = 2$. The pair of ladder operators of the $\mathfrak{C}$-family connects here the isolated ground state with the equidistant part of the spectrum, and together with the ladder operators $\mathfrak{A}_1^{\pm}$ they form the complete spectrum-generating set of operators for the system. The intertwining operators of the negative scheme are

$$A_{(-)}^{-} = \frac{d}{dx} - x - \frac{4x}{2x^2+1}, \qquad A_{(-)}^{+} \equiv (A_{(-)}^{-})^{\dagger}\,. \tag{6.5.2}$$

We also have the intertwining operators $A_{(+)}^{\pm}$ constructed on the base of the seed states of the positive scheme $(1,2)$. These four operators satisfy their respective intertwining relations of the form (6.1.2), and their alternate products (6.1.5) reduce here to polynomials $P_{n_-}(L_{(-)}) = L_{(-)}+5 \equiv H_1$, $P_{n_-}(L) = L+5 \equiv H_0$ and $P_{n_+}(L_{(+)}) = (L_{(+)}-3)(L_{(+)}-5)$, $P_{n_+}(L) = (L+3)(L+5)$, where $L = L_{\text{os}}$ is the Hamiltonian operator of the harmonic oscillator, and $L_{(+)}$ is the Hamiltonian produced by positive scheme, which is related with $L_{(-)}$, according to (5.1.4), by $L_{(+)} - L_{(-)} = 6$. Here, the eigenstate $A_{(-)}^{-}\widetilde{\psi_{-2}} = 1/\psi_{-2}$ is the isolated ground state of zero energy of the shifted Hamiltonian operator H_1.

The super-extended Hamiltonian and its spectrum are

$$\mathcal{H} = \begin{pmatrix} H_1 & 0 \\ 0 & H_0 \end{pmatrix}, \qquad \mathcal{E}_0 = 0\,, \qquad \mathcal{E}_{n+1} = 2n+6\,, \qquad n = 0, 1, \ldots\,. \tag{6.5.3}$$

The ground state of zero energy is non-degenerate and corresponds to the ground state $(A_{(-2)}^{-}\widetilde{\psi_{-2}}, 0)^t$. Other energy levels are doubly degenerate and correspond to eigenstates of the extended Hamiltonian (6.5.3) and supercharge $\mathcal{Q}_1^0$, see below:

$$\Psi_{n+1}^{+} = \begin{pmatrix} (\mathcal{E}_{n+1})^{-1/2}A_{(-)}^{-}\psi_n \\ \psi_n \end{pmatrix}, \qquad \Psi_{n+1}^{-} = \sigma_3\Psi_{n+1}^{+}\,. \tag{6.5.4}$$

The system (6.5.3) is characterized by unbroken $\mathcal{N} = 2$ Poincaré supersymmetry. Now we use the construction of Sec. 6.3 to produce generators of the extended nonlinearly deformed super-Schrödinger symmetry of the system. Following (6.3.1) and (6.3.4), we construct the odd operators $\mathcal{Q}_a^z$ with $z = -2, -1, 0, \ldots, 5$, and matrix bosonic ladder operators $\mathcal{G}_{\pm k}^{(1)}$ with $k = 1, \ldots, 5$. Also we must consider the operators $\mathcal{G}^{(0)}_{\pm k}$ with $k = 1, 2$ defined in (6.3.11). To obtain all the ingredients, we have to use the version of relation (C.2.5) for this system translated to the supersymmetric extension of $\mathfrak{C}^{\pm}_{N+k}$ which is

$$\mathcal{G}^{(1)}_{\pm(3l+n)} = (-\mathcal{G}^{(1)}_{\pm 3})^{l}\mathcal{G}^{(1)}_{\pm n}\,, \qquad n = 3, 4, 5\,, \qquad l = 0, 1, \ldots\,. \tag{6.5.5}$$

Then we generate the even part of the superalgebra:

$$[\mathcal{H}, \mathcal{G}^{(1)}_{\pm n}] = \pm 2n\mathcal{G}^{(1)}_{\pm n}\,, \qquad [\mathcal{H}, \mathcal{G}^{(0)}_{\pm l}] = \pm 2l\mathcal{G}^{(0)}_{\pm l}\,, \tag{6.5.6}$$

$$[\mathcal{G}^{(1)}_{\alpha}, \mathcal{G}^{(1)}_{\beta}] = P_{\alpha,\beta}\mathcal{G}^{(1)}_{\alpha+\beta} + M_{\alpha,\beta}\mathcal{G}^{(0)}_{\alpha+\beta}\,, \quad \alpha, \beta = \pm 1, \ldots, \pm 5\,, \tag{6.5.7}$$

$$[\mathcal{G}^{(0)}_{\alpha}, \mathcal{G}^{(1)}_{\beta}] = \Pi_{-}(F_{\alpha,\beta}\mathcal{G}^{(1)}_{\alpha+\beta} + N_{\alpha,\beta}\mathcal{G}^{(0)}_{\alpha+\beta})\,, \quad \alpha = 1, 2\,, \quad \beta = \pm 1, \ldots, \pm 5\,, \tag{6.5.8}$$

$$[\mathcal{G}^{(0)}_{-1}, \mathcal{G}^{(0)}_{+1}] = 2\Pi_{-}\,, \qquad [\mathcal{G}^{(0)}_{\pm 1}, \mathcal{G}^{(0)}_{\mp 2}] = \pm 6\mathcal{G}^{(0)}_{\pm 1}\,, \qquad [\mathcal{G}^{(0)}_{-2}, \mathcal{G}^{(0)}_{+2}] = 8\Pi_{-}(\mathcal{H} - 5)\,, \tag{6.5.9}$$

where we put $\mathcal{G}^{(1)}_{0} = \mathcal{G}^{(0)}_{0} = 1$ and $P_{\alpha,\beta}$, $F_{\alpha,\beta}$, $M_{\alpha,\beta}$ and $N_{\alpha,\beta}$ are some polynomials in $\mathcal{H}$ and $\Pi_{-} = \frac{1}{2}(1 - \sigma_3)$, some of which are numerical coefficients, whose explicit form is listed in Appendix C.4. We note that in Eqs. (6.5.7) and (6.5.8), the operators $\mathcal{G}^{(1)}_{\pm n}$ with $1 < n \leq 7$ can appear, where for $n > 5$ we use relation (6.5.5) (admitting $\mathcal{G}^{(0)}_{\pm 3}$ as coefficients in the algebra). Additionally we note that the operators $\mathcal{G}^{(0)}_{\pm m}$ with $m > 2$ in both equations where they appear are absorbed in generators $\mathcal{G}^{(1)}_{\pm m}$.

For eigenstates we have the relations

$$\Psi^{\pm}_{3j+k} = (\mathcal{G}^{(1)}_{+3})^{j}\Psi^{\pm}_{k}\,, \qquad \Psi_0 = \mathcal{G}^{(1)}_{-3}\Psi^{\pm}_{1}, \qquad j = 1, 2, \ldots\,, \qquad k = 1, 2, 3\,, \tag{6.5.10}$$

$$\Psi^{\pm}_{j} = (\mathcal{G}^{(1)}_{+1})^{j}\Psi^{\pm}_{1}\,, \qquad \mathcal{G}^{(1)}_{\pm 1}\Psi_0 = \mathcal{G}^{(1)}_{-1}\Psi^{\pm}_{1} = 0\,. \tag{6.5.11}$$

Eq. (6.5.10) shows that we can connect the isolated ground state with the equidistant part of the spectrum using $\mathcal{G}^{(1)}_{\pm 3}$, which are not spectrum-generating operators. Eq. (6.5.11) indicates that the states in the equidistant part of the spectrum can be connected by $\mathcal{G}^{(1)}_{\pm 1}$, but this part of the spectrum cannot be connected by them with the ground state. Thus we have to use a combination of both pairs of these operators. On the other hand, the odd operators $\mathcal{Q}^{z}_{a}$ satisfy relations (6.3.2), where $\mathbb{P}_0 = \mathcal{H}$, and, therefore, we have again the linear $\mathcal{N} = 2$ Poincaré supersymmetry as a sub-superalgebra generated by $\mathcal{H}$, $\mathcal{Q}^{0}_{a}$ and Σ. The general anti-commutation structure is given by

$$\{\mathcal{Q}^{n}_{a}, \mathcal{Q}^{m}_{b}\} = \delta_{ab}(\mathbb{C}_{nm} + (\mathbb{C}_{nm})^{\dagger}) + i\epsilon_{ab}(\mathbb{C}_{nm} - (\mathbb{C}_{nm})^{\dagger})\,, \tag{6.5.12}$$

where $\mathbb{C}_{n,m} = \mathbb{C}_{n,m}(\mathcal{G}^{(1)}_{|n-m|}, \mathcal{G}^{(0)}_{|n-m|})$ in general are some linear combinations of the indicated ladder operators with coefficients to be polynomials in $\mathcal{H}$, $\mathcal{G}^{(0)}_{\pm 3}$ and σ_3. Some of these relations define ladder operators, see Eq. (6.3.3). For $n = N = 3$ and $m = -1, -2$ we can use (6.3.7) knowing that $\mathcal{Q}'^{z}_a = \mathcal{Q}^{3-z}_a$, see Sec. 6.3. For structure of anti-commutation relations with other combinations of indexes, see Appendix C.4. To complete the description of the generated nonlinear supersymmetric structure, we write down the commutators between the independent lowering operators and supercharges:

$$[\mathcal{G}^{(1)}_{-m}, \mathcal{Q}^n_a] = \mathbb{Q}^1_{m,n}(\mathcal{Q}^{n-m}_a + i\epsilon_{ab}\mathcal{Q}^{n-m}_b) + \mathbb{Q}^2_{m,n}(\mathcal{Q}^{m+n}_a - i\epsilon_{ab}\mathcal{Q}^{m+n}_b)\,, \tag{6.5.13}$$

$$[\mathcal{G}^{(0)}_{-m}, \mathcal{Q}^n_a] = \mathbb{G}^1_{m,n}(\mathcal{Q}^{n-m}_a + i\epsilon_{ab}\mathcal{Q}^{n-m}_b) + \mathbb{G}^2_{m,n}(\mathcal{Q}^{m+n}_a - i\epsilon_{ab}\mathcal{Q}^{m+n}_b)\,. \tag{6.5.14}$$

Here $\mathbb{Q}^j_{m,n}$ and $\mathbb{G}^j_{m,n}$ with $j = 1, 2$ are polynomials in $\mathcal{H}$ or numerical coefficients, some of which are listed in the sets of general commutation relations in Appendix C.3, while other are given explicitly in Appendix C.4. As the odd fermionic operators are Hermitian, then $[\mathcal{G}^{(1)}_{+m}, \mathcal{Q}^z_a] = -([\mathcal{G}^{(1)}_{-m}, \mathcal{Q}^z_a])^\dagger$, and we do not write them explicitly. In matrix language, Eq. (6.5.13) can be written as

$$[\mathcal{G}^{(1)}_{-m}, \mathcal{Q}^n_a] = \begin{pmatrix} 0 & \mathfrak{S}^-_{n+m} \\ \mathfrak{S}^+_{n-m} & 0 \end{pmatrix}, \tag{6.5.15}$$

and an important point here is that the number $n-m$ could take values less than -2 and $n+m$ could be greater than 5, but fermionic operators are defined with the index z taking integer values in the interval $I = [-2, +5]$. It is necessary to remember that we cut the series of $\mathfrak{S}^\pm_z$ because operators outside the defined interval are reduced to combinations (products) of other basic operators. In this way, we formally apply the definition of $\mathfrak{S}^\pm_z$ outside of the indicated interval and use the relation in Appendix C.2 to show that these "new" generated operators reduce to combinations of operators with index values in the interval I and of the generators $\mathfrak{C}_{\pm 3}$.

Finally, the subsets which produce closed sub-superalgebras here are those defined by $\mathcal{U}^{(1)}_{0,z}$ in (6.3.10), with $z = 1, \ldots, 5$ in addition to $\mathcal{I}^{(1)}_{N,-k}$ given in (8.1.16) with $k = 1, 2$.

With respect to the positive scheme, the super-Hamiltonian is given by $\mathcal{H}' = \mathrm{diag}\,(L_{(+)} - 3, L_0 - 3)$. It has two positive energy singlet states of the form $(0, \psi_n)$ with $n = 1, 2$; besides, there are two ground states $\Psi^+_0 = (\phi_0, \psi_0)$ and $\Psi^-_0 = \sigma_3\Psi^+_0$ of energy -2. According to the construction from the previous section, the fermionic operators here are $\mathcal{Q}'^{z}_a = \mathcal{Q}^{3-z}_a$, and the basic subsets which generate closed sub-superalgebras are $\mathcal{U}'^{(1)}_{0,k}$ and $\mathcal{I}'^{(1-2\theta(l))}_{N,l}$ with $k = 3, 4, 5$ and $l = -1, -2, 4, 5$.

One can note that considering $\mathcal{G}^{(1)}_{\pm 3}$ as coefficients, the subset $\{\mathcal{H}, \mathcal{G}^{(1)}_{\pm 3}, \sigma_3, \mathcal{Q}^{-2}_a, \mathcal{Q}^1_a, \mathcal{Q}^4_a, \mathbb{I}\}$ also generates a closed nonlinear superalgebraic structure.

6.6 Remarks

In fact, the construction in Sec. 6.3 offers more possibilities: in principle, the choice of the constant λ_* in the Hamiltonian (1.2.8) can be modified in such a way that another pair of fermionic operators in the scheme 6.3.1 will be the true integrals of the motion. As a result, the super-extended system will have a different spectrum. We schematically discussed this picture in the original work [Inzunza and Plyushchay (2019a)]. Another possibility is to choose $L_0 = L_{(-)}$ and $L_{[n]} = L_{(+)}$ and, as a consequence, the intertwining operators will be the ladder operators in (8.2.6), and one can expect that the use of intermediate systems in the DCKA procedure will provide lower order intertwining operators, however this is still an open problem.

Finally, the discussion in these last two chapters involved AFF models with integer coupling constant $m(m+1)$, so the next natural step is to try to generalize for the case $\nu(\nu+1)$ with ν real equal to or greater than $-1/2$. This is the objective of the next chapter.

Chapter 7

The Klein four-group and Darboux duality

The invariance of the QHO eigenvalue problem to the discrete transformation $(x, E) \to (ix, -E)$ was the basis of the construction presented in the last two chapters. The presence of nonphysical eigenstates gives rise to the so-called Darboux duality, which was the key to building the spectrum-generating ladder operators for extended rational systems. In this chapter we demonstrate that the Schrödinger equation for the AFF model with $\nu \geq -1/2$ has an even larger discrete symmetry group, which will be responsible for the generalization of Darboux duality for these systems. Such a discrete group has its particular consequences when it acts on eigenstates and (super) symmetry generators.

With the generalization of the Darboux duality at hand, constructing spectrum-generating ladder operators for rational deformations of the general AFF models, as well as their nonlinear algebras, is straightforward. It is interesting to recall that when ν is a half-integer number, the Jordan states associated with confluent Darboux transformations naturally enter in the framework. In particular, some deformed systems undergo structural changes when we set $\nu = \ell - 1/2$ with $\ell = 0, 1 \ldots$. The results contained in this chapter were reported in our work [Inzunza and Plyushchay (2019b)].

7.1 The Klein four-group in AFF model

Parameterizing the coupling constant in parabolic form $g = \nu(\nu+1)$, which is symmetric with respect to $\nu = -\frac{1}{2}$, we artificially induce the invariance of the equation

$$\left(-\frac{\partial^2}{\partial x^2} + x^2 + \frac{\nu(\nu+1)}{x^2}\right)\psi = i\frac{\partial}{\partial t}\psi \tag{7.1.1}$$

with respect to the transformation $\rho_1 : \nu \to -\nu-1$. Equation (7.7.1) is also invariant with respect to the transformation $\rho_2 : (x,t) \to (ix,-t)$. These two transformations generate the Klein four-group as a symmetry of equation (7.7.1): $K_4 \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 = (1, \rho_1, \rho_2, \rho_1\rho_2 = \rho_2\rho_1)$, where each element is its own inverse. At the level of the stationary Schrödinger equation, the action of ρ_2 reduces to the transformation $\rho_2 : (x, E_{\nu,n}) \to (ix, -E_{\nu,n})$, which means that ρ_2 is a completely broken $\mathbb{Z}_2$ symmetry, for which the transformed eigenstates $\rho_2(\psi_{\nu,n}) = \psi_{\nu,n}(ix)$ with eigenvalues $-E_{\nu,n}$ are nonphysical solutions. The transformation ρ_1 at the same level of the stationary Schrödinger equation implies that the energy eigenvalues change as $E_{\nu,n} \to \rho_1(E_{\nu,n}) = E_{-\nu-1,n} = 4n - 2\nu + 1$. The difference between the original energy level and the transformed one is $E_{\nu,n} - E_{-\nu-1,n} = \Delta E \cdot (\nu + 1/2)$, where $\Delta E = 4$ is the distance between two consecutive levels. So, if we take $\nu = \ell - 1/2$ with $\ell = 0, 1, \ldots$, we obtain $\rho_1(E_{\ell-1/2,n}) = E_{\ell-1/2,n-\ell}$, and find that physical energy levels with $n \geq \ell$ are transformed into physical energy levels but lowered by 4ℓ. Under the action of ρ_1, the eigenstates in (4.1.3) are transformed into the functions

$$\rho_1(\psi_{\nu,n}) = \sqrt{\tfrac{n!}{\Gamma(n-\nu+1/2)}}\, x^{-\nu} L_n^{(-\nu-1/2)}(x^2) e^{-x^2/2} := \psi_{-\nu-1,n} \,. \tag{7.1.2}$$

In the case of $\nu \neq \ell - 1/2$, functions (7.1.2) do not satisfy boundary condition at $x = 0$ because of the presence of the factor $x^{-\nu}$, and they are nonphysical, formal eigenstates of $\mathcal{H}_\nu$. The case of $\nu = \ell - 1/2$ requires, however, a separate consideration. To analyze this case, we observe that

$$\rho_1(\psi_{\ell-1/2,n}) = \sqrt{\tfrac{n!}{\Gamma(n-\ell+1)}}\, x^{-\ell+1/2} L_n^{(-\ell)}(x^2) e^{-x^2/2} \,. \tag{7.1.3}$$

Due to the poles of Gamma function, this expression vanishes when $n < \ell$, i.e., ρ_1 annihilates the first ℓ eigenstates of the system. On the other hand, the identity

$$\frac{(-\eta)^m}{m!} L_n^{(m-n)}(\eta) = \frac{(-\eta)^n}{n!} L_m^{(n-m)}(\eta) \,, \tag{7.1.4}$$

with integer m and n, which follows from (2.1.27), allows us to write $\rho_1(\psi_{\ell-1/2,n}) = (-1)^\ell \psi_{\ell-1/2,n-\ell}$ when $n \geq \ell$, and this is coherent with the change of the energy eigenvalues under application to them of transformation ρ_1. In conclusion, ρ_1 corresponds to a symmetry which is just the identity operator when $\ell = 0$, while for $\ell \geq 1$ this symmetry annihilates the ℓ lowest physical eigenstates, but restores them by acting on the higher eigenstates [1]. From this point of view, in the case of half-integer ν, transformation ρ_1 does not produce anything new. Nevertheless, we can also construct a

[1] This is similar to a picture of a Hilbert's hotel under departure of clients from first ℓ rooms with numbers $n = 0, \ldots, \ell-1$ with simultaneous translation of the clients from rooms with numbers $n = \ell, \ell+1, \ldots$, into the rooms with numbers $n - \ell$. Note that the power $(\mathcal{C}_\nu^-)^\ell$ of lowering generator of conformal symmetry with $\nu = \ell - \frac{1}{2}$ acts on physical eigenstates in a way similar to ρ_1, but violating normalization of the states.

finite set of nonphysical solutions of the same nonphysical nature as in (7.1.2) given by the functions

$$\psi_{-\ell-1/2,k} := \rho_1\left(\sqrt{\tfrac{\Gamma(k+l+1)}{k!}}\psi_{\ell-1/2,k}\right) = x^{-\ell+1/2}L_n^{(-\ell)}(x^2)e^{-x^2/2}, \quad k=0,\ldots,\ell-1, \tag{7.1.5}$$

singular at $x=0$, whose corresponding eigenvalues are $E_{-\ell-1/2,n} = 4n-2\ell+2$.

We note that the combined transformation $\rho_1\rho_2(\psi_{\nu,n})$ always produces nonphysical solutions for all values of ν due to the presence of ρ_2. Wave eigenfunctions transformed by the K_4 generators ρ_2 and $\rho_1\rho_2$ diverge exponentially at infinity, and for the following consideration it is convenient to introduce a special common notation for them: $\psi_{r(\nu),n}(x)$, with $r(\nu) = -\nu-1$ for functions that vanish at infinity and $\psi_{r(\nu),-n}(x) = \psi_{r(\nu),n}(ix)$ for functions that diverge when $x\to\infty$. In the case of $\nu = \ell-1/2$, $\ell \geq 1$, we have $E_{-\ell-1/2,\ell-n-1} = -E_{-\ell-1/2,n}$ for $n<\ell$, and one finds that (7.1.5) and their partners in the sense of Eq. (1.1.7) are related with nonphysical eigenstates produced by ρ_2 and their partners,

$$\psi_{-\ell-1/2,\ell-1-n} \propto \widetilde{\psi}_{-\ell-1/2,-n}\,, \qquad \widetilde{\psi}_{-\ell-1/2,n} \propto \psi_{-\ell-1/2,-\ell+1-n}\,. \tag{7.1.6}$$

Now, let us study the quantum conformal symmetry of the AFF model from the perspective of the discrete Klein four-group. Keep in mind that under these transformations, $\mathfrak{sl}(2,\mathbb{R})$ ladder operators $\mathcal{C}_\nu^\pm$ introduced in (4.1.6) change as

$$\rho_1(\mathcal{C}_\nu^\pm) = \mathcal{C}_\nu^\pm\,, \qquad \rho_2(\mathcal{C}_\nu^\pm) = \rho_3(\mathcal{C}_\nu^\pm) = -\mathcal{C}_\nu^\mp\,, \tag{7.1.7}$$

so what we have here is a group of automorphisms of the conformal algebra. Knowing that $\mathcal{C}_\nu^-$ annihilates the ground state, we can use the K_4 group to obtain the kernels of $\mathcal{C}_\nu^\pm$ in the case $\nu \geq -1/2$,

$$\ker\,\mathcal{C}_\nu^- = \operatorname{span}\{\psi_{\nu,0},\psi_{-\nu-1,0}\}\,, \qquad \ker\,\mathcal{C}_\nu^+ = \operatorname{span}\{\psi_{\nu,-0},\psi_{-\nu-1,-0}\}\,. \tag{7.1.8}$$

For $\nu=-1/2$, the kernels of $\mathcal{C}_{-1/2}^\pm$ are similar to (7.1.8) but with the states $\psi_{-\nu-1,0}$ and $\psi_{-\nu-1,-0}$ are replaced, respectively, by the Jordan states

$$\Omega_{-1/2,0} = \left(a-\tfrac{1}{2}\ln x\right)\psi_{-1/2,0}\,, \qquad \Omega_{-1/2,-0} = \left(b-\tfrac{1}{2}\ln x\right)\psi_{-1/2,-0}\,, \tag{7.1.9}$$

where a and b are constants.

In the context of the Darboux transformations, the equations in (7.1.8) indicate that the second order differential operators $-\mathcal{C}_\nu^\pm$ are generated by the choice of the seed states $(\psi_{\nu,\mp 0},\psi_{-\nu-1,\pm 0})$, and by means of Eq. (1.2.4) we can write the equalities

$$\mathcal{C}_{\nu}^{\mp}\phi_{r(\nu),z}=-\frac{W(\psi_{\nu,\pm 0},\psi_{-\nu-1,\pm 0},\phi_{r(\nu),z})}{W(\psi_{\nu,\pm 0},\psi_{-\nu-1,\pm 0})}\,, \tag{7.1.10}$$

where $\phi_{r(\nu),z}$ with $z=\pm n$, $n\in\mathbb{N}$, corresponds to an eigenstate or a Jordan state of L_{ν}. The Wronskian form of these equalities is useful to find the action of the ladder operators on the states $\widetilde{\psi}_{r(\nu),\pm 0}$ and $\breve{\Omega}_{-1/2,0}$. Using some Wronskian identities from the Appendix A, specifically the Eqs. (A.1.2) and (A.1.4), as wells as the relations

$$W(\psi_{\nu,\pm 0},\psi_{-\nu-1,\pm 0})=-(2\nu+1)e^{\mp x^2}\,,\qquad W(\psi_{-1/2,\pm 0},\Omega_{-1/2,\pm 0})=e^{\mp x^2}\,, \tag{7.1.11}$$

one can find that

$$\mathcal{C}_{\nu}^{-}\widetilde{\psi}_{r(\nu),0}\propto\psi_{r(-\nu-1),-0}\,,\qquad \mathcal{C}_{\nu}^{+}\widetilde{\psi}_{r(\nu),-0}\propto\psi_{r(-\nu-1),0}\,, \tag{7.1.12}$$

$$\mathcal{C}_{-1/2}^{\mp}\widetilde{\psi}_{-1/2,\pm 0}\propto\Omega_{-1/2,\mp 0}\,,\qquad \mathcal{C}_{-1/2}^{\mp}\breve{\Omega}_{-1/2,\pm 0}\propto\psi_{-1/2,\mp 0}\,. \tag{7.1.13}$$

So far, we realize that the states of Jordan should play some role in the case of half-integer ν, however, let us first consider the general case. For this, we use (1.3.5) and the $\mathfrak{sl}(2,\mathbb{R})$ algebra to prove the relations

$$\Omega_{r(\nu),\pm n}\propto(\mathcal{C}_{\nu}^{\pm})^{n}\Omega_{r(\nu),\pm 0}\,,\qquad \breve{\Omega}_{r(\nu),\pm n}\propto(\mathcal{C}_{\nu}^{\pm})^{n}\breve{\Omega}_{r(\nu),\pm 0}\,. \tag{7.1.14}$$

Thus, the ladder operators act in a similar way as they act on eigenstates of L_{ν}, but with a difference when $n=0$. When $\nu\neq-1/2$, we obtain the relations $\mathcal{C}_{\nu}^{\pm}\Omega_{r(\nu),\mp 0}\propto\widetilde{\psi}_{r(-\nu-1),\pm 0}$ and $\mathcal{C}_{\nu}^{\pm}\breve{\Omega}_{r(\nu),\mp 0}\propto\Omega_{r(-\nu-1),\pm 0}$. Due to (7.1.6) one can make the identification $\breve{\Omega}_{-\ell-1/2,\pm 0}=\Omega_{-\ell-1/2,\mp(\ell-1)}$, so in the half-integer case $\nu=\ell-1/2$ with $\ell\geq 1$ we obtain

$$\mathcal{C}_{\ell-1/2}^{\pm}\Omega_{\ell-1/2,\mp 0}\propto\psi_{-\ell-1/2,\mp(\ell-1)}\,,\qquad \mathcal{C}_{\ell-1/2}^{\pm}\Omega_{-\ell-1/2,\mp 0}\propto\psi_{\ell-1/2,\mp(\ell-1)}\,. \tag{7.1.15}$$

Acting on these relations by $(\mathcal{C}_{\ell-1/2}^{\pm})^{\ell}$, we obtain zero, and conclude that

$$\begin{aligned}\ker(\mathcal{C}_{\ell-1/2}^{\pm})^{\ell+k}\; &=\mathrm{span}\{\psi_{\ell-1/2,\mp 0},\ldots,\psi_{\ell-1/2,\mp(\ell+k-1)},\psi_{-(\ell-1/2)-1,\mp 0},\ldots,\\ &\qquad\psi_{-(\ell-1/2)-1,\mp(\ell-1)},\Omega_{\ell-1/2,\mp 0},\ldots,\Omega_{\ell-1/2,\mp(k-1)}\}\end{aligned} \tag{7.1.16}$$

for $k=1,2,\ldots$. The whole picture is summarized in Figure 7.1.

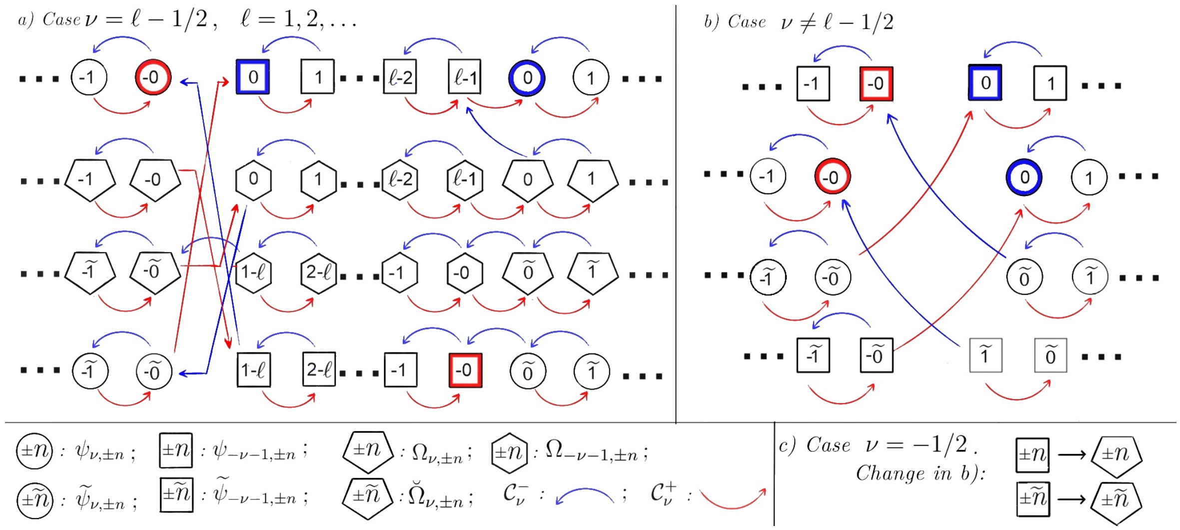

Figure 7.1: The action of the ladder operators in dependence on the value of ν. Diagram *a)* illustrates the case of half-integer $\nu = \ell - 1/2$ with $\ell = 1, \ldots,$ where it is shown how Jordan states can be related to eigenstates by the action of $\mathcal{C}_\nu^\pm$. Diagram *b)* corresponds to non-half-integer values of ν. In *c)*, it is indicated how the case with $\nu = -1/2$ can be obtained from *b)* by changing the corresponding states. The shapes with borders highlighted in blue (red) represent the states annihilated by $\mathcal{C}_\nu^-$ ($\mathcal{C}_\nu^+$).

7.2 Superconformal symmetry and the Klein four-group

Here, we inspect the action of the Klein four-group on a supersymmetric extension of the AFF model. To do so, we must pay attention to the intertwining operators $A_\nu^\pm$ and $B_\nu^\pm$ introduced in Chap. 2, Eqs. (2.2.1) and (2.2.6) (with $\omega = 1$). Acting on them, the group produces

$$\rho_1(A_\nu^\mp) = -B_{\nu-1}^\pm\,, \qquad \rho_1(B_\nu^\mp) = -A_{\nu-1}^\pm\,, \tag{7.2.1}$$

$$\rho_2(A_\nu^\pm) = -iB_\nu^\pm\,, \qquad \rho_2(B_\nu^\pm) = -iA_\nu^\pm\,. \tag{7.2.2}$$

These relations are valid for $\nu > -1/2$, while for $\nu = -1/2$ the transformation ρ_1 reduces to the identity.

The symmetry generators of the super-extended AFF model, namely $\{\mathcal{H}_\nu^e, \mathcal{R}_\nu, \mathcal{C}_\nu^\pm, \mathcal{Q}_\nu^a, \mathcal{S}_\nu^b\}$, were defined in Eqs. (2.2.4), (2.2.10), (2.2.11) and (2.2.15). The basic blocks to construct these objects are the intertwining operators $A_\nu^\pm$ and $B_\nu^\pm$, so the role of the Klein four-group at the supersymmetric level is at hand. Nevertheless, before to apply the relations (7.2.1)-(7.2.2) in the supersymmetric generators, it is convenient to remember that the corresponding superalgebra (2.2.16)-(2.2.22) has the automorphism $f = f^{-1}$, which corresponds to the transformations $\mathcal{H}_\nu^e \to \mathcal{H}_\nu^e - 4\mathcal{R}_\nu = \mathcal{H}_\nu^b$, $\mathcal{R}_\nu \to -\mathcal{R}_\nu$, $\mathcal{G}_\nu^\pm \to \mathcal{G}_\nu^\pm$, $\mathcal{Q}_\nu^1 \to -\mathcal{S}_\nu^1$, $\mathcal{Q}_\nu^2 \to \mathcal{S}_\nu^2$, $\mathcal{S}_\nu^1 \to -\mathcal{Q}_\nu^1$ $\mathcal{S}_\nu^2 \to \mathcal{Q}_\nu^2$. Then, the action of ρ_1 gives us

$$\rho_1(\mathcal{H}^e_\nu)=\sigma_1(\mathcal{H}^e_{\nu-1}-4\mathcal{R}_{\nu-1})\sigma_1\,,\qquad \rho_1(\mathcal{G}^\pm_\nu)=\sigma_1(\mathcal{G}^\pm_{\nu-1})\sigma_1\,, \tag{7.2.3}$$

$$\rho_1(\mathcal{R}_\nu)=\sigma_1(-\mathcal{R}_{\nu-1})\sigma_1\,, \tag{7.2.4}$$

$$\rho_1(\mathcal{Q}^1_\nu)=\sigma_1(-\mathcal{S}^1_{\nu-1})\sigma_1\,,\qquad \rho_1(\mathcal{Q}^2_\nu)=\sigma_1(\mathcal{S}^2_{\nu-1})\sigma_1\,, \tag{7.2.5}$$

$$\rho_1(\mathcal{S}^1_\nu)=\sigma_1(-\mathcal{Q}^1_{\nu-1})\sigma_1\,,\qquad \rho_1(\mathcal{S}^2_\nu)=\sigma_1(\mathcal{Q}^2_{\nu-1})\sigma_1\,, \tag{7.2.6}$$

which in fact is a combination of the shift $\nu\to\nu-1$, the action of f and the unitary rotation. The transformed generators (7.2.3)-(7.2.6) still satisfy the same superconformal algebra, i.e. ρ_1 is an automorphism of the $\mathfrak{osp}(2|2)$ symmetry, however the new generators describe another super-extended system: Unlike the initial system $\mathcal{H}^e_\nu$, in the transformed one the $\mathcal{N}=2$ Poincaré supersymmetry is spontaneously broken in the case of $\nu>-1/2$, see Chap. 2. The only exception from this rule corresponds to the case $\nu=-1/2$, where the transformed Hamiltonian reduces to $\sigma_1\mathcal{H}^e_{-1/2}\sigma_1$, and represents a unitarily transformed super-Hamiltonian with the unbroken $\mathcal{N}=2$ Poincaré supersymmetry.

On the other hand, one can verify that when ρ_1 acts on the Hamiltonian $\mathcal{H}^b_\nu$, it produces $\sigma_1(\mathcal{H}^e_{\nu-1})\sigma_1$, and this time the $\mathcal{N}=2$ Poincaré supersymmetry of the system is changed from the spontaneously broken phase (in the case of $\nu>-1/2$) to the phase of unbroken supersymmetry, with the only exception of the system $\mathcal{H}^b_{-1/2}$ with unbroken supersymmetry, which unitary transforms into $\sigma_1\mathcal{H}^b_{-1/2}\sigma_1$. This action of transformation ρ_1 on super-extended systems can be compared with the case of the non-extended AFF system, where ρ_1 acts identically on its Hamiltonian and generators of the conformal symmetry, though, as we saw, it acts nontrivially on eigenstates of the system.

On the other hand, the action of ρ_2 produces

$$\rho_2(\mathcal{H}^e_\nu)=-\mathcal{H}^b_\nu\,,\quad \rho_2(\mathcal{G}^\pm_\nu)=-\mathcal{G}^\mp_\nu\,,\quad \rho_2(\mathcal{R}_\nu)=\mathcal{R}_\nu\,, \tag{7.2.7}$$

$$\rho_2(\mathcal{Q}^1_\nu)=-i\mathcal{S}^1_\nu\,,\qquad \rho_2(\mathcal{Q}^2_\nu)=-i\mathcal{S}^2_\nu\,, \tag{7.2.8}$$

$$\rho_2(\mathcal{S}^1_\nu)=-i\mathcal{Q}^1_\nu\,,\qquad \rho_2(\mathcal{S}^2_\nu)=-i\mathcal{Q}^2_\nu\,. \tag{7.2.9}$$

Transformed Hamiltonian operator is similar here to the Hamiltonian produced by the automorphism f but multiplied by -1. This correlates with the anti-Hermitian nature of the transformed fermion generators of superalgebra. Accordingly, the spectrum of the transformed matrix Hamiltonian is negative, not bounded from below, and each of its level is doubly degenerate for $\nu\geq-1/2$.

In correspondence with the described picture, the application of the combined transformation $\rho_2\rho_1$ is just another automorphism of the superconformal algebra (2.2.16)-(2.2.22), which produces anti-Hermitian odd generators, and $\rho_2\rho_1(\mathcal{H}^e_\nu)=\sigma_1(-\mathcal{H}^e_{\nu-1})\sigma_1$. The discrete spectrum of the transformed Hamiltonian is not restricted from below and is given by the numbers $\mathcal{E}_n = -4n$, $n = 0, 1, \ldots$, where each negative energy level is doubly degenerate, while non-degenerate zero energy level corresponds to the state $(\psi_{\nu,0}, 0)^t$.

7.3 Dual Darboux schemes

With the new set of nonphysical solutions, in this section we extend the idea of dual schemes for the AFF model with $\nu \geq -1/2$. As we have shown in Sec. 7.1, the case in which ν takes half-integer values is special, because the Jordan states take relevance through the properties of the conformal symmetry generators[2], which are simultaneously the ladder operators for corresponding AFF systems, see equation (7.1.15). For this reason, we start first with the case where ν is not a half-integer. Let us choose a generic set of physical and nonphysical eigenstates of L_ν as seed states,

$$\{\alpha\} = (\psi_{\nu,k_1}, \ldots, \psi_{\nu,k_{N_1}}, \psi_{-\nu-1,l_1}, \ldots, \psi_{-\nu-1,l_{N_2}}) , \qquad k_i, l_j = \pm 0, \pm 1, \ldots , \tag{7.3.1}$$

where $i = 1, \ldots, N_1$ and $j = 1, \ldots, N_2$, and, for simplicity, we suppose that $|k_1| < \ldots < |k_{N_1}|$ and $|l_1| < \ldots < |l_{N_2}|$. Let us assume that in the scheme (7.3.1) there are no repeated states and both k_i and l_j carry the same sign for all i and j. Also let us define the index number

$$n_N = \max\left(|k_1|, \ldots, |k_{N_1}|, |l_1|, \ldots, |l_{N_2}|\right) . \tag{7.3.2}$$

which can correspond to a state with index ν or $-\nu-1$. By means of the algorithm described in Appendix B.2 one can show that

$$W(\{\alpha\}) = e^{-(n_N+1)x^2} W(\{\Delta_-\}) , \tag{7.3.3}$$
$$\{\Delta_-\} := (\psi_{-\nu-1,-0}, \psi_{\nu,-0}, \ldots, \check{\psi}_{-\nu-1,-r_i}, \check{\psi}_{\nu,-s_i}, \ldots, \psi_{-\nu-1,-n_N}, \psi_{\nu,-n_N}) ,$$

is satisfied, where the marked states $\check{\psi}_{-\nu-1,-r_i}$ and $\check{\psi}_{\nu,-s_i}$, with $r_i = n_N - k_i$ and $s_j = n_N - l_j$, are omitted from the set $\{\Delta_-\}$. On the contrary, if k_i and l_j carry the minus sign, we have the equality

$$W(\{\alpha\}) = e^{(n_N+1)x^2} W(\{\Delta_+\}) , \tag{7.3.4}$$
$$\{\Delta_+\} := (\psi_{-\nu-1,0}, \psi_{\nu,0}, \ldots, \check{\psi}_{-\nu-1,r_i}, \check{\psi}_{\nu,s_j}, \ldots, \psi_{-\nu-1,n_N}, \psi_{\nu,n_N}) ,$$

[2] Operators $\mathcal{C}_\nu^\pm$ can be interpreted as the second order intertwining operators associated with the seed states $(\psi_{-\nu-1,0}, \psi_{\nu,0})$ for $\nu > 1/2$, and to the confluent scheme $(\Omega_{-1/2,0}, \psi_{-1/2,0})$, when $\nu = 1/2$.

where now $r_i = n_N - |k_i|$ and $s_j = n_N - |l_j|$. These relations are also valid if one of the numbers N_1 or N_2 is equal to zero, which means that in the corresponding scheme there are only states of the same kind with respect to the first index, $-\nu-1$ or ν, respectively.

When considering $\nu = \ell - 1/2$ with $\ell = 0, 1, 2, \ldots$, some repeated states could appear due to $\rho_1(\psi_{\ell-1/2,n}) = (-1)^{\ell}\psi_{\ell-1/2,n-\ell}$. This means that the Wronskian must vanish, however, that happens because, in the general case, this object takes the form $\Lambda(\nu)f(x;\nu)$, where $\Lambda(\nu)$ disappears in these special cases (see the example (7.3.7) below). To obtain a deformed AFF system with the potential modified by $-2\ln(f(x;\nu))''$ for half-integer ν, as well as its dual scheme, we will have relations analogous to (7.3.3) and (7.3.4), but changing each state of the form $\psi_{-\nu-1,\pm(\ell+k)}$ by $\Omega_{\ell-1/2,\pm k}$, which means that we are dealing with the confluent Darboux transformation, see Appendix B.3 for a detailed derivation. The general rules of the Darboux duality can be summarized and better understood with the examples presented diagrammatically in Fig. 7.2.

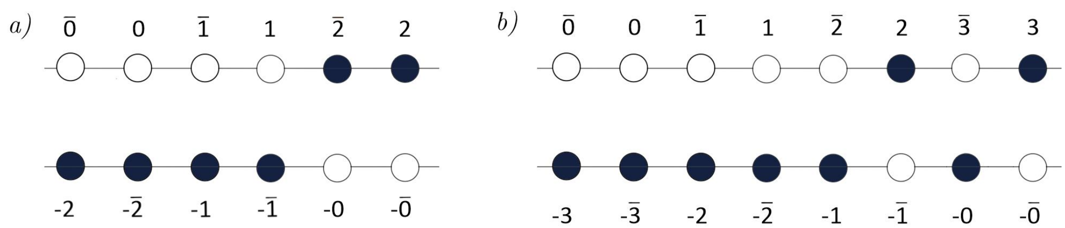

Figure 7.2: Two "mirror diagrams" corresponding to dual schemes for the conformal mechanics model. The numbers $\pm n$ indicate the states $\psi_{\nu,\pm n}$, and symbols $\pm\bar{n}$ correspond to the states $\psi_{-\nu-1,\pm n}$.

These types of diagrams are read in the same way as for the harmonic oscillator mirror diagram presented in Chap. 5 and in this case they correspond to the following Wronskian relations:

$$W(\psi_{-\nu-1,2}, \psi_{\nu,2}) = e^{-3x^2} W(\psi_{-\nu-1,-1}, \psi_{\nu,-1}, \psi_{-\nu-1,-2}, \psi_{\nu,-2}), \tag{7.3.5}$$

$$W(\psi_{\nu,2}, \psi_{\nu,3}) = e^{-4x^2} W(\psi_{\nu,-0}, \psi_{\nu,-1}, \psi_{-\nu-1,-2}, \psi_{\nu,-2}, , \psi_{-\nu-1,-3}, \psi_{\nu,-3}), \tag{7.3.6}$$

whose explicit forms are

$$\begin{aligned} W(\psi_{\nu,2}, \psi_{-\nu-1,2}) = \ & (2\nu+1)e^{-x^2}\big(45 - 72\nu + 16(-4x^6 + x^8) \\ & +8x^4(15 - 4\nu(1+\nu)) + \nu^2(-7 + 2\nu(2+\nu))\big), \end{aligned} \tag{7.3.7}$$

$$\begin{aligned} W(\psi_{\nu,2}, \psi_{\nu,3}) = \ & e^{-x^2}x^{3+2\nu}\big(16x^8 - 32x^6(5+2\nu) + 24x^4(5+2\nu)^2 - \\ & 8x^2(3+2\nu)(5+2\nu)(7+2\nu) + (3+2\nu)(5+2\nu)^2(7+2\nu)\big). \end{aligned} \tag{7.3.8}$$

The transformation which relates the AFF systems described by L_ν with $L_{\nu+m}$ can also be understood within this picture. Furthermore, using a diagram similar to those in Fig. 7.2, one can show that the schemes $\{\Delta_+\} = (\psi_{r(\nu),0}, \ldots, \psi_{r(\nu),m-1})$ and $\{\Delta_-\} = (\psi_{r(\nu),-0}, \ldots, \psi_{r(\nu),-(m-1)})$ are dual.

7.4 Rationally deformed AFF systems

A rational deformation of the AFF model can be generated by taking a set of the seed states

$$\{\alpha_{KA}\} = (\psi_{\nu,l_1}, \psi_{\nu,l_1+1}, \ldots, \psi_{\nu,l_m}, \psi_{\nu,l_m+1})\,, \tag{7.4.1}$$

composed from m pairs of neighbour physical states. Krein-Adler theorem [Krein (1957); Adler (1994)] guarantees that the resulting system described by the Hamiltonian operator of the form

$$L^{KA}_{(\nu,m)} = L_{\nu+m} + 4m + \tfrac{F_\nu(x)}{Q_\nu(x)} \tag{7.4.2}$$

is nonsingular on $\mathbb{R}^+$. Here $F_\nu(x)$ and $Q_\nu(x)$ are real-valued polynomials, $Q_\nu(x)$ has no zeroes on $\mathbb{R}^+$, its degree is two more than that of $F_\nu(x)$, and so, the last rational term in (7.4.2) vanishes at infinity. The spectrum of the system (7.4.2) is the equidistant spectrum of the AFF model with the removed energy levels corresponding to the seed states. Consequently, any gap in the resulting system has a size $12+8k$, where $k=0,1,\ldots$ correspond to k adjacent pairs in the set (7.4.1) which produce a given gap. An example of this kind of systems is generated by the scheme $(\psi_{\nu,2}, \psi_{\nu,3})$, whose dual negative scheme is given by equation (7.3.6).

Another class of rationally extended AFF systems is provided by isospectral deformations generated by the schemes of the form

$$\{\alpha_{iso}\} = (\psi_{\nu,-s_1}, \ldots, \psi_{\nu,-s_m})\,, \tag{7.4.3}$$

which contains the states of the form $\rho_2(\psi_{\nu,n}(x)) = \psi_{\nu,n}(ix)$. As the functions used in this scheme are proportional to $x^{\nu+1}$ and do not have real zeros other than $x=0$, one obtains a regular on $\mathbb{R}^+$ system of the form

$$L^{iso}_{(\nu,m)} = L_{\nu+m} + 2m + f_\nu(x)\,, \tag{7.4.4}$$

where $f_\nu(x)$ is a rational function disappearing at infinity [Grandati (2012)], and one can find that potential of the system (7.4.4) is a convex on $\mathbb{R}^+$ function. In this case the transformation does not remove or add energy levels, and, consequently, the initial system $\mathcal{H}_\nu$ and the deformed system (7.4.4) are completely isospectral superpartners. Some concrete examples of the systems (7.4.4) with integer values of ν were considered in the two previous chapters, see also [Cariñena et al. (2018)] .

Consider yet another generalized Darboux scheme which allows us to interpolate between different rationally deformed AFF systems. For this we assume that the initial AFF system is characterized by the parameter $\nu = \mu + m$, where $-1/2 < \mu \leq 1/2$ and m can take any non-negative integer value. For these ranges of values of the parameter ν, real zeros of the functions $\psi_{\mu+m,n-m}$ are located between zeros of $\psi_{-(\mu+m)-1,n}$, so that we can rethink the Krein-Adler theorem and consider the scheme

$$\{\gamma_\mu\} = (\psi_{-(\mu+m)-1,n_1}, \psi_{(\mu+m),n_1-m}, \ldots, \psi_{-(\mu+m)-1,n_N}, \psi_{(\mu+m),n_N-m}), \tag{7.4.5}$$

which includes $2N$ states and where we suppose that $n_i - m \geq 0$ for all $i = 1, \ldots, N$. The DCKA transformation based on the set (7.4.5) produces the system

$$L^{def}_{\mu+m} := L_{\mu+m} - 2(\ln W(\gamma_\nu))'' = L_{\mu+m} + 4N + h_{\mu+m}(x)/q_{\mu+m}(x), \tag{7.4.6}$$

where the constant $4N$ is provided by the Gaussian factor in the Wronskian, and the last term is a rational function vanishing at infinity and having no zeros on the whole real line, including the origin, if an only if $-1/2 < \mu \leq 1/2$, see Appendix A.2. Let us analyze now some special values of μ.

The case $\mu = 0$: by virtue of relations between Laguerre and Hermite polynomials mentioned in Chap 2, see equation (2.1.28), in this case we obtain those systems which were generated in [Cariñena et al. (2018)] and discussed in Chap. 5, we refer to systems (5.1.2).

The case $\mu = 1/2$: we have here the relation

$$\rho_1(\psi_{m+1/2,n_i}) = \psi_{-m-3/2,n_i} = (-1)^{m+1}\psi_{m+1/2,n_i-m-1}, \tag{7.4.7}$$

due to which the scheme (7.4.5) transforms into

$$\{\gamma_{1/2}\} = (\psi_{1/2+m,n_1-m-1}, \psi_{1/2+m,n_1-m}, \ldots, \psi_{1/2+m,n_N-m-1}, \psi_{1/2+m,n_N-m}), \tag{7.4.8}$$

which corresponds to (7.4.1) with $l_i = n_i - m - 1$. We additionally suppose that $n_i - m - 1 \neq n_{i-1} - m$, otherwise the Wronskian vanishes. Note that when $\mu \neq 1/2$, the image of the states $\psi_{\mu+m,n_i-m-1}$ under Darboux mapping (1.2.4) is a physical state, but in the case $\mu = 1/2$ such states are mapped into zero since the argument $\psi_{1/2+m,n_i-m-1}$ appears twice in the Wronskian of the numerator.

The case $\mu = -1/2$: this case was not included in the range of μ from the beginning due to relation $\rho_1(\psi_{m-1/2,n_i}) = \psi_{-m-1/2,n_i} = (-1)^m\psi_{m-1/2,n_i-m}$ which would mean the appearance of the repeated states in the scheme (7.4.5) and vanishing of the corresponding Wronskian. However, in Appendix A.2 we show that the limit relation $\lim_{\mu\to-1/2} W(\{\gamma_\mu\})/(\mu+\frac{1}{2})^N \propto W(\{\gamma\})$ is valid, where the scheme $\{\gamma\}$ is

$$\{\gamma\} = (\psi_{m-1/2,n_1-m}, \Omega_{m-1/2,n_1-m}, \ldots, \psi_{m-1/2,n_N-m}, \Omega_{m-1/2,n_N-m}), \tag{7.4.9}$$

which corresponds to a non-singular confluent Darboux transformation, [Correa et al. (2015)].

By considering this last comment, in conclusion we have that when $-1/2 \leq \mu < 1/2$, the states $\psi_{-(\mu+m)-1,n_i}$ (and $\Omega_{m-1/2,n_i-m}$ in the case of $\mu = -1/2$) are nonphysical states. This means that only the physical states $\psi_{\nu+m,n_i-m}$ indicate the energy levels removed under the corresponding Darboux transformation, i.e., there are gaps of the minimum size $2\Delta E = 8$, where $\Delta E = 4$ is the distance between energy levels of the AFF model, which can merge to produce energy gaps of the size $8 + 4k$. On the other hand, when $\mu = 1/2$, we have a typical Krein-Adler scheme with gaps of the size $12 + 4k$.

To give an example, we put $m = 0$, that means $\nu = \mu$, and consider the scheme $(\psi_{-\nu-1,2}, \psi_{\nu,2})$ given in (7.3.7) with $-1/2 < \nu \leq 1/2$, and in the case of $\nu = -1/2$ we have the scheme $(\psi_{-1/2,2}, \Omega_{-1/2,2})$. The potential of the rationally deformed AFF system generated by the corresponding Darboux transformation is shown in Fig. 7.3 and Fig. 7.4.

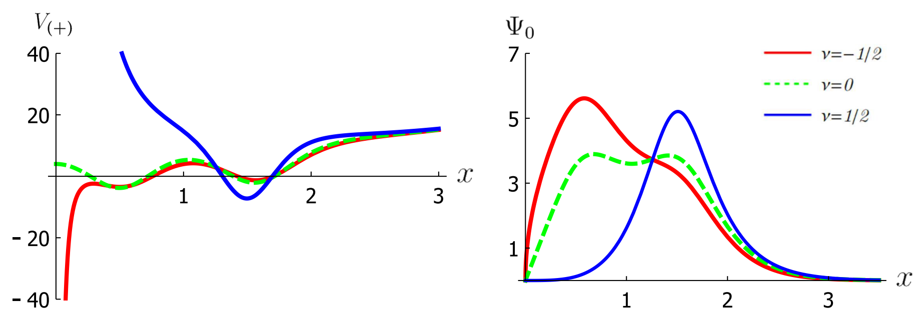

Figure 7.3: On the left, a graph of the corresponding potential is shown which is produced by the associated Darboux transformation applied to the AFF model with three indicated values of the parameter ν versus the dimensionless coordinate x. For $\nu = -1/2$, the corresponding limit is taken, and the resulting system has an attractive potential with a (not shown) potential barrier at $x = 0$. For $\nu = 0$, we obtain a rationally extended half-harmonic oscillator. The case $\nu = 1/2$ corresponds to the Krein-Adler scheme $(\psi_{1/2,1}, \psi_{1/2,2})$ with a gap equal to 12. On the right, the ground states of the corresponding generated systems are shown as functions of dimensionless coordinate x.

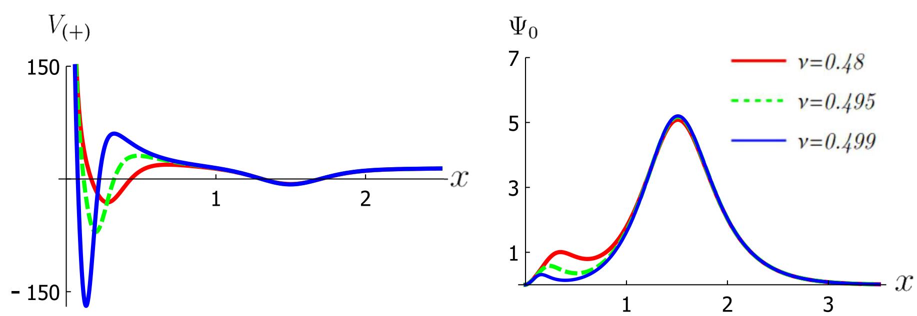

Figure 7.4: On the left, the potential of deformed systems with ν close to $1/2$ is shown. On the right, the ground states of the corresponding systems are displayed.

As it is seen from the figures, the first minimum of the potential grows in its absolute value, its position moves to 0, and it disappears at $\nu = 1/2$, while the local maximum near zero also grows, its position approaches zero, and it goes to infinity in the limit. Besides, the first maximum of the ground state vanishes when ν approximates the limit value $1/2$. Coherently with the described behavior of the potential, the image of the Darboux-transformed state $\psi_{\nu,1}$, which is the first excited state of the new system when $-1/2 \leq \nu < 1/2$, vanishes when $\nu \to 1/2$, the corresponding energy level disappears from the spectrum at $\nu = 1/2$, and the size of the gap increases from 8 to 12.

The described three possible selection rules to choose the seed states correspond to the negative scheme (7.4.3), which generates isospectral deformations, the positive Krein-Adler scheme (7.4.1), and the positive interpolating scheme (7.4.5). Then we can apply the Darboux duality to obtain the corresponding dual schemes for them. The positive and negative dual schemes will be used in the next subsection to construct complete sets of the spectrum-generating ladder operators for the rationally deformed conformal mechanics systems.

7.5 Intertwining and ladder operators

In this paragraph we proceed to construct the intertwining and ladder operators of rational deformed system obtained by means of the seed states selection rules detailed above. For simplicity we do not touch here the schemes that contain Jordan states. However, we have relations (1.3.8) and (7.1.16), and relations (7.3.3) and (7.3.4) which were extended to such cases with the corresponding substitutions. This means that the properties summarized below are also valid for the schemes containing Jordan states. Suppose that the positive (negative) scheme possesses n_+ (n_-) seed states. Then the generated Hamiltonian $L_{(\pm)}$ satisfy the relation

$$L(+) - L_{(-)} = \Delta E(n_{n_+} + 1) = 2(n_+ + n_-)\,, \qquad \Delta E = 4\,, \tag{7.5.1}$$

where n_{n_+} is the bigest quantum number in the positive scheme. Let us denote by $A^{\pm}_{(+)}$ and $A^{\pm}_{(-)}$ the intertwining operators of the positive and negative schemes being differential operators of the orders n_+ and n_-, respectively. They satisfy the intertwining relations

$$A^-_{(\pm)} L_\nu = L_{(\pm)} A^-_{(\pm)}\,, \qquad A^+_{(\pm)} L_{(\pm)} = L_\nu A^+_{(\pm)}\,. \tag{7.5.2}$$

As the states $\widetilde{\psi}_{r(\nu),\pm n}$ behave asymptotically as $e^{\pm x^2/2}$, the states produced from them by application of differential operators $A^-_{(\pm)}$ will carry the same exponential factor. Having this asymptotic behavior in mind, let us suppose that $\psi_{r(\nu),-l_*}$ and $\psi_{r(\nu),n_*}$ are some arbitrary states from the negative and positive scheme, respectively. By using (7.5.2), we obtain the relations

$$A^-_{(-)} \widetilde{\psi}_{r(\nu),-l_*} = A^-_{(+)} \rho_1(\psi_{r(\nu),n_{n+}-l_*})\,, \qquad A^-_{(+)} \widetilde{\psi}_{r(\nu),n_*} = A^-_{(-)} \rho_1(\psi_{r(\nu),-(n_{n+}-n_*)})\,, \tag{7.5.3}$$

in both sides of which the functions satisfy the same second order differential equation and have the same behaviour at infinity. Note that in the dual schemes in (7.3.3) and (7.3.4), the indexes $n_{n_+}-l_*$ and $-(n_{n_+}-n_*)$ are in correspondence with the indexes r_i, and s_i of the states omitted from the positive and negative scheme, respectively. This helps us to find that

$$\ker\left(A^+_{(-)}A^-_{(+)}\right)=(\psi_{\nu,0},\psi_{-\nu-1,0},\ldots,\psi_{\nu,n_{n_+}},\psi_{-\nu-1,n_{n_+}})=\ker\,(\mathcal{C}^-_\nu)^{n_{n_+}+1}\,, \tag{7.5.4}$$

from where we obtain the identities

$$A^+_{(-)}A^-_{(+)}=(-1)^{n_{n_+}+1-n_+}(\mathcal{C}^-_\nu)^{n_{n_+}+1}\,,\qquad A^+_{(+)}A^-_{(-)}=(-1)^{n_{n_+}+1-n_+}(\mathcal{C}^+_\nu)^{n_{n_+}+1}\,. \tag{7.5.5}$$

Finally, to have a complete picture we write the relations

$$A^-_{(-)}\psi_{r(\nu),k}=A^-_{(+)}\psi_{r(\nu),n_{n+}+1+k'}\,,\qquad A^-_{(+)}\psi_{r(\nu),-k'}=A^-_{(-)}\psi_{r(\nu),-(n_{n+}+1+k')}\,. \tag{7.5.6}$$

Note that in the case $\nu=0$, first equation reduces to (6.1.3).

In the case of the dual schemes where $\nu=m-1/2$, similar relations are obtained but with $\psi_{-\mu-m-1,\pm n_i}$ and $\widetilde{\psi}_{-\mu-m-1,\pm n_i}$ replaced by $\Omega_{m-\frac{1}{2},\pm(n_i-m)}$ and $\breve{\Omega}_{m-\frac{1}{2},\pm(n_i-m)}$ when is required.

With the help of the described intertwining operators, we can construct three types of ladder operators for $L_{(\pm)}$ which are given by:

$$\mathcal{A}^\pm=A^-_{(-)}\mathcal{C}^\pm_\nu A^+_{(-)}\,,\quad \mathcal{B}^\pm=A^-_{(+)}\mathcal{C}^\pm_\nu A^+_{(+)}\,,\quad \mathcal{C}^+=A^-_{(-)}A^+_{(+)}\,,\quad \mathcal{C}^-=A^-_{(+)}A^+_{(-)}\,. \tag{7.5.7}$$

Let us denote these operators in the compact form $\mathcal{F}^\pm_a=(\mathcal{A}^\pm,\mathcal{B}^\pm,\mathcal{C}^\pm)$, $a=1,2,3$, and use (7.5.1) and (7.5.2) to obtain the commutation relations

$$\begin{aligned}
&[L_{(\pm)},\mathcal{F}^\pm_a]=\pm R_a\mathcal{F}^\pm_a\,,\qquad [\mathcal{F}^-_a,\mathcal{F}^+_a]=\mathcal{P}_a(L_{(\pm)})\,,\\
&R_1=R_2=4\,,\qquad \mathcal{P}_1=(\eta+2\nu+3)(\eta-2\nu+1)P_{n_-}(\eta)P_{n_-}(\eta+4)\big|^{\eta=L_{(-)}}_{\eta=L_{(-)}-4}\,,\\
&\qquad\qquad\qquad\quad \mathcal{P}_2=(\eta+2\nu+3)(\eta+2\nu+1)P_{n_+}(\eta)P_{n_+}(\eta+4)\big|^{\eta=L_{(+)}}_{\eta=L_{(+)}-4}\,,\\
&R_3=4(n_{n_+}+1)\,,\quad \mathcal{P}_3=P_{n_+}(\eta)P_{n_-}(\eta)\big|^{\eta=L_{(-)}}_{\eta=L_{(+)}-4}\,,
\end{aligned} \tag{7.5.8}$$

where

$$P_{n_-}(y)=\prod_{i=1}^{n_-}(y-\lambda^-_i)\,,\qquad P_{n_+}(y)=\prod_{i=1}^{n_+}(y-\lambda^+_i)\,, \tag{7.5.9}$$

and $\lambda^\pm_i$ are the corresponding eigenvalues of the seed states in the positive and negative schemes. Equations (7.5.8) are three different but related copies of the nonlinearly deformed conformal algebra $\mathfrak{sl}(2,\mathbb{R})$. One can verify the commutators between generators with different values of index a

do not vanish, and therefore the complete structure is rather complicated.

Similarly to the non-deformed case, be means of a unitary transformation produced by $U = e^{-itL_{(\pm)}}$ we obtain the integrals of motion ${}_{H}\mathcal{F}_a^{\pm}(t) = e^{\mp R_a}\mathcal{F}_a^{\pm}$, and by linear combinations of them construct the Hermitian generators $\mathfrak{D}_a(t) = (\mathcal{F}_a^{-}(t) - \mathcal{F}_a^{+}(t))/(i2R_a)$ and $\mathfrak{K}_a(t) = (\mathcal{F}_a^{+}(t) + \mathcal{F}_a^{-}(t) + 2L_{(\pm)})/R_a^2$, which generate three copies of a nonlinear deformation of the Newton-Hooke algebra,

$$
[L_{(\pm)}, \mathfrak{D}_a] = -i\left(L_{(\pm)} - \frac{(R_a)^2}{2}\mathfrak{K}_a\right)\,, \qquad [L_{(\pm)}, \mathfrak{K}_a] = -2i\mathfrak{D}_a\,, \tag{7.5.10}
$$
$$
[\mathfrak{D}_a, \mathfrak{K}_a] = \frac{1}{iR_a^3}\left(\mathcal{P}_a(L_{(\pm)}) - 2R_a L_{(\pm)} + R_a^3\mathfrak{K}_a\right)\,,
$$

which are hidden symmetries of the system described by $L_{(\pm)}$.

In the isospectral case, the operators $\mathcal{A}^{\pm}$ are the spectrum generating ladder operators, where their action on physical eigenstates of $L_{(\pm)}$ is similar to that of $\mathcal{C}_{\nu}^{\pm}$ in the AFF model. On the other hand, in rationally extended gapped systems obtained by Darboux transfromations based on the schemes not containing Jordan states, the separated states have the form $A_{(-)}^{-}\widetilde{\psi}_{-\nu-1,-l_j} = A_{(+)}^{-}\psi_{\nu,n_{n+}-l_j}$, where the states $\psi_{-\nu-1,-l_j}$ belong to the negative scheme and $\psi_{\nu,n_{n+}-l_j}$ are the omitted states in the corresponding dual positive scheme. Since by construction the separated states belong to the kernel of $A_{(-)}^{+}$, the operators $\mathcal{A}^{\pm}$ and $\mathcal{C}^{-}$ will always annihilate all them.

In summary, the resulting picture is more or less the same as we had for the cases analyzed in the previous chapters. We have three pairs of ladder operators; $\mathcal{B}^{\pm}$ detect the upper and lower energy levels of each isolated valence band, $\mathcal{A}^{\pm}$ operators annihilate all the isolated states, and $\mathcal{C}^{\pm}$ operators connect isolated states with the equidistant part of the spectrum.

7.6 An Example

In this section we will apply the machinery of the dual schemes and the construction of nonlinear deformations of the conformal algebra to a nontrivial example of rationally extended systems with gaps. Remember that if we take $\nu = \mu + m$, we replace $\psi_{-(\mu+m)-1,\pm n}$ by $\Omega_{-(\mu+m)-1,\pm(n-m)}$ with $n > m$ when $\mu \to -1/2$ in each of the relations that we have in the following, see Sec. 7.3.

Consider a system generated on the base of the Darboux-dual schemes

$$
(\psi_{\nu,2}, \psi_{\nu,3}) \sim (\psi_{\nu,-0}, \psi_{\nu,-1}, \psi_{\nu,-2}, \psi_{-\nu-1,-2}, \psi_{\nu,-3}, \psi_{-\nu-1,-3})\,. \tag{7.6.1}
$$

Here, $n_- = 2$, $n_+ = 6$, $n_{n_+} = n_{n_-} = 3$ and $n_- + n_+ = 2(n_{n_+} + 1) = 8 = 2\Delta E$. The positive scheme, whose Wronskian is given explicitly in (7.3.8), corresponds to the Krein-Adler scheme that provides us with the system

$$
L_{(+)} = -\tfrac{d^2}{dx^2} + V_{(+)}(x)\,, \tag{7.6.2}
$$

whose potential $V_{(+)}$ is plotted in Figure 7.5. The spectrum of the system, $\mathcal{E}_{\nu,0} = 2\nu + 3$, $\mathcal{E}_{\nu,1} = 2\nu + 7$, $\mathcal{E}_{\nu,n} = 2\nu + 4(n+2) + 3$, $n = 2, \ldots$, is characterized by the presence of the gap of the size $3\Delta E = 12$, which appears between the first and second excited states. The negative scheme generates the shifted Hamiltonian operator $L_{(-)} = L_{(+)} - 4\Delta E$. In terms of the intertwining operators $A^{\pm}_{(+)}$ and $A^{\pm}_{(-)}$ of the respective positive and negative schemes, the physical eigenstates of (7.6.2) are given by

$$\Psi_j = A^-_{(+)}\psi_{\nu,j} = A^-_{(-)}\widetilde{\psi}_{-\nu-1,j-3}\,, \qquad j = 0, 1\,, \tag{7.6.3}$$

$$\Psi_j = A^-_{(+)}\psi_{\nu,j+2} = A^-_{(-)}\psi_{\nu,j-2}\,, \qquad j = 2, 3, \ldots\,. \tag{7.6.4}$$

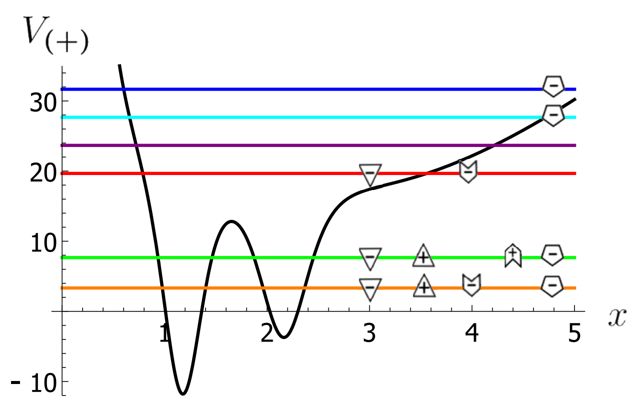

Figure 7.5: The resulting potential with $\nu = 1/3$ and energy levels of the system. The energy levels of the physical states annihilated by the ladder operators $\mathcal{A}^-$, $\mathcal{A}^+$, $\mathcal{B}^-$, $\mathcal{B}^+$, and $\mathcal{C}^-$ are indicated from left to right.

The explicit form of the polynomials (7.5.9) for the system is

$$P_{n_+}(\eta) = (\eta - 11 - 2\nu)(\eta - 15 - 2\nu)\,, \tag{7.6.5}$$

$$P_{n_-}(\eta) = (\eta + 9 - 2\nu)(\eta + 13 - 2\nu)\prod_{i=0}^{3}(\eta + 4n + 3 + 2\nu)\,, \tag{7.6.6}$$

and so, $A^-_{(\pm)}A^+_{(\pm)} = P_{n_\pm}(\mathcal{H}_\nu)$ and $A^-_{(\pm)}A^+_{(\pm)} = P_{n_\pm}(L_{(\pm)})$.

The spectrum-generating ladder operators are given by Eq. (7.5.7), and the nonlinearly deformed conformal algebras generated by each corresponding pair of the ladder operators and the Hamiltonian $L_{(+)}$ are obtained from (7.5.8) by using polynomials (7.6.5) and (7.6.6). To clarify physical nature of the ladder operators, one can inspect their corresponding kernels by using relations (7.1.12) and (7.5.5). As a result, one gets that the physical eigenstates annihilated by these operators are indicated in figure 7.5 and all other functions in the respective kernels are nonphysical solutions.

7.7 Remarks

Note that the group K_4 can also be discussed in the context of Schrödinger equation

$$\left(-\frac{\partial^2}{\partial x^2} + \frac{\nu(\nu+1)}{x^2}\right)\psi = i\frac{\partial}{\partial t}\psi\,, \tag{7.7.1}$$

the stationary solutions of which are $\Psi_\nu(x,t;\kappa) = \psi_\nu(x;\kappa)e^{-i\kappa^2 t}$, where $\psi_\nu(x;\kappa) = \sqrt{x}J_{\nu+\frac{1}{2}}(\kappa x)$. The transformation ρ_2 gives us the modified Bessel functions, besides ρ_1 produces singular solutions when ν is not a half-integer number. In the case $\nu = \ell - 1/2$ with $\ell = 0, 1, 2\dots$, we have that $\rho_1(\psi_{\ell-1/2}(x,\kappa)) = \sqrt{x}J_{-\ell}(\kappa x) = (-1)^\ell \psi_{\ell-1/2}(x,\kappa)$.

Chapter 8

Three-dimensional conformal mechanics in a monopole background

The conformal algebra shown in Chap. 2 can be realized in higher-dimensional models. In the same sense, the conformal bridge is an algebraic construction, independent of the realization. This means that it also works for these higher-dimensional generalizations.

In this chapter, we will study a direct generalization of the AFF model in three dimensions, whose Hamiltonian corresponds to

$$H = \frac{\boldsymbol{\pi}^2}{2m} + \frac{m\omega^2 r^2}{2} + \frac{\alpha}{2mr^2}\,, \tag{8.0.1}$$

where $\omega > 0$, $\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}$, $\boldsymbol{A}$ is a U(1) gauge potential of a Dirac magnetic monopole at the origin with charge g, $\nabla \times \boldsymbol{A} = \boldsymbol{B} = g\boldsymbol{r}/r^3$, and the coupling α should be chosen appropriately to prevent a fall to the center, see below. We solve the Hamiltonian equations, study the conformal Newton-Hooke symmetry of the system, and investigate a hidden symmetry which appears in a special case $\alpha = \nu^2$, $\nu = eg$. The results of this chapter are based on the article [Inzunza et al. (2020a)] which was inspired by the line of reasoning used in [Plyushchay and Wipf (2014)] to identify the hidden symmetry and characterize the particle's trajectories.

8.1 Classical case

The particle's coordinates and kinetic momenta obey the Poisson brackets relations $\{r_i, \pi_j\} = \delta_{ij}$, $\{r_i, r_j\} = 0$, and $\{\pi_i, \pi_j\} = e\epsilon_{ijk}B_k$, which give rise to the equations of motion

$$\dot{\boldsymbol{r}} = \frac{1}{m}\boldsymbol{\pi}\,, \qquad \dot{\boldsymbol{\pi}} = \frac{1}{mr^3}(\alpha\boldsymbol{n} - \nu\,\boldsymbol{r}\times\boldsymbol{\pi}) - m\omega^2\boldsymbol{r}\,, \tag{8.1.1}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$. From (8.1.1) we derive the equations $\frac{dr}{dt} = \frac{1}{m}\pi_r$, and $\dot{\boldsymbol{n}} = \frac{1}{mr^2}\boldsymbol{J}\times\boldsymbol{n}$, where we denote $\pi_r = \boldsymbol{n}\cdot\boldsymbol{\pi}$, and

$$\boldsymbol{J} = \boldsymbol{r}\times\boldsymbol{\pi} - \nu\boldsymbol{n} \tag{8.1.2}$$

is the conserved Poincaré vector identified as the angular momentum of the system,

$$\{J_i, J_j\} = \epsilon_{ijk}J_k\,, \qquad \{J_i, r_j\} = \epsilon_{ijk}r_k\,, \qquad \{J_i, \pi_j\} = \epsilon_{ijk}\pi_k\,. \tag{8.1.3}$$

In terms of this conserved quantity the Hamiltonian can be presented in the form

$$H = \frac{\pi_r^2}{2m} + \frac{\mathscr{L}^2}{2mr^2} + \frac{m\omega^2 r^2}{2}\,, \qquad \mathscr{L}^2 := \boldsymbol{J}^2 - \nu^2 + \alpha\,, \tag{8.1.4}$$

which reveals that the variables r and π_r, $\{r, \pi_r\} = 1$, behave like y and p in the one-dimensional AFF model (2.1.16). From (8.1.4) one also reads the following assertions:

- There is no fall to the center if $\mathscr{L}^2 > 0$, i.e. $\alpha > 0$, that we will assume from now on.
- The possible values of the angular momentum J and energy obey the relation $\frac{\mathscr{L}\omega}{H} \le 1$.
- The turning points for the radius are

$$r_\pm^2 = \frac{H}{m\omega^2}(1\pm\rho)\,, \qquad 0 \le \rho = \sqrt{1 - \frac{\mathscr{L}^2\omega^2}{H^2}} < 1\,, \qquad r_+r_- = \frac{\mathscr{L}}{m\omega}\,. \tag{8.1.5}$$

On the other hand, to solve the equations of motion it is worth parameterizing $\boldsymbol{n}$ as

$$\boldsymbol{n}(t) = \boldsymbol{n}_\parallel + \boldsymbol{n}_\perp(t) = -\nu\,\tfrac{\boldsymbol{J}}{J^2} + \boldsymbol{n}_\perp(t), \qquad \boldsymbol{J}\cdot\boldsymbol{n}_\perp(t) = 0\,, \qquad \boldsymbol{J}\cdot\boldsymbol{n} = \boldsymbol{J}\cdot\boldsymbol{n}_\parallel = -\nu. \tag{8.1.6}$$

$$\boldsymbol{n}_\perp(t) = \boldsymbol{n}_\perp(0)\cos\varphi(t) + \hat{\boldsymbol{J}}\times\boldsymbol{n}_\perp(0)\sin\varphi(t)\,. \tag{8.1.7}$$

From (8.1.7) and the equation of motion for $\boldsymbol{n}$ we get $\dot{\varphi} = \frac{J}{mr^2}$. These relations involve a clockwise rotation of $\boldsymbol{n}_\perp$ from the perspective of vector $\boldsymbol{J}$. Thus, if $\boldsymbol{J}$ is oriented along $\boldsymbol{e}_z$, and $\nu < 0$, $0 < \theta < \pi/2$, where $\theta = \arccos(-\nu/J)$, the path of the particle is on the upper sheet of the cone and $\boldsymbol{n}_\perp$ rotates clockwise in the horizontal plane. If on the other hand $\boldsymbol{J}$ is oriented along $-\boldsymbol{e}_z$ and $\nu > 0$, $\pi/2 < \theta < \pi$, then the path is again on the upper sheet of the cone, but the vector $\boldsymbol{n}_\perp$ rotates counterclockwise in the (x, y) plane looking from $\boldsymbol{e}_z$. We also note that when $J = \nu$, then $\theta = \pi$ so $\boldsymbol{n}$ is co-linear to $\boldsymbol{J}$ and there is no rotation at all. In the following we exclude that case.

The corresponding solutions for the angular and radial variables are

$$r^2(t) = \tfrac{H}{m\omega^2}(1 - \rho\cos(2\omega t))\,, \qquad \varphi(t) = \tfrac{J}{\mathscr{L}}\arctan\!\Big(\tfrac{r_{\max}}{r_{\min}}\tan(\omega t)\Big)\,, \tag{8.1.8}$$

where the initial conditions $r(t=0) = r_- := r_{\min}$ and $\varphi(t=0) = 0$ are assumed (also we redefine

$r_+ := r_{\max}$). By expressing time thought $\varphi(t)$ and introducing it in to $r^2(t)$, we obtain the trajectory equation

$$\frac{1}{r^2(\varphi)} = \frac{mH}{\mathscr{L}^2}\left[1 + \gamma\cos\left(\frac{2\mathscr{L}}{J}\varphi\right)\right], \tag{8.1.9}$$

which shows us that the angular period is $\pi J/\mathscr{L}$. The condition for a periodic trajectory is

$$\frac{2\mathscr{L}}{J}2\pi l_r = 2\pi l_a \quad\Longleftrightarrow\quad \frac{2\mathscr{L}}{J} = \frac{l_a}{l_r}, \qquad l_r, l_a = 1, 2, \dots . \tag{8.1.10}$$

From the definition of $\mathscr{L}$ in (8.1.4) we find that the trajectories are closed for arbitrary values of J if and only if $\alpha = \nu^2$. On the other hand, when $\alpha \neq \nu^2$, the trajectory will be closed only for special values of the angular momentum given by the condition

$$\alpha = \nu^2 + \left(\frac{1}{4}\frac{l_a^2}{l_r^2} - 1\right)J^2, \tag{8.1.11}$$

and in this case the condition $\frac{\mathscr{L}\omega}{H} \leq 1$ takes the form $\frac{l_a}{l_r} \leq \frac{2H}{\omega J}$. Figure 8.1 illustrates several particular orbits lying on the corresponding conical surface in a general case $\alpha \neq \nu^2$ and in the special case $\alpha = \nu^2$. Trajectories $\boldsymbol{r}(\varphi)$ are shown there for fixed values of H, $\boldsymbol{J}$ and ν, but for different values of α.

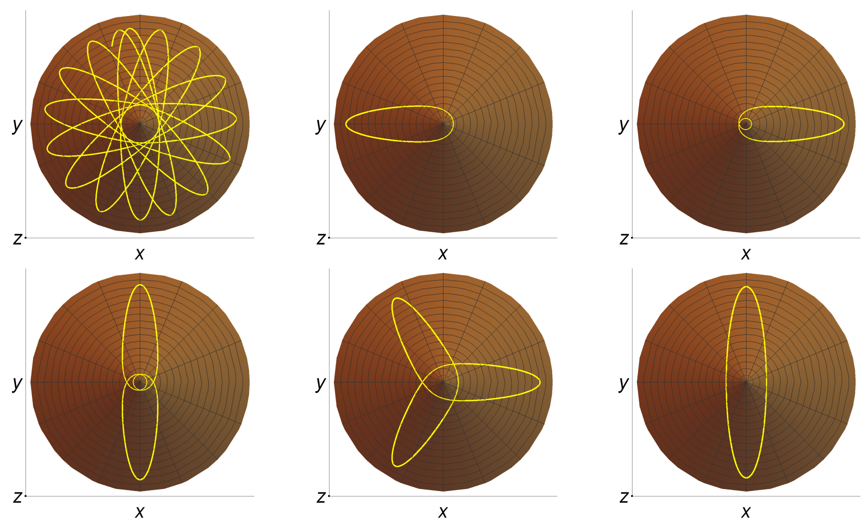

Figure 8.1: The depicted trajectories correspond to the vector $\boldsymbol{J}$ oriented along $\boldsymbol{e}_z$. The first figure in the top row represents the generic case with non-closed trajectory. The other figures are examples of closed trajectories with parameters satisfying the relation (8.1.11), with quotients $l_a/l_r = \{1,\ 1/2,\ 2/3,\ 3/2,\ 2\}$ are sequentially shown. The last relation $l_a/l_r = 2$ corresponds to the special case $\alpha = \nu^2$.

Below we shall see that when $\alpha = \nu^2$, the projection to the plane orthogonal to $\boldsymbol{J}$ of the trajectory shown on the last plot is an ellipse centered at the origin of the coordinate system similarly to the case of the three-dimensional isotropic harmonic oscillator. This corresponds to a fundamental universal property of the magnetic monopole background which we discuss in the last section. Since the center of the projected elliptical trajectory is in the center of an ellipse, the angular period P_a is twice the radial period P_r, $P_a/P_r = 2$, similarly to the isotropic harmonic oscillator. This is different from the picture of the finite orbits in Kepler problem where the force center is in one of the foci, and as a result $P_a = P_r$. This similarity with the isotropic oscillator and contrast to the Kepler problem are also reflected in the spectra of the systems at the quantum level.

As we have the AFF model form of the Hamiltonian in (8.1.4), we can intermediately write the rest of the Newton-Hooke conformal algebra generators. They are given by

$$\mathcal{D} = \tfrac{1}{2}\left(rp_r\cos(2\omega\tau) + \left(m\omega r^2 - H\omega^{-1}\right)\sin(2\omega\tau)\right)\,, \tag{8.1.12}$$

$$\mathcal{K} = \cos(2\omega\tau)m\tfrac{r^2}{2} - \tfrac{H}{\omega^2}\sin^2(\omega\tau) - \tfrac{\sin(2\omega\tau)}{2\omega}rp_r\,. \tag{8.1.13}$$

Together with H they satisfy the algebra (2.1.30). The Casimir invariant corresponds to $\mathscr{F} = \frac{\mathscr{L}^2}{4}$.

To conclude this part of the analysis, we comment on the limit $\omega \to 0$. In this case the generators H, D and K take the form

$$H_0 = \frac{\pi_r^2}{2m} + \frac{\mathscr{L}^2}{2mr^2}\,, \qquad D_0 = \frac{1}{2}r\pi_r - H_0 t\,, \qquad K_0 = \frac{mr^2}{2} - Dt - H_0 t^2\,, \tag{8.1.14}$$

and satisfy the conformal algebra.

The case $\alpha = 0$ of the system H_0 corresponds to a geodesic motion on the dynamical cone [Plyushchay (2000b, 2001)]. The special case of $\alpha = \nu^2$, on the other hand, was studied in [Plyushchay and Wipf (2014)]. It was shown there that the trajectory of the particle, projected to the plane orthogonal to $\boldsymbol{J}$, is a straight line along which the projected particle's motion takes place with constant velocity. Consistently with these peculiar properties, in the special case $\alpha = \nu^2$ the system with H_0 possesses a hidden symmetry described by the integral of motion $\boldsymbol{V} = \boldsymbol{\pi} \times \boldsymbol{J}$ being a sort of Laplace-Runge-Lenz vector, in the plane orthogonal to which and parallel to $\boldsymbol{J}$ the particle's trajectory lies [Plyushchay and Wipf (2014)]. In Fig. 8.3 some plots of the trajectories are shown for the system (8.1.14).

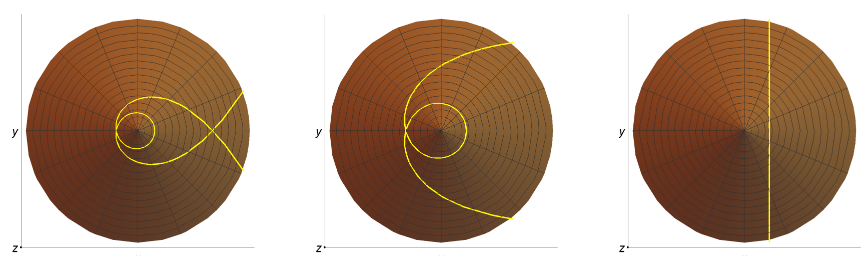

Figure 8.2: Each plot represents a trajectory for a specific value of α chosen according to (8.1.11) with the vector $\boldsymbol{J}$ oriented along $\boldsymbol{e}_z$. From left to right the cases $l_a/l_r = \{3/2,\, 1/2,\, 2\}$ are shown, where the last plot corresponds to the special case $\alpha = \nu^2$.

8.1.1 The case $\alpha = \nu^2$: hidden symmetries

In the case $\alpha = \nu^2$ the particle described by the Hamiltonian (8.0.1), which now is

$$H = \frac{\boldsymbol{\pi}^2}{2m} + \frac{m\omega^2}{2} r^2 + \frac{\nu^2}{2mr^2}\,, \tag{8.1.15}$$

admits the vector integrals of motion responsible for the closed nature of the trajectories for arbitrary choice of initial conditions. The integrals are derived by an algebraic approach as in Fradkin's construction for the isotropic three-dimensional harmonic oscillator [Fradkin (1965)].

Let us first introduce the vector quantities

$$\boldsymbol{I}_1 = \boldsymbol{\pi} \times \boldsymbol{J} \cos(\omega t) + \omega m \boldsymbol{r} \times \boldsymbol{J} \sin(\omega t)\,, \tag{8.1.16}$$

$$\boldsymbol{I}_2 = \boldsymbol{\pi} \times \boldsymbol{J} \sin(\omega t) - \omega m \boldsymbol{r} \times \boldsymbol{J} \cos(\omega t)\,. \tag{8.1.17}$$

Using the corresponding equations of motion for $\boldsymbol{r}$ and $\boldsymbol{\pi}$ is not difficult to show that $\dot{I}_i = 0$ so they are dynamical integrals of motion.

The evaluation of these integrals in the initial conditions give us

$$\boldsymbol{I}_1(0) = \frac{J^2}{r_{\text{min}}} \boldsymbol{n}_{\perp}(0)\,, \qquad \boldsymbol{I}_2(0) = m\omega r_{\text{min}} \boldsymbol{J} \times \boldsymbol{n}_{\perp}(0)\,, \tag{8.1.18}$$

thus, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are orthogonal to each other. On the other hand, the lengths of these vectors are also dynamical integrals which for the initial conditions take the form

$$|\boldsymbol{I}_1| = m\omega\sqrt{J^2 - \nu^2}\, r_{\text{max}}, \qquad |\boldsymbol{I}_2| = m\omega\sqrt{J^2 - \nu^2}\, r_{\text{min}}\,, \tag{8.1.19}$$

where we have taken into account Eqs. (8.1.18) and the second equation in (8.1.5). The sum of their squares, however, is a true integral of motion whose value is a function of H and J,

$$\boldsymbol{I}_1^2 + \boldsymbol{I}_2^2 = 2mH(J^2 - \nu^2)\,. \tag{8.1.20}$$

These vectors point in the direction of the semi-axes of the elliptic trajectory in the plane orthogonal to $\boldsymbol{J}$. The lengths of semi-major and semi-minor axes correspond to those of the vectors $r\boldsymbol{n}_{\perp}(0)$ and $r\hat{\boldsymbol{J}} \times \boldsymbol{n}_{\perp}(0)$, and are equal to $r_{\text{max}}\sqrt{1 - \nu^2/J^2}$, and $r_{\text{min}}\sqrt{1 - \nu^2/J^2}$. As it is shown in [Inzunza et al. (2020a)], in a general case of $\alpha \neq \nu^2$, the periodic change of the scalar product of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, which would not be integrals, implies a precession of the orbit, see Fig. 8.1.

Using the definition of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ in (8.1.16) and (8.1.17), we can express the position $\boldsymbol{r}(t)$ of the particle as follows,

$$\boldsymbol{r}(t) = \frac{1}{m\omega J^2}\left(\boldsymbol{J}\times\boldsymbol{I}_1\sin\omega t - \boldsymbol{J}\times\boldsymbol{I}_2\cos\omega t - \nu\frac{\sqrt{I_1^2\sin^2\omega t + I_2^2\cos^2\omega t}}{\sqrt{J^2-\nu^2}}\boldsymbol{J}\right), \tag{8.1.21}$$

where we again see that $\boldsymbol{I}_1 = \boldsymbol{I}_1(0)$ and $\boldsymbol{I}_2 = \boldsymbol{I}_2(0)$ correspond to the orthogonal set that define the elliptic trajectory in the plane.

Alternatively, one can follow a more algebraic approach to extract information on the trajectories without explicitly solving the equations of motion. It is well known from the seminal paper [Fradkin (1965)] that for the three-dimensional isotropic harmonic oscillator all symmetries of the trajectories are encoded in a tensor integral of motion. During the rest of this subsection we construct an analogous tensor for the system at hand to find the trajectories by a linear algebra techniques. We begin with the tensor integrals

$$T^{ij} = T^{(ij)} + T^{[ij]}\,, \qquad T^{(ij)} = \frac{1}{2}(I_1^iI_1^j + I_2^iI_2^j)\,, \qquad T^{[ij]} = \frac{1}{2}(I_1^iI_2^j - I_1^jI_2^i)\,. \tag{8.1.22}$$

They, unlike the vectors $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$, but like the quadratic expression (8.1.20) are the true, not depending explicitly on time integrals of motion, $\frac{d}{dt}T^{ij} = \{T^{ij}, H\} = 0$.

In accordance with (8.1.20), their components satisfy relations

$$\mathrm{tr}(T) = m(J^2-\nu^2)H\,, \qquad \epsilon_{ijk}T^{[jk]} = m\omega(J^2-\nu^2)J_i\,. \tag{8.1.23}$$

As the anti-symmetric part of T^{ij} is related to the Poincaré integral, we only need to use the symmetric part $T^{(ij)}$, which is related but not identical to Fradkin's tensor. Since the vectors (8.1.16), (8.1.17) are orthogonal to each other and to $\boldsymbol{J}$, we immediately conclude that $\boldsymbol{J}, \boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are eigenvectors of $T^{(ij)}$ with eigenvalues equal, respectively, to zero and

$$\lambda_1 = |\boldsymbol{I}_1|^2 = \frac{1}{2}m^2\omega^2(J^2-\nu^2)r^2_{\mathrm{max}}\,, \tag{8.1.24}$$

$$\lambda_2 = |\boldsymbol{I}_2|^2 = \frac{1}{2}m^2\omega^2(J^2-\nu^2)r^2_{\mathrm{min}}\,, \tag{8.1.25}$$

Also one can show that the quadratic form $\boldsymbol{r}^TT\boldsymbol{r}$ is time-independent,

$$2r_iT^{ij}r_j = (\boldsymbol{I}_1\cdot\boldsymbol{r})^2 + (\boldsymbol{I}_2\cdot\boldsymbol{r})^2 = (J^2-\nu^2)^2\,. \tag{8.1.26}$$

In a coordinate system with orthonormal base $\boldsymbol{e}_x = \hat{\boldsymbol{I}}_1, \boldsymbol{e}_y = \hat{\boldsymbol{I}}_2$ and $\boldsymbol{e}_z = \hat{\boldsymbol{J}}$, the quadratic form (8.1.26) simplifies to

$$\lambda_1x^2 + \lambda_2y^2 = (J^2-\nu^2)^2\,. \tag{8.1.27}$$

With $r_{\max} r_{\min} = J/(m\omega)$ one ends up with the equation for an ellipse in the plane orthogonal to $\boldsymbol{J}$:

$$\frac{x^2}{r_{\min}^2} + \frac{y^2}{r_{\max}^2} = \frac{J^2 - \nu^2}{J^2} \,. \tag{8.1.28}$$

The lengths of the semi-major axis and semi-minor axis of the ellipse are $r_{\max}\sqrt{1-\nu^2/J^2}$ and $r_{\min}\sqrt{1-\nu^2/J^2}$, in accordance with that was found above.

Finally, the symmetric tensor components integral $T_{(ij)}$ satisfy the Poisson bracket relations

$$\{J_i, T_{(jk)}\} = \epsilon_{ijl} T_{(lk)} + \epsilon_{ikl} T_{(jl)} \,, \tag{8.1.29}$$

$$\{T_{(ij)}, T_{(lk)}\} = m(\epsilon_{ils}\mathcal{F}_{jk} + \epsilon_{iks}\mathcal{F}_{jl} + \epsilon_{jls}\mathcal{F}_{ik} + \epsilon_{jks}\mathcal{F}_{im}) J_s \,, \tag{8.1.30}$$

where $\mathcal{F}_{ij} = \frac{1}{4} m\omega^2 (J^2 - \nu^2)^2 \delta_{ij} - H T_{(ij)}$.

In fact, the quantum version of the tensor $T_{(ij)}$ was already considered in [Labelle et al. (1991)], but this is the first time that it has been obtained and used at the classical level.

8.2 Quantum theory of the model with $\alpha = \nu^2$

The quantum theory of the system with Hamiltonian (8.1.15) is discussed in details in [McIntosh and Cisneros (1970); Labelle et al. (1991); Inzunza et al. (2020a)] and here we summarize the results. We shall use the units in which $m = 1$ and $\hbar = 1$.

In coordinate representation the basic commutation relations are

$$[\hat{r}_i, \hat{r}_j] = 0 \,, \qquad [\hat{r}_i, \hat{\Pi}_j] = i\delta_{ij} \,, \qquad [\hat{\Pi}_i, \hat{\Pi}_j] = i\nu\epsilon_{ijk}\frac{\hat{r}_k}{r^3} \,. \tag{8.2.1}$$

In what follows we shall skip the hat symbol $\hat{}$ to simplify the notation. The Hamiltonian (8.1.15) can be written as

$$H = \frac{1}{2}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\boldsymbol{J}^2 + \omega^2 r^2\right] \,, \tag{8.2.2}$$

where $\boldsymbol{J}$ is just the quantum version of the Poincaré integral (8.1.2), the components of which generate the $\mathfrak{su}(2)$ symmetry. The Dirac quantization condition implies that $\nu = eg$ must take an integer or half integer value [Plyushchay (2000b, 2001)]. Using the angular momentum treatment we obtain

$$\boldsymbol{J}^2\mathcal{Y}_j^{j_3} = j(j+1)\mathcal{Y}_j^{j_3} \,, \quad J_3\mathcal{Y}_j^{j_3} = j_3\mathcal{Y}_j^{j_3} \,, \quad J_\pm\mathcal{Y}_j^{j_3} = c_{jj_3}^{\pm}\mathcal{Y}_j^{j_3\pm 1} \,, \tag{8.2.3}$$

with $J_\pm = J_1 \pm iJ_2$, and

$$j = |\nu|, |\nu|+1, \ldots \,, \qquad j_3 = -j, \ldots, j \,, \qquad c_{jj_3}^{\pm} = \sqrt{(j \pm j_3 + 1)(j \mp j_3)} \,, \tag{8.2.4}$$

where the indicated values for j correspond to a super-selection rule. The case $\nu = 0$ corresponds just to the quantum harmonic isotropic oscillator. Excluding the zero value for ν, i.e. implying that $|\nu|$ takes any nonzero integer or half-integer value, the first relation in (8.2.3) automatically provides the necessary inequality $\boldsymbol{J}^2 = j(j+1) > \nu^2$. The functions $\mathcal{Y}_j^{j_3} = \mathcal{Y}_j^{j_3}(\theta, \varphi; \nu)$ are the (normalized) monopole harmonics [Wu and Yang (1976); Lochak (1985); Plyushchay (2000b, 2001)], which are well defined functions if and only if the combination $j \pm \nu$ is in $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Then, the eigenstates and the spectrum of H are given by

$$
\begin{aligned}
\psi_{n,j}^{j_3}(\boldsymbol{r}) &= f_{n,j}(\sqrt{\omega} r)\mathcal{Y}_j^{j_3}(\theta, \varphi)\,, \\
f_{n,j}(x) &= \left(\frac{2n!}{\Gamma(n+j+3/2)}\right)^{1/2} \omega^{3/4}\, x^j L_n^{(j+1/2)}(x^2)\, e^{-x^2/2}\,, \\
E_{n,j} &= \left(2n + j + \tfrac{3}{2}\right)\omega\,,
\end{aligned}
\tag{8.2.5}
$$

where $L_n^{(j+1/2)}(y)$ are the generalized Laguerre polynomials. The degeneracy of each level depends on ν and corresponds to

$$
\mathfrak{g}(\nu, N) = \left\{
\begin{array}{ll}
\frac{1}{2}(N+\nu+1)(N-\nu+2)\,, & j-\nu \quad \text{even} \\
 & \\
\frac{1}{2}(N-\nu+1)(N+\nu+2)\,, & j-\nu \quad \text{odd}
\end{array}
\right. ,\qquad N = 2n + j\,.
\tag{8.2.6}
$$

It is remarkable that the system possesses $2|\nu| + 1$ degenerate ground states. The ground states here are not invariant under the action of the total angular momentum $\boldsymbol{J}$, although the Hamiltonian operator commutes with $\boldsymbol{J}$ and hence is spherically symmetric. Thus we see some analog of spontaneous breaking of rotational symmetry in the magnetic monopole background. This is of course in contrast to the isotropic harmonic oscillator in three dimensions which has a unique spherically symmetric ground state and symmetry algebra $\mathfrak{su}(3)$. According to [Labelle et al. (1991)] the symmetry algebra for the system under investigation is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. We do not further dwell on these interesting aspects of symmetry but rather turn to the construction of spectrum-generating ladder operators.

Note that the coefficients at radial, n, and angular momentum, j, quantum numbers in the energy eigenvalue $E_{n,j} = (2n + j + \frac{3}{2})\,\omega$ corresponds to the ratio $P_a/P_r = l_a/l_r = 2$ between the classical angular and radial periods in the special case $\alpha = \nu^2$ under investigation. This can be compared with the structure of the principle quantum number $N = n_r + l + 1$ defining the spectrum in the quantum model of the hydrogen atom, where the corresponding classical periods are equal.

The explicit wave functions in (8.2.5) are specified by the discrete quantum numbers n, j and j_3. Our target now is to identify the ladder operators for radial, n, and angular momentum, j, quantum numbers (we already have the ladders operators for j_3), which are based on the conformal and hidden symmetries of the system.

In the algebraic approach we do not fix the representation for the position and momentum operators and thus use Dirac's ket notation for eigenstates.

Ladder operators for n. Let us first consider the quantum version of the $\mathfrak{sl}(2,\mathbb{R})$ symmetry,

$$[H,\mathcal{C}]=-2\omega\,\mathcal{C}\,,\qquad [H,\mathcal{C}^\dagger]=2\omega\,\mathcal{C}^\dagger\,,\qquad [\mathcal{C},\mathcal{C}^\dagger]=4\omega H\,, \tag{8.2.7}$$

where the generators $\mathcal{C}$, $\mathcal{C}^\dagger$ are the quantum versions of combinations of Newton-Hooke symmetry generators in the Schrödinger picture at $t=0$, i.e.,

$$\mathcal{C}=H-\omega^2r^2-\frac{i\omega}{2}(\boldsymbol{r}\cdot\boldsymbol{\pi}+\boldsymbol{\pi}\cdot\boldsymbol{r})\,, \tag{8.2.8}$$

and their action on the eigenstates is

$$\mathcal{C}\,|n,j,j_3\rangle=\omega\,d_{n,j}\,|n-1,j,j_3\rangle\quad,\quad \mathcal{C}^\dagger\,|n,j,j_3\rangle=\omega\,d_{n+1,j}\,|n+1,j,j_3\rangle\,, \tag{8.2.9}$$

$$d_{n,j}=\sqrt{2n(2n+2j+1)}\,. \tag{8.2.10}$$

Ladder operators for j. We introduce the complex vector operator

$$\boldsymbol{a}=\frac{1}{2}(\boldsymbol{b}\times\boldsymbol{J}-\boldsymbol{J}\times\boldsymbol{b})=(\boldsymbol{b}\times\boldsymbol{J}-i\boldsymbol{b})\,,\qquad \boldsymbol{b}=\frac{1}{\sqrt{2}}(\boldsymbol{\pi}-i\omega m\boldsymbol{r})\,, \tag{8.2.11}$$

together with its Hermitian conjugation. The vector operator $\boldsymbol{a}$ is the quantum version of the complex classical quantity $\frac{1}{\sqrt{2}}(\boldsymbol{I}_1+i\boldsymbol{I}_2)$ in Schrödinger picture at $t=0$, and its components satisfy the relations

$$[H,a_i]=-\omega a_i\,,\qquad [J_i,a_j]=i\epsilon_{ijk}a_k\,,\qquad [a_i,a_j]=-i\epsilon_{ijk}\,\mathcal{C}J_k\,, \tag{8.2.12}$$

$$[a_i^\dagger,a_j]=-\omega[(2\boldsymbol{J}^2+1-\nu^2)\delta_{ij}-J_iJ_j)]-iH\epsilon_{ijk}J_k\,, \tag{8.2.13}$$

The action of these operators is computed algebraically in [Inzunza et al. (2020a)] and for us is sufficient to consider a_3 and $a_3^\dagger$ and their actions on the ket-states

$$a_3\,|n,j,j_3\rangle=A_{n,j,j_3}\,|n,j-1,j_3\rangle+B_{n,j,j_3}\,|n-1,j+1,j_3\rangle\,, \tag{8.2.14}$$

$$a_3^\dagger\,|n,j,j_3\rangle=A_{n,j+1,j_3}\,|n,j+1,j_3\rangle+B_{n+1,j-1,j_3}\,|n+1,j-1,j_3\rangle\,, \tag{8.2.15}$$

where the squares of the positive coefficients are

$$\left(A_{n,j,j_3}\right)^2=\omega(2n+2j+1)\,\frac{(j^2-j_3^2)(j^2-\nu^2)}{(2j)^2-1}\,,\quad \left(B_{n,j,j_3}\right)^2=\frac{2n}{2n+2j+3}\left(A_{n,j+1,j_3}\right)^2. \tag{8.2.16}$$

We see that the operators a_3 and $a_3^\dagger$ change the quantum numbers n and j, but the result is a superposition of the two eigenstate vectors. Their action is depicted in Fig. 8.3.

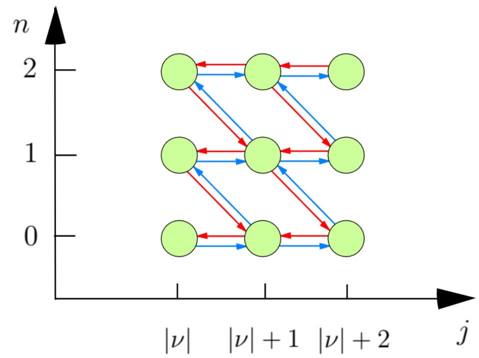

Figure 8.3: The circles represent the first two quantum numbers of the eigenstates $|n, j, j_3\rangle$. Red arrows indicate the action of a_3 and blue arrows correspond to the action of $a_3^\dagger$. Note that some circles have two emergent arrows of the same color, which means that the action of the rising/lowering operator on that states produce a superposition of two states.

Clearly, if we are working in a representation where H, $\boldsymbol{J}^2$ and J_3 are simultaneously diagonalized, it would be rather natural to try to find ladder operators that map a given eigenstate into just one eigenstate with a different quantum number j and not a superposition of eigenstates (having in mind the picture of the usual harmonic oscillator). To find such operators we introduce the nonlocal operator

$$\mathscr{J} = \sqrt{\boldsymbol{J}^2 + \tfrac{1}{4}} - \tfrac{1}{2}\,, \qquad \mathscr{J}\,|n, j, j_3\rangle = j\,|n, j, j_3\rangle\,, \tag{8.2.17}$$

and construct the operators

$$\mathscr{T}_\pm = \omega(\mathscr{J} + \tfrac{1}{2})a_3 \pm (H - \omega)a_3 \mp a_3^\dagger\,\mathcal{C} \tag{8.2.18}$$

together with their Hermitean conjugate. Actually $\mathscr{T}_\pm$ and $\mathscr{T}_\pm^\dagger$ are the third components of the vector operators $\boldsymbol{\mathcal{T}}_\pm$ and $\boldsymbol{\mathcal{T}}_\pm^\dagger$ which are given by (8.2.18) wherein a_3 and $a_3^\dagger$ are replaced by $\boldsymbol{a}$ and $\boldsymbol{a}^\dagger$ on the right hand side. But in what follows it suffices to consider $\mathscr{T}_\pm$ and $\mathscr{T}_\pm^\dagger$ which are ladder operators for the energy,

$$[H, \mathscr{T}_\pm] = \omega\mathscr{T}_\pm\,, \qquad [H, \mathscr{T}_\pm^\dagger] = -\omega\mathscr{T}_\pm^\dagger\,. \tag{8.2.19}$$

They decrease and increase the angular momentum according to

$$\mathscr{T}_+\,|n, j, j_3\rangle = \omega(2j+1)A_{n,j,j_3}\,|n, j-1, j_3\rangle\,, \tag{8.2.20}$$

$$\mathscr{T}_-\,|n, j, j_3\rangle = \omega(2j+3)B_{n,j,j_3}\,|n-1, j+1, j_3\rangle\,, \tag{8.2.21}$$

and the analogous Hermitian conjugate relations. These nonlocal objects were inspired by a similar construction presented in [Quesne and Moshinsky (1990)] for the three-dimensional isotropic harmonic oscillator.

Now one can generate in a simple way all eigenstates of the commuting observables $H, \boldsymbol{J}^2$ and J_3 by acting with the local ladder operators $\mathcal{C}$, $\mathcal{C}^\dagger$, $J_\pm$ and with the nonlocal ladder operators $\mathscr{T}_+, \mathscr{T}_+^\dagger$ on just one eigenstate. The same can be achieved with local ladder operators when one uses $a, a^\dagger$ instead of $\mathscr{T}_+, \mathscr{T}_+^\dagger$, but then the recursive construction gets more involved, since $a, a^\dagger$ map into a superposition of eigenstates.

8.2.1 The conformal bridge in monopole background

Here we show how the generators of the conformal symmetry as well as the hidden symmetry of the quantum system (8.2.2) can be obtained from generators of the corresponding symmetries of the quantum system studied in [Plyushchay and Wipf (2014)]. This will be realized by means the conformal bridge transformation introduced in Chap. 3 .

Similarly to the classical case, in the limit $\omega \to 0$ the quantum version of the generators (8.1.14) has the form

$$H_0 = \tfrac{1}{2}\left(\boldsymbol{\pi}^2 + \tfrac{\nu^2}{r^2}\right) = \tfrac{1}{2}\left(-\tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial}{\partial r}\right) + \tfrac{1}{r^2}\boldsymbol{J}^2\right) , \tag{8.2.22}$$

$$D_0 = \tfrac{1}{4}(\boldsymbol{r}\cdot\boldsymbol{\pi} + \boldsymbol{\pi}\cdot\boldsymbol{r}) - H_0 t\,, \qquad K_0 = \tfrac{1}{2}r^2 - Dt - H_0 t^2\,. \tag{8.2.23}$$

They produce the quantum conformal algebra

$$[D_0, H_0] = iH_0\,, \qquad [D_0, K_0] = -iK_0\,, \qquad [K_0, H_0] = 2iD_0\,. \tag{8.2.24}$$

The Hamiltonian H_0 is a non-compact generator of the conformal algebra $\mathfrak{sl}(2,\mathbb{R})$ with a continuous spectrum $(0,\infty)$. In the same limit and in the quantum version, the vector integrals $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ transform into vectors

$$\boldsymbol{I}_1 \to \frac{1}{2}\left(\boldsymbol{\pi}\times\boldsymbol{J} - \boldsymbol{J}\times\boldsymbol{\pi}\right) := \boldsymbol{V}\,, \qquad \frac{\boldsymbol{I}_2}{\omega} \to \frac{1}{2}\big((\boldsymbol{\pi}t - \boldsymbol{r})\times\boldsymbol{J} - \boldsymbol{J}\times(\boldsymbol{\pi}t - \boldsymbol{r})\big) := \boldsymbol{G}\,, \tag{8.2.25}$$

which we identify, respectively, as the Laplace-Runge-Lentz vector and the Galilei boost generator for the system H_0 [Plyushchay and Wipf (2014)] in the Weyl-ordered form. The commutator relations of the vectors $\boldsymbol{V}$ and $\boldsymbol{G}$ with the generators of the conformal algebra are

$$[H_0, G_i] = -iV_i\,, \qquad [K_0, V_i] = iG_i\,, \qquad [H_0, V_i] = [K_0, G_i] = 0\,, \tag{8.2.26}$$

$$[D_0, V_i] = \tfrac{i}{2}V_i\,, \qquad [D_0, G_i] = -\tfrac{i}{2}G_i\,. \tag{8.2.27}$$

In order to go in the opposite direction, i.e., to recover our system H and its symmetry generators starting from the generators (8.2.22), (8.2.23) and (8.2.25), we implement the conformal bridge transformation [Inzunza et al. (2020b)],

$$\mathfrak{S} = e^{-\omega K_0} e^{\frac{1}{2\omega} H_0} e^{i \ln 2 D_0} \,, \tag{8.2.28}$$

where generators are fixed at $t = 0$. A similarity transformation generated by $\mathfrak{S}$ yields

$$\mathfrak{S}(\boldsymbol{J})\mathfrak{S}^{-1} = \boldsymbol{J} \,, \qquad \mathfrak{S}(\boldsymbol{V})\mathfrak{S}^{-1} = \boldsymbol{a} \,, \qquad \mathfrak{S}(\omega \boldsymbol{G})\mathfrak{S}^{-1} = -i\boldsymbol{a}^\dagger \,, \tag{8.2.29}$$

$$\mathfrak{S}(H_0)\mathfrak{S}^{-1} = \tfrac{1}{2}\mathcal{C} \qquad \mathfrak{S}(2i\omega D_0)\mathfrak{S}^{-1} = H \,, \qquad \mathfrak{S}(\omega^2 K_0)\mathfrak{S}^{-1} = -\tfrac{1}{2}\mathcal{C}^\dagger \,, \tag{8.2.30}$$

where $H = H_0 + \omega^2 K_0$ is the quantum Hamiltonian (8.2.2). Then, as we know from Chap. 3, the eigenstates of H are mapped from the rank n Jordan states of zero energy of H_0, which also satisfy the equation $2i\omega D_0 \chi_{n,j}^{j_3} = \omega(2n + j + 3/2)\chi_{n,j}^{j_3}$. Besides, the coherent states are obtained from the wave-type eigenstates of H_0. On one hand, the mentioned Jordan states are

$$\chi_{n,j}^{j_3}(r, \theta, \phi) = r^{j+2n} \mathcal{Y}_j^{j_3}(\theta, \phi) \,, \tag{8.2.31}$$

and after the transformation we get

$$\mathfrak{S}\chi_{n,j}^{j_3} = \tfrac{(-1)^n}{\sqrt{2}} \left(\tfrac{2}{\omega}\right)^{n+\frac{j}{2}+\frac{3}{4}} [n!\Gamma(n + j + 3/2)]^{\frac{1}{2}} \, \psi_{n,j}^{j_3} \,. \tag{8.2.32}$$

On the other hand, the corresponding eigenstates of H_0 are

$$\phi_j^{j3}(\boldsymbol{r}; \kappa) = \tfrac{1}{\sqrt{r}} J_{j+\frac{1}{2}}(\kappa r) \mathcal{Y}_j^{j_3} = \textstyle\sum_{n=0}^{\infty} \tfrac{(-1)^n (\kappa/2)^{2n+j+1/2}}{n!\Gamma(n+j+3/2)} \chi_{n,j}^{j_3}(\boldsymbol{r}) \,, \tag{8.2.33}$$

and the normalized coherent states of H are

$$\begin{aligned} \zeta_j^{j_3}(\boldsymbol{r}; \kappa) &= N\mathfrak{S}\phi_j^{j3}(\boldsymbol{r}; \tfrac{\kappa}{\sqrt{2}}) = \sqrt{2} N e^{-\frac{\omega x^2}{2} + \frac{\kappa^2}{4\omega}} \phi_j^{j3}(\boldsymbol{r}; \kappa) \\ &= \tfrac{N}{\omega^{1/2}} \textstyle\sum_{n=0}^{\infty} \tfrac{1}{\sqrt{n!\Gamma(n+j+\frac{3}{2})}} \left(\tfrac{\kappa}{2\sqrt{\omega}}\right)^{2n+j+1/2} \psi_{n,j}^{j_3}(\boldsymbol{r}) \,, \end{aligned} \tag{8.2.34}$$

where $N^2 = \sqrt{\omega}/(I_{j+\frac{1}{2}}(\frac{|\kappa|^2}{2\omega}))$, the term $I_{j+\frac{1}{2}}(z)$ is the modified Bessel function of the first kind, and we have put the modulus in its argument because κ admits an analytic extension for complex values, as is usual for coherent states.

8.3 Remarks

As we have shown, hidden symmetries appear only when $\alpha = \nu^2$. In this case, one always has closed trajectories, the angular period is twice the radial period, and even more, the projected dynamics in the plane orthogonal to the Poincaré vector turns out to be similar to that of the three-dimensional isotropic harmonic oscillator trajectory. In fact, such an interesting "coincidence" is actually an universal property of the monopole background: Consider the system described by the Hamiltonian

$$H = \frac{\boldsymbol{\pi}^2}{2m} + \frac{\nu^2}{2mr^2} + U(r)\,, \tag{8.3.1}$$

where $U(r)$ is an arbitrary central potential. The dynamical variables $\boldsymbol{r} \times \boldsymbol{J}$ and $\boldsymbol{\pi} \times \boldsymbol{J}$ satisfy the same equations of motion as the vector variables $\boldsymbol{r} \times \boldsymbol{L}$ and $\boldsymbol{p} \times \boldsymbol{L}$ when $\nu = eg = 0$, where $\boldsymbol{L}$ is the usual angular momentum:

$\nu \neq 0$	$\nu = 0$
$\frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{J}) = \frac{1}{m}\boldsymbol{\pi} \times \boldsymbol{J}$	$\frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{L}) = \frac{1}{m}\boldsymbol{p} \times \boldsymbol{L}$
$\frac{d}{dt}(\boldsymbol{\pi} \times \boldsymbol{J}) = U'(r)\,\boldsymbol{n} \times \boldsymbol{J}$	$\frac{d}{dt}(\boldsymbol{p} \times \boldsymbol{L}) = U'(r)\,\boldsymbol{n} \times \boldsymbol{L}$

Table 8.1: Comparison of dynamics in the presence and absence of the monopole charge.

Therefore, the movement in the plane orthogonal to $\boldsymbol{J}$ is equivalent to the dynamics obtained in the absence of the magnetic monopole source, and if we know the solutions $\boldsymbol{r} = \boldsymbol{r}(t)$ and $\boldsymbol{p} = \boldsymbol{p}(t)$ in the case $\nu = 0$, the dynamic for $\boldsymbol{\pi} \times \boldsymbol{J}$ and $\boldsymbol{r} \times \boldsymbol{J}$ is at hand,

$$\boldsymbol{r}(t) = \frac{1}{J^2}\left(\boldsymbol{J} \times (\boldsymbol{r}(t) \times \boldsymbol{J}) + \sqrt{\frac{|\boldsymbol{r}(t) \times \boldsymbol{J}|}{J^2 - \nu^2}}\boldsymbol{J}\right). \tag{8.3.2}$$

On the other hand, if we take the system $\widetilde{H}_\nu = \frac{1}{2m}\boldsymbol{\pi}^2 + \widetilde{U}(r)$ with arbitrary central potential $\widetilde{U}(r)$, the corresponding dynamical problem is reduced to that of the system (8.3.1) with central potential $U(r) = \widetilde{U}(r) - \nu^2/2mr^2$. The indicated similarities and relations allow, in particular, to identify immediately the analog of the Laplace-Runge-Lenz vector (8.2.25) for a particle in the monopole background in the cases $\widetilde{U} = 0$, $U = 0$ and for the Kepler problem with $U = q/r$. This was done previously in [Plyushchay (2001); Plyushchay and Wipf (2014)] and [Labelle et al. (1991)] using different approaches.

In the next chapter we will study how to extend this picture to supersymmetric quantum mechanics.

Chapter 9

A charge-monopole superconformal model

In this chapter we extend our system by means of an additional contribution in the Hamiltonian (8.2.2) that involves spin degrees of freedom. The supplemented term describes a strong long-range spin-orbit coupling and one of its direct consequences is the appearance of two independent subsets of energy levels. In one of these subsets or towers, infinitely degenerate energy levels appear, while in the other, the levels have finite degeneration. The system is studied in detail in Sec. 9.1.

In the Sec. 9.2, we show that thanks to this term, the system introduced earlier supports a factorization in terms of intertwining operators that naturally leads us to a supersymmetric extension, which is nothing more than a three-dimensional realization of the superalgebra $\mathfrak{osp}(2|2)$. Finally, in Sec. 9.3, it is shown that by means of certain dimensional reductions, it is possible to obtain supersymmetric AFF models in their exact and spontaneously broken supersymmetric phase. Something special about the models obtained in this way is that the coupling constant in the potential is $j(j+1)$, where j can takes integer or half-integer values, starting from $\nu=(eg)^2$.

9.1 Introducing spin degrees of freedom: Spin-orbit coupling

Let us consider the following two Hamiltonians with strong spin-orbit coupling

$$H_{\pm\omega}=\frac{1}{2}\left(\boldsymbol{\pi}^2+\omega^2r^2+\frac{\nu^2}{r^2}\right)\pm\omega\,\boldsymbol{\sigma}\cdot\boldsymbol{J}=H\pm\omega\,\boldsymbol{\sigma}\cdot\boldsymbol{J}\,. \tag{9.1.1}$$

The Hamiltonians $H_{\pm\omega}$ are similar to those which appear as subsystems of the nonrelativistic limit of the supersymmetric Dirac oscillator discussed in [Moshinsky and Szczepaniak (1989); Bentez et al. (1990)]. Thus the eigenvalue problems can be solved similarly as in those references, but the usual spherical harmonics are replaced by the monopole harmonics. Actually, if we choose a spin-orbit coupling $\omega' \boldsymbol{\sigma}\cdot\boldsymbol{J}$ with $0 \le \omega < \omega'$, then the spectra of both Hamiltonians would be unbounded from below. On the other hand, for $0 \le \omega' < \omega$ all energies will have finite degeneracy. Only in the very particular case $\omega' = \omega$, which we consider here, the spectra are bounded from below and half of the energies have a finite degeneracy whereas the other half have infinite degeneracy. This reminds us the BPS-limits in field theory, where different interactions balance and supersymmetry is observed.

The operators H and $\boldsymbol{\sigma}\cdot\boldsymbol{J}$ commute and as a consequence $H_{\pm\omega}$ commute with the "total angular momentum"

$$\boldsymbol{K} = \boldsymbol{J} + \boldsymbol{s} = \boldsymbol{J} + \tfrac{1}{2}\,\boldsymbol{\sigma}\,, \qquad [K_i, K_j] = i\epsilon_{ijk}K_k\,. \tag{9.1.2}$$

The possible eigenvalues of $\boldsymbol{K}^2$ are $k(k+1)$. It is well-known how to construct the simultaneous eigenstates of $\boldsymbol{K}^2$ and K_3:

$$|n,k,k_3,\pm\rangle = \sum_{m_s} C^{kk_3}_{jj_3\frac{1}{2}m_s}\,|n,j,j_3\rangle \otimes \left|\tfrac{1}{2},m_s\right\rangle_{k=j\pm\frac{1}{2}}\ , \tag{9.1.3}$$

where the Clebsch-Gordan coefficients

$$C^{kk_3}_{jj_3\frac{1}{2}m_s} = \left\langle j,j_3,\tfrac{1}{2},m_s\right|k,k_3\rangle \tag{9.1.4}$$

on the right hand side are nonzero only if $j_3 + m_s = k_3$ and if the triangle-rule holds, which means that the total angular momentum k is either $j+\frac{1}{2}$ or $j-\frac{1}{2}$. In the first case the eigenstates of the total angular momentum are denoted by $|\dots,k,k_3,+\rangle$ and in the second case by $|\dots,k,k_3,-\rangle$. The sums (9.1.3) contain just two terms, since the eigenvalue m_s of the third spin-component $s_3 = \frac{1}{2}\sigma_3$ is either $\frac{1}{2}$ or $-\frac{1}{2}$. In the coordinate representation the wavefunctions corresponding to these kets are given by

$$\langle \boldsymbol{r}|n,k,k_3,\pm\rangle = f_{n,j}(\sqrt{\omega}r)\,\langle \boldsymbol{n}|k,k_3,\pm\rangle\ , \tag{9.1.5}$$

$$\langle \boldsymbol{n}|k,k_3,\pm\rangle = \tfrac{1}{\sqrt{2k+1\mp 1}}\begin{pmatrix} \pm\sqrt{k\pm k_3 + (1\mp 1)/2}\,\mathcal{Y}^{k_3-1/2}_{k\mp 1/2}(\theta,\varphi;\nu) \\ \sqrt{k\mp k_3 + (1\mp 1)/2}\,\mathcal{Y}^{k_3+1/2}_{k\mp 1/2}(\theta,\varphi;\nu) \end{pmatrix} := \Omega^{k_3\,\pm}_{k}\,. \tag{9.1.6}$$

If $\nu = eg$ is integer-valued then j is a non-negative integer and k a positive half-integer. If eg is half-integer, then j is a positive half-integer and k is in $\mathbb{N}_0$.

The vector in (9.1.3) is a simultaneous eigenstate of $\boldsymbol{J}^2$ with eigenvalue $j(j+1)$, of $\boldsymbol{K}^2$ with eigenvalue $k(k+1)$, of H with eigenvalue $(2n+j+\frac{3}{2})\omega$, where $j = k \mp 1/2$, and finally of the operator $\boldsymbol{\sigma}\cdot\boldsymbol{J}$:

$$\boldsymbol{\sigma}\cdot\boldsymbol{J}\,|n,k,k_3,\pm\rangle = \big(\pm(k+\tfrac{1}{2}) - 1\big)\,|n,k,k_3,\pm\rangle\ . \tag{9.1.7}$$

As a consequence the action of the Hamiltonians in (9.1.1) on these states is

$$H_{+\omega}\,|n,k,k_3,\pm\rangle = \omega\left(2n+k+\tfrac{1}{2}\pm k\right)|n,k,k_3,\pm\rangle\ , \tag{9.1.8}$$

$$H_{-\omega}\,|n,k,k_3,\pm\rangle = \omega\left(2n+k+\tfrac{5}{2}\mp(k+1)\right)|n,k,k_3,\pm\rangle\ . \tag{9.1.9}$$

We see that the discrete eigenvalues of both Hamiltonians $H_{\pm\omega}$ fall into two families: in one family all energies are infinitely degenerate and in the other family they all have finite degeneracy (due to their dependence on the quantum number k). More explicitly, for $k=j\mp\frac{1}{2}$ the eigenvalues of $H_{\mp\omega}$ have infinite degeneracy and for $k=j\pm\frac{1}{2}$ they have finite degeneracy $\mathfrak{g}(N,\nu)=N^2-\nu^2$, where $N=n+j+1$. A similar peculiar behavior is observed in the Dirac oscillator spectrum [Moshinsky and Szczepaniak (1989)].

Operators $K_\pm = K_1 \pm iK_2$ are the ladder operators for the magnetic quantum number k_3. The ladder operators for the radial quantum number are given in (8.2.8), and their action on the simultaneous eigenstates reads

$$\mathcal{C}\,|n,k,k_3,\pm\rangle = \omega d_{n,j}\,|n-1,k,k_3,\pm\rangle\ , \tag{9.1.10}$$

$$\mathcal{C}^\dagger\,|n,k,k_3,\pm\rangle = \omega d_{n+1,j}\,|n+1,k,k_3,\pm\rangle\ , \tag{9.1.11}$$

with coefficients defined in (8.2.10). Thus, as for the spin-zero particle system in monopole background, we can easily construct local ladder operators for n and k_3. But again, finding ladder operators for k is more difficult. One way to proceed is to follow the ideas employed for the Dirac oscillator in [Moshinsky and Szczepaniak (1989); Bentez et al. (1990); Quesne and Moshinsky (1990)]. First we decompose the total Hilbert space in two subspaces, $\mathscr{H}=\mathscr{H}^{(+)}\oplus\mathscr{H}^{(-)}$, where each $\mathscr{H}^{(\pm)}$ is spanned by the states $|n,k,k_3,\pm\rangle$. Actually we can construct nonlocal operators which project orthonormally onto these subspaces,

$$\mathscr{P}_+ = \tfrac{1}{2}+\sqrt{\boldsymbol{K}^2+\tfrac{1}{4}}-\sqrt{\boldsymbol{J}^2+\tfrac{1}{4}}\,, \tag{9.1.12}$$

$$\mathscr{P}_- = \tfrac{1}{2}-\sqrt{\boldsymbol{K}^2+\tfrac{1}{4}}+\sqrt{\boldsymbol{J}^2+\tfrac{1}{4}}\,, \tag{9.1.13}$$

and reproduce or annihilate the eigenstates,

$$\mathscr{P}_{(\pm)}\big|_{\mathscr{H}^{(\pm)}} = \mathbb{1}\big|_{\mathscr{H}^{(\pm)}}\,, \qquad \mathscr{P}_{(\pm)}\big|_{\mathscr{H}^{(\mp)}} = 0\,. \tag{9.1.14}$$

In next step we introduce the operators

$$\mathcal{A}_\pm = \mathscr{P}_\pm \mathscr{T}_\pm \mathscr{P}_\pm\,, \tag{9.1.15}$$

where the nonlocal $\mathscr{T}_{\pm}$ have been defined in (8.2.18). The presence of the projectors will ensure that $\mathcal{A}_{\pm}$ only acts on eigenstates in $\mathscr{H}^{(\pm)}$, and its action on these eigenstates can be computed straightforwardly using the relations (8.2.20) and (8.2.21):

$$\mathcal{A}_{+}\,|n,k,k_3,+\rangle = (k-1)\sqrt{n+k}\,\Lambda_{k,k_3,j}\;|n,k-1,k_3,+\rangle\;, \tag{9.1.16}$$

$$\mathcal{A}_{-}\,|n,k-1,k_3,-\rangle = (k+1)\sqrt{n}\,\Lambda_{k,k_3,j}\;|n-1,k,k_3,-\rangle\;, \tag{9.1.17}$$

with

$$\Lambda_{k,k_3,j} = \frac{\omega^{3/2}}{k}\sqrt{2(k^2-k_3^2)(j^2-\nu^2)}\,.$$

These relations mean that the operators $\mathcal{A}_{\pm}$ and their adjoint act as ladder operators for the quantum number k. Together with operators $K_{\pm}$, $\mathcal{C}$, $\mathcal{C}^{\dagger}$ they generate all eigenstates in the full Hilbert space from just two eigenstates, one from each subspace $\mathscr{H}^{(\pm)}$.

9.2 The $\mathfrak{osp}(2|2)$ superconformal extension

In this subsection we construct and analyze supersymmetric partners of the Hamiltonians $H_{\pm\omega}$ by introducing factorizing operators. From these we obtain two $\mathcal{N}=2$ super-Poincaré quantum systems which are related to each other by a common integral of motion which generates an R-symmetry. Supplementing the supercharges of one of these systems by supercharges of another, we extend the $\mathcal{N}=2$ super-Poincaré symmetry up to the $\mathfrak{osp}(2|2)$ superconformal symmetry realized by a three-dimensional system of spin-$1/2$ particle in a monopole background.

Consider the first-order scalar operators

$$\Theta = i\,\boldsymbol{\sigma}\cdot\boldsymbol{b} - \frac{1}{\sqrt{2}}\frac{\nu}{r}\,, \qquad \Xi = i\,\boldsymbol{\sigma}\cdot\boldsymbol{b}^{\dagger} - \frac{1}{\sqrt{2}}\frac{\nu}{r}\,, \tag{9.2.1}$$

and their adjoint $\Theta^{\dagger}$ and $\Xi^{\dagger}$. The products of these operators with their adjoint are

$$H_{[1]} := \Theta\Theta^{\dagger} = H_{+\omega} + \tfrac{3}{2}\omega\,, \qquad \breve{H}_{[1]} := \Xi\Xi^{\dagger} = H_{-\omega} - \tfrac{3}{2}\omega\,, \tag{9.2.2}$$

where $H_{\pm\omega}$ are given in (9.1.1). The associated superpartners take the form

$$H_{[0]} := \Theta^{\dagger}\Theta = \breve{H}_{[1]} - \nu\left(\tfrac{1}{r^2}+2\omega\right)\sigma_r\,, \tag{9.2.3}$$

$$\breve{H}_{[0]} := \Xi^{\dagger}\Xi = H_{[1]} - \nu\left(\tfrac{1}{r^2}-2\omega\right)\sigma_r\,, \tag{9.2.4}$$

wherein the projection of $\boldsymbol{\sigma}$ to the normal unit vector appears,

$$\sigma_r = \boldsymbol{n}\cdot\boldsymbol{\sigma} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} . \tag{9.2.5}$$

The first order operators satisfy the intertwining relations

$$\Theta H_{[0]} = H_{[1]}\Theta\,, \qquad \Theta^\dagger H_{[1]} = H_{[0]}\Theta^\dagger\,, \tag{9.2.6}$$

$$\Xi\breve{H}_{[0]} = \breve{H}_{[0]}\Xi\,, \qquad \Xi^\dagger\breve{H}_{[1]} = \breve{H}_{[0]}\Xi^\dagger\,. \tag{9.2.7}$$

To compute the action of the intertwining operators $\Theta^\dagger$ and $\Xi^\dagger$ in eigenstates of $H_{\pm\omega}$ is useful to express them in the form

$$\Theta^\dagger = \frac{\sigma_r}{\sqrt{2}}\left(-\frac{1}{r}\frac{\partial}{\partial r}r + \omega r + \frac{1+\boldsymbol{\sigma}\cdot\boldsymbol{J}}{r}\right)\,, \tag{9.2.8}$$

$$\Xi^\dagger = \frac{\sigma_r}{\sqrt{2}}\left(-\frac{1}{r}\frac{\partial}{\partial r}r - \omega r + \frac{1}{r}(1+\boldsymbol{\sigma}\cdot\boldsymbol{J})\right)\,. \tag{9.2.9}$$

Then the strategy is to apply directly this operators on the eigenstates of H_ω in their coordinate representation (9.1.5), obtaining in this way the eigenstates of systems $H_{[0]}$ and $\breve{H}_{[0]}$. The action of operators Θ and Ξ in these new eigenvectors follows from the intertwining relations (9.2.6)-(9.2.7). The final result is

$$\Theta^\dagger\,|n,k,k_3,\pm\rangle = \pm\sqrt{2\omega(n+1+\beta_\pm k)}\,\|n+\beta_\mp,k,k_3,\pm\rangle\,, \quad \beta_\pm = \tfrac{1}{2}(1\pm 1)\,, \tag{9.2.10}$$

$$\Theta\|n,k,k_3,\pm\rangle = \pm\sqrt{2\omega(n+\beta_\pm(k+1))}\;|n-\beta_\mp,k,k_3,\pm\rangle\,, \tag{9.2.11}$$

$$\Xi^\dagger\,|n,k,k_3,\pm\rangle = \pm\sqrt{2\omega(n+\beta_\mp(k+1)}\,\|n-\beta_\pm,k,k_3,\pm\rangle\,, \tag{9.2.12}$$

$$\Xi\,\|n,k,k_3,\pm\rangle = \pm\sqrt{2\omega(n+1+\beta_\mp k)}\;\,|n+\beta_\pm,k,k_3,\pm\rangle\,. \tag{9.2.13}$$

Where in coordinate representation the normalized spinors $\|n,k,k_3,\pm\rangle$ have the explicit form

$$\langle\boldsymbol{r}\|n,k,k_3,\pm\rangle = f_{n,j\pm1}\sigma_r\Omega_k^{k_3\,\pm}\,, \tag{9.2.14}$$

and $\Omega_k^{k_3\,\pm}$ are given in (9.1.6).

From these equations it is easy to show that

$$H_{[0]}\|n,k,k_3,\pm\rangle = 2\omega(n+\beta_\pm(k+1))\|n,k,k_3,\pm\rangle\,, \tag{9.2.15}$$

$$\breve{H}_{[0]}\,\|n,k,k_3,\pm\rangle = 2\omega(n+1+k\beta_\mp)\,\|n,k,k_3,\pm\rangle\,, \tag{9.2.16}$$

and note that in one hand, $\|0,k,k_3,-\rangle$ are zero-modes of $H_{[0]}$ since they are annihilated by Θ, and on the other hand $\Xi^\dagger$ as well as $\breve{H}_{[1]}$ annihilate the set of states $|0,k,k_3,+\rangle$.

Having at hand the eigenstates $\|n,k,k_3,\pm\rangle$, one may find spectrum-generating ladder operators. In this context Eqs. (9.2.10), (9.2.11), (9.2.12) and (9.2.13) can be used to construct such operators for the quantum number n. They read

$$\tilde{\mathcal{C}} = \Xi^\dagger\Theta\,, \qquad \tilde{\mathcal{C}}^\dagger = \Theta^\dagger\Xi\,, \tag{9.2.17}$$

and act on the eigenvectors $\|\ldots\rangle$ as follows:

$$\begin{aligned} \tilde{\mathcal{C}}^\dagger\,\|n,k,k_3,\pm\rangle &= 2\omega d_{n+1,j\pm1}\,\|n+1,k,k_3,\pm\rangle\,,\\ \tilde{\mathcal{C}}\,\|n,k,k_3,\pm\rangle &= 2\omega d_{n,j\pm1}\,\|n-1,k,k_3,\pm\rangle\,. \end{aligned} \tag{9.2.18}$$

Actually, the first order operators Θ and $\Xi^\dagger$ factorize the earlier considered second order ladder operator (8.2.8) according to $\mathcal{C} = \Theta\Xi^\dagger$.

Having constructed lowering and raising operators for n, we are still missing ladder operators for k and k_3. For the latter we may of course use $K_\pm$, since Θ, Ξ and their adjoint are scalar operators with respect to $\boldsymbol{K}$. But once more, for the angular momentum quantum number k we can introduce nonlocal "dressed" operators

$$\tilde{\mathcal{A}}_- = \Theta\sqrt{\tfrac{1}{H_{[1]}}}\mathcal{A}_-\sqrt{\tfrac{1}{H_{[1]}}}\Theta^\dagger\,, \qquad \tilde{\mathcal{A}}_+ = \Xi\sqrt{\tfrac{1}{\breve{H}_{[1]}}}\mathcal{A}_+\sqrt{\tfrac{1}{\breve{H}_{[1]}}}\Xi^\dagger\,, \tag{9.2.19}$$

and their adjoint operators, where $\mathcal{A}_\pm$ have been given in (9.1.15). The operators $\tilde{\mathcal{A}}_\pm$ are the analogs to $\mathcal{A}_\pm$ for the vectors $\|n,k,k_3,\pm\rangle$, as we can see from the equations

$$\tilde{\mathcal{A}}_+\|n,k,k_3,+\rangle = (k-1)\sqrt{n+k}\,\Lambda_{k,k_3,j}\,\|n,k-1,k_3,+\rangle\,, \tag{9.2.20}$$

$$\tilde{\mathcal{A}}_-\|n,k-1,k_3,-\rangle = (k+1)\sqrt{n}\,\Lambda_{k,k_3,j}\,\|n-1,k,k_3,-\rangle\,. \tag{9.2.21}$$

In a final step we combine the four 2×2 matrix Hamiltonians introduced above into two 4×4 matrix super-Hamiltonians as follows:

$$\mathcal{H} = \begin{pmatrix} H_{[1]} & 0 \\ 0 & H_{[0]} \end{pmatrix}\,, \qquad \breve{\mathcal{H}} = \begin{pmatrix} \breve{H}_{[1]} & 0 \\ 0 & \breve{H}_{[0]} \end{pmatrix}\,. \tag{9.2.22}$$

In the limit $\nu \to 0$ they turn into different versions of the Dirac oscillator in the nonrelativistic limit, see [Moshinsky and Szczepaniak (1989)]. Both operators commute with the $\mathbb{Z}_2$-grading operator

$\Gamma=\sigma_3\otimes\mathbb{I}_{2\times 2}$, $[\Gamma,\mathcal{H}]=[\Gamma,\breve{\mathcal{H}}]=0$, and their difference is the (bosonic) integral of motion

$$\mathcal{R}=\frac{1}{2\omega}(\mathcal{H}-\breve{\mathcal{H}})=(\boldsymbol{J}\cdot\boldsymbol{\sigma}+\tfrac{3}{2})\Gamma-2\nu\sigma_r\Pi_-=\left(\begin{array}{cc}\boldsymbol{\sigma}\cdot\boldsymbol{J}+\frac{3}{2} & 0\\ 0 & -(\boldsymbol{\sigma}\cdot\boldsymbol{J}+2\nu\sigma_r+\frac{3}{2})\end{array}\right), \quad (9.2.23)$$

where Π_- is a projector,

$$\Pi_\pm=\tfrac{1}{2}(1\pm\Gamma)\,. \quad (9.2.24)$$

In the fermionic sectors of the systems $\mathcal{H}$ and $\breve{\mathcal{H}}$ we have the nilpotent operators

$$\mathcal{Q}=\left(\begin{array}{cc}0 & \Theta\\ 0 & 0\end{array}\right), \qquad \mathcal{W}=\left(\begin{array}{cc}0 & 0\\ \Xi^\dagger & 0\end{array}\right), \quad (9.2.25)$$

$\{\Gamma,\mathcal{Q}\}=\{\Gamma,\mathcal{W}\}=0$, and their adjoint operators.

The even integral $\mathcal{R}$ in (9.2.23) generates an R-symmetry for both systems. Having in mind that $\mathcal{H}$ and $\breve{\mathcal{H}}$ can be diagonalized simultaneously, from now on we treat $\mathcal{H}$ as the Hamiltonian of the super-extended system and $\breve{\mathcal{H}}=\mathcal{H}-2\omega\mathcal{R}$ as its integral. Then, by anti-commuting $\mathcal{Q}$ and $\mathcal{W}$ we obtain the bosonic generator

$$\mathcal{G}=\{\mathcal{W},\mathcal{Q}\}=\left(\begin{array}{cc}\mathcal{C} & 0\\ 0 & \tilde{\mathcal{C}}\end{array}\right), \qquad [\Gamma,\mathcal{G}]=0\,, \quad (9.2.26)$$

together with its adjoint. They are composed from the ladder operators of sub-systems $H_{[1]}$ and $H_{[0]}$ of our system $\mathcal{H}$.

Taking together, these scalar generators with respect to

$$\mathcal{K}_i=\left(\begin{array}{cc}K_i & 0\\ 0 & K_i\end{array}\right), \qquad i=1,2,3\,, \quad (9.2.27)$$

obey the $\mathfrak{osp}(2,2)$ superalgebra mentioned in Chap. 2, Eqs. (2.2.24)-(2.2.27), and therefore this construction maybe considered as generalization of the super-extended AFF model to three dimensions.

The common eigenstates of $\mathcal{H}$, $\mathcal{R}$, Γ, $\mathcal{K}_3$ and $\boldsymbol{\mathcal{K}}^2$ are given by

$$|n,k,k_3,\pm,1\rangle=\left(\begin{array}{c}|n,k,k_3,\pm\rangle\\ 0\end{array}\right), \qquad |n,k,k_3,\pm,-1\rangle=\left(\begin{array}{c}0\\ \|n,k,k_3,\pm\rangle\end{array}\right), \quad (9.2.28)$$

which satisfy the eigenvalue equations

$$\mathcal{H}\,|n,k,k_3,\pm,\gamma\rangle = 2\omega\big(n+\tfrac{1}{2}(1+\gamma)+\beta_\pm(k+\tfrac{1}{2}(1-\gamma))\big)\,|n,k,k_3,\pm,\gamma\rangle\ , \tag{9.2.29}$$

$$\Gamma\,|n,k,k_3,\pm,\gamma\rangle = \gamma\,|n,k,k_3,\pm,\gamma\rangle\ , \qquad \gamma=\pm 1\,, \tag{9.2.30}$$

$$\mathcal{R}\,|n,k,k_3,\pm,\gamma\rangle = [\pm(k+\tfrac{1}{2})+\tfrac{\gamma}{2}]\,|n,k,k_3,\pm,\gamma\rangle\ , \tag{9.2.31}$$

$$\boldsymbol{\mathcal{K}}^2\,|n,k,k_3,\pm,\gamma\rangle = k(k+1)\,|n,k,k_3,\pm,\gamma\rangle\ , \tag{9.2.32}$$

$$\mathcal{K}_3\,|n,k,k_3,\pm,\gamma\rangle = k_3\,|n,k,k_3,\pm,\gamma\rangle\ . \tag{9.2.33}$$

The operators $\mathcal{Q}$ and $\mathcal{Q}^\dagger$ ($\mathcal{W}$ and $\mathcal{W}^\dagger$) defined in (9.2.25), interchange the state vectors $|n,k,k_3,\pm,\gamma\rangle$ and $|n,k,k_3,\pm,-\gamma\rangle$ according to the rules in (9.2.10), (9.2.11) and (9.2.12), (9.2.13). The ground states of $\mathcal{H}$ ($\breve{\mathcal{H}}$) which are given by $|n,k,k_3,-,-1\rangle$ ($|n,k,k_3,+,+1\rangle$) are invariant under transformations generated by these fermionic operators, therefore the quantum system $\mathcal{H}$ exhibits the unbroken $\mathcal{N}=2$ Poincaré supersymmetry.

Finally, the spectrum-generating ladder operators for the supersymmetric system correspond to operators $\mathcal{G}$ and $\mathcal{G}^\dagger$ (associated with n), $\mathcal{K}_\pm$ that change k_3, and the matrix nonlocal operators

$$\begin{pmatrix} \mathcal{A}_\pm & 0 \\ 0 & \tilde{\mathcal{A}}_\pm \end{pmatrix}, \qquad \begin{pmatrix} \mathcal{A}_\pm^\dagger & 0 \\ 0 & \tilde{\mathcal{A}}_\pm^\dagger \end{pmatrix}. \tag{9.2.34}$$

related to the angular quantum number k.

9.3 Dimensional reductions

The system studied in the last section and the one presented in Chap 2, Sec. 2.2 share the same symmetry, and in this paragraph we will show that they are related by a dimensional reduction. For the sake of simplicity, we put $\omega=1$ here, and denote $\sqrt{\omega}r=r$ as x.

The first step is to note that the Hamiltonian $\mathcal{H}$ can be presented in the following form

$$\mathcal{H}=\frac{1}{2}\left[-\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^2\frac{\partial}{\partial x}\right)+x^2\right]\mathbb{I}_{4\times 4}+\frac{1}{2x^2}(\boldsymbol{\mathcal{K}}^2-\Gamma\mathcal{R}+\tfrac{3}{4})+\mathcal{R}\,. \tag{9.3.1}$$

Then, to do the reduction we introduce the set of equations

$$(\boldsymbol{\mathcal{K}}^2-k(k+1))\,|\chi,\pm\rangle=0\,, \qquad (\mathcal{K}_3-k_3)\,|\chi,\pm\rangle=0\,, \tag{9.3.2}$$

$$\mathcal{P}_\pm\,|\chi,\pm\rangle=0\,, \qquad \mathcal{P}_\pm=\tfrac{1}{2k+1}(\Pi_\pm+k\mp\mathcal{R})\,, \tag{9.3.3}$$

where $k=j\pm\frac{1}{2}$, and $k_3=j_3\pm\frac{1}{2}$. $k=j\pm\frac{1}{2}$, and $k_3=j_3\pm\frac{1}{2}$. Here, the most general form of

$|\chi,\pm\rangle$ is

$$|\chi,\pm\rangle=\sum_{n=0}^{\infty}a_n^{\pm}\,|n,k,k_3,\pm,1\rangle+b_n^{\pm}\,|n,k,k_3,\pm,-1\rangle=\sum_{n=0}^{\infty}\begin{pmatrix}a_n^{\pm}\,|n,k,k_3,\pm\rangle\\ b_n^{\pm}\|n,k,k_3,\pm\rangle\end{pmatrix}\,, \tag{9.3.4}$$

and effectively, operators $\mathcal{P}_{\pm}$ are projectors onto the orthogonal subspaces $|\chi,-\rangle$ and $|\chi,+\rangle$. These states satisfy

$$\mathcal{H}\,|\chi,-\rangle=\frac{1}{x}\mathcal{H}_j^{-}x\otimes\mathbb{I}_{2\times2}\,|\chi,-\rangle\ ,\qquad \mathcal{H}\,|\chi,+\rangle=\sigma_1(\frac{1}{x}\mathcal{H}_{j+1}^{+}x)\sigma_1\otimes\mathbb{I}_{2\times2}\,|\chi,+\rangle\ , \tag{9.3.5}$$

where $\mathcal{H}_j^{-}=\mathcal{H}_j^{e}$ and $\mathcal{H}_j^{+}=\mathcal{H}_j^{b}$ are the one-dimensional supersymmetric extension of the AFF model in exact and spontaneously broken phase, see Chap 2, Sec. 2.2. Moreover, if we call as $\mathcal{B}_a$ and $\mathcal{F}_b$ (where index $\mathcal{B}_1$ is the Hamiltonian and so on) the bosonic and fermionic generators of the three-dimensional system, respectively, and in the same vein $\mathscr{B}_{j,a}^{\pm}$ and $\mathscr{F}_{j,b}^{\pm}$ are their analogs for one-dimensional system in their respective supersymmetric phases, we get

$$\mathcal{B}_a\,|\chi,-\rangle=\tfrac{1}{x}\mathscr{B}_{j,a}^{-}x\otimes\mathbb{I}_{2\times2}\,|\chi,-\rangle\ ,\qquad \mathcal{F}_b\,|\chi,-\rangle=\tfrac{1}{x}\mathscr{F}_{j,b}^{-}x\otimes\sigma_r\,|\chi,-\rangle\ , \tag{9.3.6}$$

$$\mathcal{B}_a\,|\chi,+\rangle=\sigma_1(\tfrac{1}{x}\mathscr{B}_{j+1,a}^{+}x)\sigma_1\otimes\mathbb{I}_{2\times2}\,|\chi,+\rangle\,,\quad \mathcal{F}_b\,|\chi,+\rangle=\sigma_1(\tfrac{1}{x}\mathscr{F}_{j,b}^{+}x)\sigma_1\otimes\sigma_r\,|\chi,+\rangle\,. \tag{9.3.7}$$

In these equations the generators take the form of a direct product of two matrix operators: In case of bosonic (fermionic) operators one has $x^{-1}B\otimes x\mathbb{I}_{2\times2}$ ($x^{-1}F\otimes x\sigma_r$), where B (F) is a particular bosonic (fermionic) operator of the one-dimensional AFF model in its corresponding supersymmetric phase. Note that in the odd sector we still have angular dependence due to σ_r.

To complete the reduction we introduce the operators

$$\mathcal{O}_{\pm}=\begin{pmatrix}|v\rangle\,\langle k,k_3,\pm| & 0\\ 0 & |v\rangle\,\langle k,k_3,\pm|\,\sigma_r\end{pmatrix}\,,\qquad |v\rangle=\begin{pmatrix}1\\1\end{pmatrix}, \tag{9.3.8}$$

and their adjoint, as well as the unitary operator

$$U=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\,,\qquad UU^{\dagger}=1\,,\qquad \det U=-1\,. \tag{9.3.9}$$

Operators $\mathcal{O}_{\pm}$ effectively integrate the angular variables, so the bosonic generators do not change, but the fermionic generators are transformed into

$$\mathcal{O}_-\mathcal{F}_b\mathcal{O}_-^\dagger\,|\Psi,-\rangle = \tfrac{1}{x}\mathscr{F}^-_{j,b}x\otimes\sigma_1\,|\Psi,-\rangle\;, \tag{9.3.10}$$

$$\mathcal{O}_+\mathcal{F}_b\mathcal{O}_+^\dagger\,|\Psi,+\rangle = \sigma_1(\tfrac{1}{x}\mathscr{F}^+_{j+1,b}x)\sigma_1\otimes\sigma_1\,|\Psi,+\rangle\;, \tag{9.3.11}$$

where $\mathcal{O}_\pm\,|\chi,\pm\rangle = |\Psi,\pm\rangle$. On the other hand, by means of the unitary transformation produced by U, we are able to present the bosonic and fermionic generators, already transformed by $\mathcal{O}_\pm$, in the form $\mathbb{I}_{2\times2}\otimes x^{-1}Bx$ and $\sigma_1\otimes x^{-1}Fx$, respectively. From these expressions one simply extracts the one-dimensional generators by means of the projectors $\Pi_\pm$, and it is also easy to show that the objects $\Pi_\pm U\,|\Psi,\pm\rangle$ take the form of the eigenstates of the AFF supersymmetric model divided by x.

In summary, we have two schemes of dimensional reductions made up by a projection on a subspace with fixed k, the integration of the remaining angular variables, and a unitary transformation. Let us denote these two schemes as $\delta_\pm = \{\mathcal{P}_\pm,\mathcal{O}_\pm,U\}$. Then, by applying the scheme δ_- (δ_+) in our three-dimensional $\mathcal{N}=2$ $\mathfrak{osp}(2|2)$ superconformal system we obtain a super-extension of the AFF model in the exact (spontaneously broken) supersymmetric phase, and there is a one-to-one correspondence between bosonic and fermionic generators of the three-dimensional model with those associated with the one-dimensional model.

9.4 Remarks

We end this chapter with a comment related to supersymmetry and Dirac Hamiltonian. Taking the nilpotent operators $\mathcal{Q}^\pm$ given in (9.2.25), a Hermitian supercharge can be constructed, and this has the form

$$\mathcal{Q}_0 = -\sqrt{2}(\mathcal{Q}^+ + \mathcal{Q}^-) = \gamma^i(p_i - e\mathscr{A}_i) + e\gamma^0\mathscr{A}_0\,, \tag{9.4.1}$$

where $\mathscr{A}_0 = \frac{g}{r}$, $\mathscr{A}_i = A_i - i\frac{\omega}{e}\gamma^5 r_i$, and $\gamma^5 = \Gamma$ is our grading operator in Sec. 9.2. Then the operator (9.4.1) can be viewed as a parity breaking Euclidean Dirac operator with components of the gauge potential satisfying the relations $-\partial_i\mathscr{A}_0 = \epsilon_{ijk}\partial_j\mathscr{A}_k = gr_i/r^3$. Hence we are dealing with a new type of parity breaking dyon background. Actually, the γ^5 terms do not allow for an $\mathcal{N}=4$ supersymmetric extension and we only have $\mathcal{N}=2$ supersymmetry, with the second supercharge given by $i\sqrt{2}(\mathcal{Q}^+ - \mathcal{Q}^-) = i\gamma^5\mathcal{Q}_0$. It is interesting to relate a parity-breaking Dirac operators with supersymmetric quantum mechanics. In this context it is not clear whether a (pseudo)classical supersymmetric system exists whose quantization would produce our three-dimensional superconformal system, or we have here a kind of a classical anomaly [Gamboa and Plyushchay (1998)]. Also, the fact that the ground state is infinitely degenerate is maybe due to this parity breaking term.

Conclusions and Outlook

In conclusion, we recall the problems *a) to d)* that were originally listed in the introduction, but now in the light of the obtained results. This will also allow to point out interesting problems for further research.

a) Connection between different mechanical systems through symmetries

We addressed the problem of establishing a mapping between the two forms of dynamics (in the sense of Dirac [Dirac (1949)]) associated with conformal algebra.

The indicated mapping is the conformal bridge transformation introduced in [Inzunza et al. (2020a)] (Chap. 3), that relates an asymptotically free system with an harmonically confined one. The transformation maps rank n Jordan states of the zero energy (and eigenstates) of the first system to eigenstates (coherent states) of the second. The conformal bridge also maps symmetry generators from one system to the another. From its general nature, this mapping provides a new approach to study higher dimensional (in the sense of degrees of freedom) conformal invariant systems, such as the Calogero model [Calogero (1969, 1972)]. Actually, we have already shown its applicability for the Landau problem analyzed in Chap. 3, as well as for the monopole background model in Chap. 8. A fairly natural question is whether there is any analog transformation at the level of supersymmetric quantum mechanics, in such a way we could include in this mapping fermionic integrals of motion. There could also be some relationship between this transformation and the Riemann hypothesis, since Hamiltonians of the form xp have been used in this direction [Connes (1999); Berry and Keating (1999); Regniers and Van der Jeugt (2010); Sierra and Rodriguez-Laguna (2011); Bender et al. (2017)].

b) Hidden and bosonized supersymmetry

We wanted to establish the origin of the hidden bosonized superconformal symmetry of the harmonic oscillator in one dimension [de Crombrugghe and Rittenberg (1983); Balantekin et al. (1988); Cariñena and Plyushchay (2016a); Bonezzi et al. (2017)].

It was shown that such a bosonized supersymmetry originates from a nontrivial supersymmetric system, via the nonlocal Foldy-Wouthuysen transformation [Inzunza and Plyushchay (2018)] (Chap. 4). The only fermionic true integrals of that system are the trivial Pauli matrices, and other operators are dynamical integrals, in the sense of the total Heisenberg equation. In contrast to the usual super-harmonic oscillator, the system has spontaneously broken supersymmetry. We explain the nature of this system through confluent Darboux transformation and in the scheme of free anomaly quantization for second order supersymmetry [Plyushchay (2000a); Klishevich and Plyushchay (2001); Plyushchay (2017)]. The question about what happens in higher dimensional cases remains open, however we think that the conforming bridge transformation could provide us an answer.

c) Hidden symmetries in rationally extended conformal mechanics

The objective was to find the spectrum generating ladder operators for rational deformations of the AFF model and its supersymmetric extensions.

We have used the DCKA transformation to produce a rational extension of the AFF model. The nature of the resulting Hamiltonians depends on the choice of the seed states: We can produce isospectral and non-isospectral rational deformations that have an arbitrary number of gaps of different sizes in their spectra. Starting from the harmonic oscillator [Cariñena et al. (2018)] (Chap. 5), we implemented an algorithmic procedure that takes a set of seed states for DCKA transformation (them could be physical or nonphysical, but not a mixture), and produces a new set of seed states of a different nature. Both Darboux schemes essentially generate the same system, up to an additive constant. This is what we called a Darboux duality for the harmonic oscillator, and we have used it to construct the spectrum-generating ladder operators for rational deformations of the AFF model with potential $x^2 + m(m+1)/x^2$ where $m = 0, 1, \ldots$. These ladder operators fall into three categories; Operators of the type $\mathcal{A}$ that irreducibly act on the equidistant part of the spectrum but annihilate all separate states. Operators of type $\mathcal{B}$ that act similarly to $\mathcal{A}$ on the equidistant part of the spectrum but annihilate only the upper (rising operator) and lower (lowering operator) states in each separate band. Finally, operators of type $\mathcal{C}$, that connect the separated part of the spectrum with its equidistant part. These results are analogous to what was obtained for rational extensions of the harmonic oscillator in [Cariñena and Plyushchay (2017)].

This phenomenon in which different possible options of Darboux schemes produce the same system, also appears in the context of deformations of the free particle, specifically, in the construction of the so-called reflectionless potentials, see [Matveev and Salle (1991)] for a background on the subject. The main difference between these systems and the rational deformations of the harmonic oscillator (as well as deformations of the AFF model), is that the Darboux schemes produce there the same potential without any additive constant. This implies that the Darboux dressing procedure provides there the true integrals of motion, which are the so-called Lax-Novikov integrals, see [Correa et al. (2008); Arancibia et al. (2013); Arancibia and Plyushchay (2014); Arancibia et al. (2014); Plyushchay (2020)] for more information.

The next step was to study the complete nonlinear supersymmetry that characterizes the rational super-extensions of the AFF model and the harmonic oscillator [Inzunza and Plyushchay

(2019a)] (Chap. 6). By means of a set of algebraic relations, we have obtained a large chain of new higher-order dynamical integrals that act irreducibly in the system, in a similar way as powers of the first-order ladder operators do in the case of the simplest harmonic oscillator. We stopped the generation of integrals when we realized that certain objects can be written in terms of more basic elements than they are, otherwise one would have an infinite-dimensional algebra of the W type, see [de Boer et al. (1996)] and references therein. With fermionic generators we have a similar picture. Despite having so many new operators, which we cannot avoid because they arise from the commutation relations between operators of the type $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, the role they play is not clear since the spectrum-generating set was already built. Perhaps there is a more basic structure behind this construction, hidden in the virtual systems produced by the Darboux chain, but this is still an open question. In this context, another interesting problem to investigate is whether these higher order generators can be obtained by means of a quantization prescription of a pseudo-classical system, however one must bear in mind that higher order supersymmetry presents a quantum anomaly [Klishevich and Plyushchay (2001); Plyushchay (2017)].

In [Inzunza and Plyushchay (2019b)] (Chap. 7) we extend the Darboux duality to the case of the AFF model with potential $x^2 + \nu(\nu+1)/x^2$, where $\nu \geq -1/2$. This is possible due to the Klein four-group associated to the Schrödinger equation of the model. Having the Darboux duality for this system allows us to extend the notion of the three classes of ladder operators described above, now for any possible deformation of the AFF model. We have not considered spectrum-generating algebras and supersymmetric extensions for these cases, so this remains as an open problem. Within all this, the cases in which ν is a half-integer number are really special: When this happens, the confluent Darboux transformation is involved in some of the recipes for constructing rationally extended potentials, and some rational extensions undergo significant structural changes. Such changes are reflected both in the available energy levels, such as in the number of physical states, and also in the kernels of the of spectrum-generating ladder operators, where now nonphysical states and Jordan states appear.

On the other hand, systems very similar to these, but without the harmonic term, appeared in a completely different context, through the so-called $\mathcal{PT}$ regularization [Correa and Fring (2016); Mateos Guilarte and Plyushchay (2017, 2019)]. These models are intimately related to the Korteweg-de Vries equation due to the Lax pair formalism [Matveev and Salle (1991)] and help to provide new types of solutions. It would be interesting to clarify if there is a generalization of the conformal bridge for deformed systems, that could provide us a new knowledge related to integrable models.

d) Hidden symmetries in three-dimensional conformal mechanics

For this problem, we have considered a particle with electric charge e in a Dirac monopole background, i.e., a $U(1)$ external vector potential $\boldsymbol{A}$, the curl of which gives us the spherically symmetric magnetic field produced by a monopole source with charge g, see details in [Sakurai (1994); McIntosh and Cisneros (1970); Labelle et al. (1991); Inzunza et al. (2020a)] and in the references cited there. The particle was also subjected to a central potential of the form $V(\boldsymbol{r}) = \frac{\alpha}{2m\boldsymbol{r}^2} + \frac{m\omega\boldsymbol{r}^2}{2}$. We investigated the possibility of obtaining hidden integrals of motion for this system, and we also looked for a possible supersymmetric extension of this model.

It was found that the system has hidden symmetries when $\alpha = (eg)^2$. At the classical level, they control the periodic nature of the trajectory, besides in the quantum case, these integrals reveal the nature of spectrum degeneration of the system.

To construct the hidden integrals at the classical level, we have used the fact that the projection of the particle's trajectory into the orthogonal plane to the Poincaré vector integral (the modified angular momentum of the system), is analogous to the orbit of the three-dimensional harmonic oscillator. Actually, we demonstrated that this is a universal property of this background, i.e., if we change the harmonic trap for an arbitrary central potential, the dynamics in the mentioned plane will be the same that would occur in the absence of the monopole charge.

It is also necessary to emphasize that the system has the $\mathfrak{sl}(2,\mathbb{R})$ symmetry and is connected with an $\mathfrak{so}(2,1)$ invariant system previously analyzed in [Plyushchay and Wipf (2014)], by means of the conformal bridge transformation. This brings us another way to get the integrals of the hidden symmetries.

Inspired by the so-called "Dirac oscillator" proposed in [Moshinsky and Szczepaniak (1989); Bentez et al. (1990); Quesne and Moshinsky (1990)], we introduced a special spin-orbit coupling term into the Hamiltonian of our system in the monopole background (Chap. 9), and this naturally leads us to the construction of a supersymmetric extension. The resulting model is a three-dimensional realization of the $\mathfrak{osp}(2|2)$ superconformal symmetry, and some of its interesting proprieties appear in the following list:

- In the limit $\nu \to 0$, the Hamiltonian of our system takes the form

$$\mathcal{H}_{\mathrm{DO}} = \frac{1}{2}\left(\boldsymbol{p}^2 + \omega^2\boldsymbol{r}^2\right)\mathbb{I}_{4\times4} + \omega\Gamma(\boldsymbol{\sigma}\cdot\boldsymbol{L} + \frac{3}{2})\,, \qquad \Gamma = \sigma_3\otimes\mathbb{I}_{2\times2}\,.$$

  which is identified with the mentioned Dirac oscillator Hamiltonian in the non-relativistic limit.

- In the limit $\omega \to 0$, our Hamiltonian operator is transformed into

$$\mathcal{H}_{\mathrm{dyon}} = \frac{1}{2}\left((\boldsymbol{p} - e\boldsymbol{A})^2 + \frac{\nu^2}{r^2}\right)\mathbb{I}_{4\times4} + \frac{\nu}{r^2}\,\boldsymbol{\sigma}\cdot\boldsymbol{r}\Pi_-\,, \qquad \Pi_\pm = \frac{1}{2}(1\pm\Gamma)\,,$$

  which is interpreted as the Pauli Hamiltonian of a supersymmetric dyon ($c = 1$) [Plyushchay and Wipf (2014)]. This system has the exceptional superconformal symmetry $D(2;1,\alpha)$ with $\alpha = 1/2$, which is larger than $\mathfrak{osp}(2|2)$ superalgebra, so we believe that some important structures are still missing in our construction.

- The system has two classes of energy levels organized in two independent towers. The eigenvalues associated with one of these towers are infinitely degenerate, while the energies in the other tower have finite degeneracy.

- Through the application of two different dimensional reduction schemes, the system is transformed into the super-extended AFF model. One scheme gives us the extended system in the spontaneously broken supersymmetric phase, while the other scheme produces the system in the exact supersymmetric phase.

This type of system opens an interesting line of research, which consists in exploring the supersymmetric structure of a Dirac Hamiltonian that breaks parity symmetry (since the Hermitian supercharges of our model can be interpreted in this way), and searching for applications for systems with infinitely degenerate ground energy.

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Appendix A

Wronskian identities

Here we consider the equalities between wavefunctions and Wronskians in the sense of "up to a multiplicative constant" when the corresponding constant is not essential.

A.1 Wronskian relations due to DCKA transformation

Suppose that we have two collections of (formal) eigenstates of (1.2.1), $\{\phi_n\} = (\phi_1, \ldots, \phi_n)$ and $\{\varphi_l\} = (\varphi_1, \ldots, \varphi_l)$. In the first step, we generate a Darboux transformation by taking the first collection as the set of the seed states, and obtain the intermediate Hamiltonian operator with potential $V_1 = V(x) - 2(\ln W(\{\phi_n\}))''$. In this way, the states of the second collection $\{\varphi_l\}$ will be mapped into the set of (formal in general case) eigenstates $\{\mathbb{A}_n\varphi_l\} = (\mathbb{A}_n\varphi_1, \ldots, \mathbb{A}_n\varphi_l)$. Then, employing these states as the seed states for a second Darboux transformation, we finally obtain a Schrödinger operator with a potential $V_2 = V_1(x) - 2(\ln W(\{\mathbb{A}_n\varphi_l\}))''$. Having in mind that the same result will be produced by a one-step generalized Darboux transformation based on the whole set of the chosen eigenstates of the system L, we obtain the equality

$$W(\{\phi_n\})W(\{\mathbb{A}_n\varphi_l\}) = W(\phi_1, \ldots, \phi_n, \varphi_1, \ldots, \varphi_l)\,. \tag{A.1.1}$$

Consider now the set of two states corresponding to a same eigenvalue λ_j, $\{\phi_2\} = (\phi_1 = \psi_j, \phi_2 = \widetilde{\psi}_j)$. In this case $W(\psi_j, \widetilde{\psi}_j) = 1$, and the corresponding intertwining operator reduces to $\mathbb{A}_2 = -(L - \lambda_j)$. Using this observation and Eq. (A.1.1), we derive the equality $W(\psi_j, \widetilde{\psi}_j, \varphi_1, \ldots, \varphi_l) = W(\{\varphi_l\})$, which is generalized for the relation

$$W(\psi_1, \widetilde{\psi}_1, \ldots, \psi_s, \widetilde{\psi}_s, \varphi_1, \ldots, \varphi_l) = W(\{\varphi_l\})\,. \tag{A.1.2}$$

In the case when functions $\varphi_1, \ldots, \varphi_l$ are not obligatorily to be eigenstates of the operator L, the last relation changes for

$$W(\psi_1, \widetilde{\psi}_1, \ldots, \psi_s, \widetilde{\psi}_s, \varphi_1, \ldots, \varphi_l) = W\left(\{\textstyle\prod_{k=1}^{s}(-L+\lambda_k)\varphi_l\}\right). \tag{A.1.3}$$

In the context of generalized Darboux transformations based on a mixture of eigenstates and Jordan states, a useful relation

$$W(\psi_*, \widetilde{\psi}_*, \Omega_*, \breve{\Omega}_*, \varphi_1, \ldots, \varphi_l) = W(\varphi_1, \ldots, \varphi_l) \tag{A.1.4}$$

can be obtained by employing Eq. (A.1.3) with $s = 1$, and Eqs. (1.3.2) and (A.1.2), Here we imply that φ_i with $i = 1, \ldots, l$ is the set of solutions of equation (1.2.1) with $\lambda_i \neq \lambda_*$.

A.2 Jordan states and Wronskian relations

We show here that the Wronskian (7.4.5) takes non-zero values and that it reduces to (7.4.9) in the limit $\mu \to -1/2$. For this, consider first a generic system (1.2.1) which has a set of the seed states $(\phi_1, \phi_2, \ldots, \phi_{2l-1}, \phi_{2l})$ with eigenvalues $\lambda_1 < \lambda_2 < \ldots < \lambda_{2l-1} < \lambda_{2l}$. Then the following relation

$$W(\phi_1, \phi_2, \ldots, \phi_{2l-1}, \phi_{2l}) = \prod_{i=0}^{l-1} W(\mathbb{A}_{2i}\phi_{2i+1}, \mathbb{A}_{2i}\phi_{2i+2}), \tag{A.2.1}$$

can be proved by induction, where $\mathbb{A}_0 = 1$, and $\mathbb{A}_{2i}$ with $i \geq 1$ corresponds to the intertwining operator associated with the scheme $(\phi_1, \ldots, \phi_{2i})$. From (A.2.1) it follows that if each factor $W(\mathbb{A}_{2i}\phi_{2i+1}, \mathbb{A}_{2i}\phi_{2i+2})$ does not have zeros, then the complete Wronskian neither has. To inspect the properties of the Wronskian factors, we use the relation

$$W'(\mathbb{A}_{2i}\phi_{2i+1}, \mathbb{A}_{2i}\phi_{2i+2}) = (\lambda_{2i+2} - \lambda_{2i+1})\mathbb{A}_{2i}\phi_{2i+1}\mathbb{A}_{2i}\phi_{2i+2}, \tag{A.2.2}$$

and integrate it from a to x,

$$W(\mathbb{A}_{2i}\phi_{2i+1}, \mathbb{A}_{2i}\phi_{2i+2}) = (\lambda_{2i+2} - \lambda_{2i+1}) \int_a^x \mathbb{A}_{2i}\phi_{2i+1}\mathbb{A}_{2i}\phi_{2i+2} d\zeta + \omega, \tag{A.2.3}$$

where $\omega = W(\mathbb{A}_{2i}\phi_{2i+1}, \mathbb{A}_{2i}\phi_{2i+2})|_{x=a}$. In the case when functions $\mathbb{A}_{2i}\phi_{2i+1}$, $\mathbb{A}_{2i}\phi_{2i+2}$ and their first derivatives vanish in b, we find $\omega = -(\lambda_{2i+2} - \lambda_{2i+1}) \int_a^b \mathbb{A}_{2i}\phi_{2i+1}\mathbb{A}_{2i}\phi_{2i+2} d\zeta$, and then

$$W(\mathbb{A}_{2i}\phi_{2i+1}, \mathbb{A}_{2i}\phi_{2i+2}) = -(\lambda_{2i+2} - \lambda_{2i+1}) \int_x^b \mathbb{A}_{2i}\phi_{2i+1}\mathbb{A}_{2i}\phi_{2i+2} d\zeta. \tag{A.2.4}$$

Relation (A.2.1) takes then the form

$$W(\phi_1, \phi_2, \dots, \phi_{2l-1}, \phi_{2l}) = \prod_{i=0}^{l-1} (\lambda_{2i+1} - \lambda_{2i+2}) \int_x^b \mathbb{A}_{2i}\phi_{2i+1}\mathbb{A}_{2i}\phi_{2i+2} d\zeta_i \,. \tag{A.2.5}$$

Analogously, one can consider a generic system, choose l solutions φ_i of Eq. (1.2.1), and construct l corresponding Jordan states Ω_i using Eq. (1.3.3). Assuming also that these states satisfy relations (1.3.6), one can find that

$$W(\varphi_1, \Omega_1, \dots, \varphi_l, \Omega_l) = \prod_{i=0}^{l-1} W(\mathbb{A}_{2i}^{\Omega}\varphi_{i+1}, \mathbb{A}_{2i}^{\Omega}\Omega_{i+1}) = \prod_{i=0}^{l-1} \int_x^b (\mathbb{A}_{2i}^{\Omega}\varphi_{i+1})^2 d\zeta_i \,, \qquad \mathbb{A}_0^{\Omega} = 1 \tag{A.2.6}$$

and $\mathbb{A}_{2i}^{\Omega}$ is the intertwining operator associated with the scheme $(\varphi_1, \Omega_1 \dots, \varphi_l, \Omega_l)$. Relation (A.2.6) can be proved in a way similar to that for (A.2.5).

Let us turn now to the AFF model, where $a = 0$, $b = \infty$, and choose the seed states in (A.2.1) in correspondence with our picture: for $i = 0, \dots, l-1$ we fix $\phi_{2i+1} = \psi_{-\mu-m-1,n_{i+1}}$ and $\phi_{2i+2} = \psi_{\mu+m,n_{i+1}-m}$. This identification implies that $\lambda_{2i+1} = E_{-\mu-m-1,n_{i+1}}$, $\lambda_{2i+2} = E_{\mu+m,n_{i+1}-m}$, and $\lambda_{2i+2} - \lambda_{2i+1} = 4(\mu + 1/2)$. These both functions and their first derivatives behave for large values of x as $e^{-x^2/2}$, and vanish at $x = \infty$. This behavior is not changed by application of any differential operator with which we work. On the other hand, near zero we have $\mathbb{A}_{2i}\psi_{-\mu+m+1,n_{i+1}} \sim x^{-\mu-m-i}$ and $\mathbb{A}_{2i}\psi_{\mu+m,n_{i+1}-m} \sim x^{\mu+m+1+i}$. Therefore, for small values of x, $\mathbb{A}_{2i}\psi_{-\mu+m+1,n_{i+1}}\mathbb{A}_{2i}\psi_{\mu+m,n_{i+1}-m} \sim x$, and $W(\mathbb{A}_{2i}\psi_{-\mu+m+1,n_{i+1}}, \mathbb{A}_{2i}\psi_{\mu+m,n_{i+1}-m})$ takes a finite value when $x \to 0^+$. Knowing this and Eq. (A.2.2), we employ the Adler method [Adler (1994)], and use the theorem on nodes of wave functions to show that zeros and the minima and maxima of the functions $\mathbb{A}_{2i}\psi_{-\mu+m+1,n_{i+1}}$ and $\mathbb{A}_{2i}\psi_{\mu+m,n_{i+1}-m}$ do not coincide, and that their corresponding Wronskian is non-vanishing.

In the case $\mu = -1/2$, we put $\varphi_j = \psi_{m-1/2,n_{j+1}-n}$ with $j = 0, \dots, l-1$, and then we arrive at the relations

$$\tfrac{W(\{\gamma_\mu\})}{(4\mu+2)^N} = (-1)^l \prod_{i=0}^{l-1} \int_x^\infty \mathbb{A}_{2i}\psi_{-\mu-m-1,n_{i+1}}\mathbb{A}_{2i}\psi_{\mu+m,n_{n_{i+1}}-m} d\zeta_i \,, \tag{A.2.7}$$

$$W(\{\gamma\}) = \prod_{j=0}^{l-1} \int_x^\infty (\mathbb{A}_{2j}^{\Omega}\psi_{m-1/2,n_{j+1}-m})^2 d\zeta_j \,, \tag{A.2.8}$$

where the sets $\{\gamma_\mu\}$ and $\{\gamma\}$ are defined in (7.4.5) and (7.4.9). We note that both equations are pretty similar each other, and if we suppose that $\mathbb{A}_{2i} \to \mathbb{A}_{2i}^{\Omega}$ when $\mu \to -1/2$, and take into account the relation $\psi_{m-1/2,n_j-m} \propto \psi_{-(m-1/2)-1,n_j}$, one proves by induction that

$$\lim_{\mu \to -1/2} \frac{W(\{\gamma_\mu\})}{(4\mu+2)^N} \propto W(\{\gamma\}) \,. \tag{A.2.9}$$

Appendix B

The mirror diagram

In this paragraph we prove the relations involved with mirror diagrams and Darboux duality using the Wronskian identities of Appendix A

B.1 Harmonic oscillator case

To start, we consider a positive scheme $\{\Delta_+\} = (l_1^+, \ldots, l_{n_+}^+)$, where l_i^+ with $i = 1, \ldots, n_+$ are certain positive numbers ordered from low to high, and we supose that $l_1^+ \neq 0$. By using the Wronskian identity (A.1.2) we get the relation

$$W(\{\Delta_+\}) = W(0, \widetilde{0}, \{\Delta_+\}) = e^{-x^2/2} W(-1, \{a^- \Delta_+\}) \,, \tag{B.1.1}$$

where in the last step we have used the identity (A.1.1)[1], and $\{a^- \Delta_+\}$ means that a^- acts in each state in the scheme. Let us repeat the trick a second time, obtaining $W(\{\Delta_+\}) = e^{-x^2} W(-2, -1, \{(a^-)^2 \Delta_+\})$. After l_1^+ times we get

$$\begin{aligned} W(\{\alpha\}) &= e^{-\frac{l_1^+}{2} x^2} W(-l_1^+, \ldots, -1, 0, (l_2^+ - l_1^+), \ldots, (l_{n_+}^+ - l_1^+)) \\ &= e^{-\frac{l_1^+ + 1}{2} x^2} W(\underbrace{-(l_1^+ + 1), \ldots, -2}_{\text{negative states}}, \underbrace{(l_2^+ - l_1^+ - 1), \ldots, (l_{n_+}^+ - l_1^+ - 1)}_{\text{positive states}}) \,, \end{aligned}$$

where we have used the identity (A.1.1) with the ground state denoted by zero. So now, we have to answer the question: Is $l_2^+ - l_1^+ - 1$ equal to 0?. If the answer is negative, then we continue with the trick described in (B.1.1) another $l_2^+ - l_1^+ - 1$ times. On the other hand, if the answer is affirmative, we use again the identity (A.1.1) in order to do not have a ground state in the Wronskian of the right hand side. This step is the responsible of the "missing" states in the negative scheme constructed

[1] For us, the Wronskian of a single function is the function itself, and $a^- \widetilde{\psi_0} = \psi_0(ix)$, which in our notations is -1.

in this way. We repeat the algorithm until positive eigenstates disappear in the right hand side, obtaining the relation

$$W(\{\Delta_+\}) = e^{(l^+_{n_+}+1)x^2/2} W(\Delta_-)\,, \qquad \Delta_- = (-\check{0}, \ldots, -\check{n}^-_i, \ldots, -l^+_{n_+})\,, \tag{B.1.2}$$

If one would like to start with the negative scheme, the algorithm is the same but instead of 0 and $\widetilde{0}$, is necessary to use the nonphysical states -0 and $-\widetilde{0}$ in equation (B.1.1).

B.2 The case of AFF model with $\nu \neq \ell - 1/2$

To show the mirror diagram for this case, we follow the same spirit of last subsection, but in this case we have to use second order ladder operators. As a starting point, consider the Wronskian of the set $\{\alpha\}$ defined in (7.3.1). If the states $\psi_{\nu,\pm 0}$ and $\psi_{-\nu-1,\pm 0}$ do not belong to (7.3.1), we can replace the Wronskian $W(\{\alpha\})$ by

$$W(\psi_{\nu,\pm 0}, \psi_{-\nu-1,\pm 0}, \widetilde{\psi}_{\nu,\pm 0}, \widetilde{\psi}_{-\nu-1,\pm 0}, \{\alpha\}) = e^{\mp x^2} W(\psi_{\nu,\mp 0}, \psi_{-\nu-1,\mp 0}, \{\mathcal{C}^{\mp}_{\nu}\alpha\})\,, \tag{B.2.1}$$

where we used relations (A.1.1), (A.1.2), (7.1.11) and (7.1.12), and $\{\mathcal{C}^{\mp}_{\nu}\alpha\}$ means that the ladder operators are applied to all the states in the set. On the other hand, if $\psi_{r(\nu),\pm 0}$ belong to (7.3.1), we can replace the Wronskian of the initial set of the seed states by

$$W(\psi_{r(-\nu-1),\pm 0}, \widetilde{\psi}_{r(-\nu-1),\pm 0}, \{\alpha\}) = e^{\mp x^2} W(\psi_{r(-\nu-1),\mp 0}, \{\mathcal{C}^{\mp}_{\nu}\beta_1\})\,, \tag{B.2.2}$$

where $\{\beta_1\}$ is the scheme $\{\alpha\}$ with the omitted state $\psi_{r(\nu),\pm 0}$. Finally, if $\psi_{\nu,\pm 0}$ and $\psi_{-\nu-1,\pm 0}$ belong to (7.3.1), we have

$$W(\{\alpha\}) = e^{\mp x^2} W(\{\mathcal{C}^{\mp}_{\nu}\beta_2\})\,, \tag{B.2.3}$$

where $\{\beta_2\}$ is the scheme $\{\alpha\}$ with the omitted states $\psi_{\nu,\pm 0}$ and $\psi_{-\nu-1,\pm 0}$. Note that in all these three relations we have lowered or raised the index of the states in $\{\alpha\}$, and also in the case of Eqs. (B.2.1) and (B.2.2) we have included additional states which do not belong to the initial set. Also, we note that an exponential factor has appeared. These identities can be applied to the Wronskians on the right hand side of Eqs. (B.2.1)-(B.2.3), which will contribute with new exponential factors in new Wronskians, and so on. For this reason, if we restrict the initial set $\{\alpha\}$ by the conditions described above (that every state in the set has the second index of the same sign), and we repeat this procedure n_N+1 times with positive (negative) sign of the indexes in (B.2.1)-(B.2.3), we finally obtain equation (7.3.3) or (7.3.4).

B.3 The case of AFF model with $\nu = \ell - 1/2$

To obtain the dual schemes in the half-integer case, we analyze first the relations that exist between of $\mathcal{H}_{-1/2}$ and $\mathcal{H}_{-1/2+\ell}$. The latter are given by the dual schemes $(\psi_{-1/2,\pm 0}, \ldots, \psi_{-1/2,\pm(\ell-1)})$, whose Wronskians are

$$W(\psi_{-1/2,\pm 0}, \ldots, \psi_{-1/2,\pm(\ell-1)}) = x^{\ell^2/2} e^{\mp \ell x^2/2} . \tag{B.3.1}$$

The corresponding intertwiners map eigen- and Jordan states of $\mathcal{H}_{-1/2}$ to those of $\mathcal{H}_{-1/2+\ell}$. If we choose the scheme with positive indexes, some of these mappings useful for the following are given by

$$\mathbb{A}_\ell^- \psi_{-1/2,n} = \psi_{-1/2+\ell,n-\ell} , \qquad \mathbb{A}_\ell^- \Omega_{\nu,-1/2} = \Omega_{-1/2+\ell,n-\ell} , \qquad n \geq \ell , \tag{B.3.2}$$

$$\mathbb{A}_\ell^- \Omega_{-1/2,l} = \psi_{-(-1/2+\ell)-1,l} , \qquad < \ell , \tag{B.3.3}$$

where $\mathbb{A}_\ell^-$ and its Hermitian conjugate $\mathbb{A}_\ell^+$ are the intertwining operators of the chosen Darboux transformation. On the other hand if we take the scheme with negative sign in indices, we obtain another intertwining operators $\mathbb{B}_\ell^\pm$, which satisfy the relation $\mathbb{B}_\ell^\pm = (i)^\ell \rho_2(\mathbb{A}_\ell^\pm)$, i.e, their action on eigenstates and Jordan states can be obtained by application of ρ_2 to the relations that correspond to the action of $\mathbb{A}_m^\pm$.

Now, to derive the dual schemes let us assume that we have a collection of non-repeated seed states of the form $(\psi_{-1/2,0}, \ldots, \psi_{-1/2,\ell-1}, \{\vartheta_{-1/2}\})$, where $\{\vartheta_{-1/2}\}$ contains N_1 arbitrary physical states $\psi_{-1/2,k_i}$ with $k_i > \ell - 1$ for $i = 1, \ldots, N_1$, and N_2 arbitrary Jordan states of the form $\Omega_{-1/2,l_j}$ with $j = 1, \ldots, N_1$. In the same way as we did in Sec. 7.3, we define n_N as the largest of the numbers n_{N_1} and n_{N_2}, and also we suppose for simplicity that the signs of both k_i and k_j are positive. Then we use (A.1.1) and (B.3.1) to write $W(\psi_{-1/2,0}, \ldots, \psi_{1/2,\ell-1}, \{\vartheta_{-1/2}\}) = x^{\ell^2/2} e^{-\ell x^2/2} W(\{\mathbb{A}_\ell^- \vartheta_{-1/2}\})$. The next step is to use the extension of the dual schemes for $\nu = -1/2$, i.e, we change each function of the form $\psi_{-\nu-1,n}$ by $\Omega_{-1/2,n}$ in equation (7.3.3), and use it to rewrite this last Wronskian relation as

$$W(\mathbb{A}_\ell^- \{\vartheta_{-1/2}\}) = x^{-\ell^2/2} e^{-(n_N+1-\ell/2)x^2} W(\{\Delta_-^{(-1/2)}\}) , \tag{B.3.4}$$

where $\Delta_-^{(-1/2)}$ is the dual scheme of $(\psi_{-1/2,0}, \ldots, \psi_{-1/2,\ell-1}, \{\vartheta_{-1/2}\})$ given by

$$\{\Delta_-^{(-1/2)}\} = (\psi_{-1/2,-0}, \ldots, \psi_{-1/2,-(\ell-1)}, \{\vartheta_{-1/2}^-\}) , \tag{B.3.5}$$

and

$$\{\vartheta^-_{-1/2}\} = (\psi_{-1/2,-\ell}, \Omega_{-1/2,-0}, \ldots, \check{\psi}_{-1/2,-s_j}, \check{\Omega}_{-1/2,-r_i}, \ldots, \psi_{-1/2,-n_N}, \Omega_{-1/2,-n_N}) . \qquad \text{(B.3.6)}$$

Here, as well as in the non-half-integer case, the marked functions $\check{\psi}_{-1/2,-s_j}$ and $\check{\Omega}_{-1/2,-r_i}$ indicate the omitted states with $s_j = n_N - l_j$ and $r_i = n_N - k_i$. In the last step, we use Eqs. (A.1.1) and (B.3.1) with the negative sign to write the equality $W(\{\Delta_-^{(-1/2)}\}) = x^{\ell^2/2} e^{\ell x^2/2} W(\mathbb{B}_\ell^- \{\vartheta^-_{-1/2}\})$ and as analog of (B.3.4) we obtain

$$W(\mathbb{A}_\ell^- \{\vartheta_{-1/2}\}) = e^{-(n'_N+1)x^2} W(\mathbb{B}_\ell^- \{\vartheta^-_{-1/2}\}) , \qquad n'_N = n_N - \ell . \qquad \text{(B.3.7)}$$

This relation is the dual scheme equation for the case $\nu = \ell - 1/2$. By means of (B.3.2) and its analogs for $\mathbb{B}_\ell^-$ obtained by the application of ρ_2, we conclude that in the scheme of the left hand side of the equation there are N_1 physical states of the form $\mathbb{A}_\ell^- \psi_{-1/2,k_i} = \psi_{\ell-1/2,k_i-\ell}$, and a mixture of N_2 Jordan states and formal states produced by ρ_2 distributed in the following way: we have Jordan states $\mathbb{A}_\ell^- \Omega_{-1/2,l_i} = \psi_{\ell-1/2,l_i}$ when $l_i < \ell - 1$, and formal states $\mathbb{A}_\ell^- \Omega_{-1/2,l_i} = \psi_{\ell-1/2,l_i-\ell}$ when $l_i \geq \ell$. The omitted states in the scheme on the right hand side are $\mathbb{B}_\ell^- \check{\psi}_{-1/2,-s_j} = \check{\psi}_{-1/2+\ell,-(s_j-\ell)}$ and $\mathbb{B}_\ell^- \check{\Omega}_{-1/2,-r_j} = \check{\psi}_{-\ell-1/2,-r_j}$ ($\mathbb{B}_\ell^- \check{\Omega}_{-1/2,-r_j} = \check{\psi}_{-\ell-1/2,-(r_j-\ell)}$) when $r_j \leq \ell - 1$ ($r_j > \ell$). Note that the largest index in both sides of the equation is now given by $n'_N = n_N - \ell$. In comparison with the non-half-integer case, this is the same result that we would obtain if we consider equation (7.3.3) in the non-half-integer case, and then formally change the states of the form $\psi_{-\nu-1,l_i}$ by $\Omega_{-\ell-1/2,l_i-\ell}$ when $l_i \geq \ell$ in the limit $\nu \to \ell - 1/2$.

Relation analogous to (7.3.4) would be obtained if we start from the case $\nu = -1/2$ with a scheme composed from the eigenstates and Jordan states produced by ρ_2, and then apply the same arguments employed for the case analyzed above.

Appendix C

Details for rationally extended systems

Here we show some operator identities, as well as the explicit form of some polynomials in nonlinear algebras considered in the Chap. 6.

C.1 Operator identities (6.1.8)

We have equalities $\ker\,(A^{+}_{(-)}A^{-}_{(+)}) = \Delta_{+}\cup\tilde{\delta} = \{0,1,\ldots,n\}$, where $\tilde{\delta} = \{A^{+}_{(-)}A^{-}_{(+)}\widetilde{\psi_{-l_1}},\ldots,$ $A^{+}_{(-)}A^{-}_{(+)}\widetilde{\psi_{-l_{n_-}}}\}$, and relation $\mathbb{A}^{+}_{(+)}\mathbb{A}^{-}_{(-)}\varphi_n \propto (a^+)^N\varphi_n$ following from (6.1.3) is used. The first identity from (6.1.8) follows then from equality $\ker A^{+}_{(-)}A^{-}_{(+)} = \ker(a^-)^N$ [Cariñena and Plyushchay (2017)].

In the second relation in (6.1.8), functions $f(\eta)$ and $h(\eta)$ are the polynomials

$$f(\eta)\equiv\prod_{l_k^- - n_+<0}(\eta+2l_k^-+1),\qquad h(\eta)\equiv\prod_{\check{n}_k^- - n_+\geq 0}(\eta+2\check{n}_k^-+1)\,, \tag{C.1.1}$$

where $l_k^-\in\Delta_-$ and $\check{n}_k^-$ are the absent states in Δ_-. Using the mirror diagram technique [Cariñena et al. (2018)], we obtain the equality $\ker f(L_{(-)})A^{-}_{(+)}(a^+)^{n_-} = \ker h(L_{(-)})A^{-}_{(-)}(a^-)^{n_+}$, where

$$\begin{aligned}\ker f(L_{(-)})A^{-}_{(+)}(a^+)^{n_-} = \quad & \mathrm{span}\{0,\ldots,(n_+-1),-0,\ldots,-(n_--1),\\ & \{(\underbrace{\check{n}_i^+-n_+})\},\{-(\underbrace{\check{n}_j^--n_-})\}\ \}\,.\end{aligned} \tag{C.1.2}$$

Indexes i and j are running here over the absent states of both schemes, provided the conditions $\check{n}_i^+ - n_+ \geq 0$ and $\check{n}_j^- - n_- \geq 0$ are met. A special case corresponds to the positive scheme $\Delta_+ = (l_1^+, l_1^+ + 1,\ldots,l_1^+ + q)$, for which the dual negative scheme is $\Delta_- = (-(q+1),\ldots,-(q+l_1^+))$. Here $n_+ = 1+q$ and $n_- = l_1^+$, there are no states to construct polynomials (C.1.1), and we just put

$f(\eta) = h(\eta) = 1$. Analogously, there are no tilted eigenstates in (C.1.2) in this case. In particular, if n_+ is an even number, then the DCKA transformation will produce a deformed harmonic oscillator with one-gap of size $2(l_1^- + q + 1) = 2N$ in its spectrum, while if q is an odd number and $l_1^+ = 1$, then we generate a gapless deformation of L_1^{iso} (by introducing the potential barrier at $x = 0$).

C.2 Relations between symmetry generators

We first show explicitly how the three families appear by considering the commutators

$$
\begin{aligned}
&[\mathfrak{C}_{N+l}^-, \mathfrak{A}_k^-] = P_{n_-}(\eta)|_{\eta=L_{(+)}}^{\eta=L_{(+)}+2k}\mathfrak{C}_{N+k+l}^- \,, \qquad [\mathfrak{C}_{N+l}^+, \mathfrak{B}_k^+] = P_{n_-}(\eta)|_{\eta=L_{(+)}}^{\eta=L_{(-)}-2l}\mathfrak{C}_{N+k+l}^+ \,, \\
&[\mathfrak{A}_k^+, \mathfrak{C}_{N+l}^-] = (-1)^{n_-}T_k(L_{(-)})\mathfrak{A}_{N+l-k}^- - P_{n_-}(L_{(+)})T_l(L_{(+)}+2l)\mathfrak{C}_{N+l-k}^- \,, \\
&[\mathfrak{B}_k^-, \mathfrak{C}_{N+l}^+] = (-1)^{n_-}T_k(L_{(+)}+2k)\mathfrak{B}_{N+l-k}^+ - T_l(L_{(-)})P_{n_+}(L_{(-)}-2l)\mathfrak{C}_{N+l-k}^+ \,, \\
&[\mathfrak{C}_{N+k}^+, \mathfrak{C}_N^-] = P_{n_+}(L_{(-)}-2k)\mathfrak{A}_k^- - P_{n_-}(L_{(-)}+2N)\mathfrak{B}_k^- \,, \\
&[\mathfrak{C}_{N\pm k}^\pm, \mathfrak{C}_{N\pm l}^\pm] = 0 \,, \qquad \geq 0 \,,
\end{aligned}
\tag{C.2.1}
$$

where polynomials $P_{n_\pm}(\eta)$ and $T_k(\eta)$ are defined by Eqs. (6.1.5) and (6.1.7). These commutators should be interpreted as recursive relations which generate the elements of the three families of the ladder operators proceeding from the spectrum-generating set of operators with $l = r(N, c)$ and $k = c$. On the other hand, the commutators of the ladder operators with their own conjugate counterparts are

$$
\begin{aligned}
[\mathfrak{A}_k^-, \mathfrak{A}_k^+] &= P_{n_-}(\eta - 2k)P_{n_-}(\eta)T_k(\eta)|_{\eta=L_{(-)}}^{\eta=L_{(-)}+2k} \,, \\
[\mathfrak{B}_k^-, \mathfrak{B}_k^+] &= P_{n_+}(\eta - 2k)P_{n_+}(\eta)T_k(\eta)|_{\eta=L_{(+)}}^{\eta=L_{(+)}+2k} \,, \\
[\mathfrak{C}_{N\pm k}^-, \mathfrak{C}_{N\pm k}^+] &= P_{n_-}(\eta)P_{n_+}(x - 2k)T_k(\eta)|_{\eta=L_{(-)}}^{\eta=L_{(+)}\pm 2k} \,.
\end{aligned}
\tag{C.2.2}
$$

In this way, we obtain a deformation of $sl(2,\mathbb{R})$ in (6.2.3).

Below we present some relations between lowering ladder operators, from which analogous relations for raising operators can be obtained via Hermitian conjugation.

The definitions of the three families automatically provide the following relations:

$$
\mathfrak{A}_{N+k}^- = (-1)^{n_-}P_{n_-}(L_{(-)})\mathfrak{C}_{N+k}^- \,, \qquad \mathfrak{B}_{N+k}^- = (-1)^{n_-}\mathfrak{C}_{N+k}^- P_{n_+}(L_{(+)}), \tag{C.2.3}
$$

$$
\mathfrak{C}_{N-(N+k)}^- \equiv \mathfrak{C}_{-k}^- = (-1)^{n_-}P_{n_+}(L_{(-)}+2N)\mathfrak{A}_k^+ \,, \tag{C.2.4}
$$

$$
\mathfrak{C}_{2N+l+k}^- = (-1)^{n_-}\mathfrak{C}_{N+l}^-\mathfrak{C}_{N+k}^- \,, \tag{C.2.5}
$$

$$
(\mathfrak{C}_{N+k}^-)^2 = (-1)^{n_-}\mathfrak{C}_{2N+2k}^- \,, \tag{C.2.6}
$$

where $k, l = 0, 1, \ldots$. Eq. (C.2.3) means that operators of families $\mathfrak{A}$ and $\mathfrak{B}$ with index $k \geq N$ are essentially the operators of the $\mathfrak{C}$ family. Eq. (C.2.4) shows that operators of the form $\mathfrak{C}_{-k}^\pm$ are not basic. If in (C.2.5) one fixes $l = r(N, c)$, then all the operators with index equal or greater than $N+r(N,c)$ reduce to the products of the basic elements. Finally, Eq. (C.2.6) means that the square of an operator of $\mathfrak{C}$-family with odd index $N+k$ is a physical operator, but not basic. The unique special case is when $c=2$, N is odd, and $k=0$ since there is no product of physical operators of lower order which could make the same job. From here we conclude that the basic operators are given by (6.2.1).

For one-gap systems we can use the second equation in (6.1.8) (where $f(\eta)=h(\eta)=1$) to find some relations between operators with indexes less than N:

$$
\begin{aligned}
&\mathfrak{C}^-_{n_+-k}=(-1)^{n_-}\mathfrak{A}^-_{n_+-k}T_k(L_{(-)})\,, &&\mathfrak{C}^-_{n_--k'}=(-1)^{n_-}\mathfrak{B}^-_{n_--k'}T_{k'}(L_{(-)}+2(n_+-k'))\,,\\
&\mathfrak{A}^-_{n_++k'}=(-1)^{n_-}\mathfrak{C}^-_{n_++k'}T_{k'}(L_{(-)})\,, &&\mathfrak{B}^-_{n_-+k}=(-1)^{n_-}\mathfrak{C}^-_{n_-+k}T_k(L_{(-)}+2n_+)\,,
\end{aligned}
\tag{C.2.7}
$$

where $k=0,\ldots,n_+$ and $k'=0,\ldots,n_-$. By considering the ordering relation between n_- and n_+, we can combine relations (C.2.7) to represent operators of the $\mathfrak{A}$ family in terms of $\mathfrak{B}$ family or vice-versa. For the case $n_-<n_+$ we have

$$
\mathfrak{B}^-_{n_+-k}=T_{(n_+-n_--k)}(L_{(-)}+4n_+-2k)T_k(L_{(-)}+2n_+)\mathfrak{A}^-_{n_+-k}\,,
\tag{C.2.8}
$$

where $k=0,\ldots,n_+-n_-$. In other words, only first n_--1 operators are basic. In the case $n_-=1$, there exist no basic elements in the $\mathfrak{B}$-family. As examples corresponding to this observation we have all the deformations produced by a unique nonphysical state of the form $\psi_{-n}(x)$. On the other hand, in the case $n_+<n_-$ we have

$$
\mathfrak{A}^-_{n_--k}=T_k(L_{(-)}+2N)T_{(n_--n_+-k)}(L_{(-)}+2(n_--k))\mathfrak{B}^-_{n_--k}\,,
\tag{C.2.9}
$$

where $k=0,\ldots,n_--n_+$. According to this, only first n_+-1 elements cannot be written in terms of the operators of $\mathfrak{B}$ family. The unique case in which there exist no basic elements of the families $\mathfrak{A}$ or $\mathfrak{B}$ is when $n_-=n_+=1$, which corresponds to the shape invariance of the harmonic oscillator. As a final result, the basic elements of the three families are given by (6.2.5).

We consider now the relations between Darboux generators $A^-_{(\pm)}(a^\pm)^n$ and $A^-_{(\pm)}(a^\mp)^n$. Using the first relation in (6.1.8) and the definition of operators $\mathfrak{C}^\pm_{N+l}$, we obtain relations

$$
\begin{aligned}
A^-_{(-)}(a^-)^{N+l}&=(-1)^{n_-}P_{n_-}(L_{(-)})A^-_{(+)}(a^-)^l\,, &A^-_{(+)}(a^+)^{N+l}&=(-1)^{n_-}P_{n_+}(L_{(+)})A^-_{(-)}(a^+)^l\,,\\
A^-_{(+)}(a^-)^{N+l+k}&=(-1)^{n_-}\mathfrak{C}^-_{N+l}A^-_{(+)}(a^-)^k\,, &A^-_{(-)}(a^+)^{N+l+k}&=(-1)^{n_-}\mathfrak{C}^+_{N+l}A^-_{(-)}(a^+)^k\,,
\end{aligned}
$$

where $k,l=0,1,2,\ldots$. If we fix $l=r(N,c)$, then one finds that the basic elements are just (6.2.6).

On the other hand for one-gap systems, with the help of (6.1.8) one can obtain relations

$$A^{-}_{(-)}(a^{-})^{n_{+}+k} = (-1)^{n_{-}} A_{(+)}(a^{+})^{n_{-}-k} T_{k}(L)\,, \tag{C.2.10}$$

$$A^{-}_{(+)}(a^{+})^{n_{-}+k'} = (-1)^{n_{-}} A^{-}_{(-)}(a^{-})^{n_{+}-k'} T_{k'}(L+2k')\,, \tag{C.2.11}$$

with $k = 0, \ldots, n_{-}$ and $k' = 0, \ldots, n_{+}$. These relations reduce the basic subsets of Darboux generators to (6.2.7).

C.3 (Anti)-Commutation relations for one-gap systems

In this Appendix we summarize some (anti)commutation relations for one-gap deformations of harmonic oscillator systems.

For the anticommutator of two fermionic operators in (6.3.2) we have

$$\mathbb{P}_{z} = \begin{cases} P_{n_{-}}(\eta) T_{|z|}(\eta)\big|_{\eta=\mathcal{H}+2|z|\Pi_{-}+\lambda_{-}} & -N < z \leq 0 \\ P_{n_{-}}(\eta) T_{z}(\eta+2z)\big|_{\eta=\mathcal{H}-2|z|\Pi_{-}+\lambda_{-}} & 0 < z \leq n_{+} \end{cases}, \tag{C.3.1}$$

and for the positive scheme

$$\mathbb{P}'_{z} = \begin{cases} P_{n_{+}}(\eta) T_{|z|}(\eta-2z)\big|_{\eta=\mathcal{H}'-2|z|\Pi_{-}+\lambda_{+}} & -N < z \leq 0 \\ P_{n_{+}}(\eta) T_{z}(\eta)\big|_{\eta=\mathcal{H}'+2|z|\Pi_{-}+\lambda_{+}} & 0 < z \leq n_{-} \end{cases}. \tag{C.3.2}$$

By virtue of the relation between dual schemes, the expression $\mathbb{P}'_{z}(\mathcal{H}',\sigma_3) = \mathbb{P}_{N-z}(\mathcal{H}'+N(1+\sigma_3)-\lambda_{-}+\lambda_{+},\sigma_3)$ helps to complete the set of polynomials.

For the negative scheme we also have

$$[\mathcal{G}^{(2\theta(z)-1)}_{-k}, \mathcal{Q}^{0}_{a}] = \frac{1}{2} P_{n_{-}}(\eta)\big|^{\eta=\mathcal{H}+\lambda_{-}+2|z|}_{\eta=\mathcal{H}+\lambda_{-}} (\mathcal{Q}^{z}_{a} - (2\theta(z)-1) i\epsilon_{ab}\mathcal{Q}^{z}_{b})\,, \tag{C.3.3}$$

where $z \in (-N,0)\cup(0,2N)$, while for the positive scheme, where $\mathcal{Q}'^{z}_{a} = \mathcal{Q}^{N-z}_{a}$ and $\mathcal{G}'^{(1)}_{\pm k} = \mathcal{G}^{(1)}_{\pm k}$ when $k \geq N$, we have

$$[\mathcal{G}'^{(2\theta(z)-1)}_{-z}, \mathcal{Q}'^{0}_{a}] = \frac{1}{2} P_{n_{+}}(x)\big|^{x=\mathcal{H}'+\lambda_{+}+2|z|}_{x=\mathcal{H}'+\lambda_{+}} (\mathcal{Q}'^{z}_{a} + (2\theta(z)-1) i\epsilon_{ab}\mathcal{Q}'^{z}_{b})\,, \tag{C.3.4}$$

where $z \in (-N,0)\cup(0,2N)$. On the other hand, for the negative scheme the relation $[\mathcal{G}^{(1)}_{z}, \mathcal{Q}^{z}_{a}] = \frac{\mathcal{V}_{z}(\mathcal{H})}{2}(\mathcal{Q}^{0}_{a} + i(2\theta(z)-1)\epsilon_{ab}\mathcal{Q}^{0}_{b})$ is valid, where

$$
\mathcal{V}_z = \begin{cases} P_{n_-}(\eta)T_z(\eta)\big|_{\eta=\mathcal{H}+\lambda_-}^{\eta=\mathcal{H}+\lambda_- -2z}\,, & -N<z<0\,,\\[2mm] P_{n_-}(\eta)T_z(\eta+2z)\big|_{\eta=\mathcal{H}+\lambda_- -2z}^{\eta=\mathcal{H}+\lambda_-}\,, & 0<z\le n_+\,,\\[2mm] P_{n_+}(\eta)T_{N-z}(\eta)\big|_{\eta=\mathcal{H}+\lambda_- +2(N-z)}^{\eta=\mathcal{H}+\lambda_- +2N}\,, & n_+<z\le N\,,\\[2mm] P_{n_+}(\eta)T_z(\eta+2z)\big|_{\eta=\mathcal{H}+\lambda_- -2z}^{\eta=\mathcal{H}+\lambda_- +2N}\,, & N<z<2N\,. \end{cases} \tag{C.3.5}
$$

In the positive scheme we have $[\mathcal{G}_z^{'(1)},\mathcal{Q}_a'^{z}]=\frac{\mathcal{V}_z'(\mathcal{H}')}{2}(\mathcal{Q}_a'^{0}-i(2\theta(z)-1)\epsilon_{ab}\mathcal{Q}_b'^{0})$, where $\mathcal{V}_z'(\mathcal{H}')$ are given by

$$
\mathcal{V}_z' = \begin{cases} P_{n_+}(\eta)T_z(\eta+2z)\big|_{\eta=\mathcal{H}'+\lambda_+ -2z}^{\eta=\mathcal{H}'+\lambda_+}\,, & -N<z<0\,,\\[2mm] P_{n_+}(\eta)T_z(\eta)\big|_{\eta=\mathcal{H}'+\lambda_+}^{\eta=\mathcal{H}'+\lambda_+}\,, & 0<z\le n_-\,,\\[2mm] P_{n_-}(\eta-2N)T_{N-z}(\eta-2z)\big|_{\eta=\mathcal{H}'+\lambda_+}^{\eta=\mathcal{H}'+\lambda_+ +2z}\,, & n_-<z\le N\,,\\[2mm] P_{n_-}(\eta)T_z(\eta)\big|_{\eta=\mathcal{H}'+\lambda_+ -2N}^{\eta=\mathcal{H}'+\lambda_+ +2z}\,, & N<z<2N\,. \end{cases} \tag{C.3.6}
$$

These are the missing relations which prove that the subsets $\mathcal{U}_{0,z}^{(2\theta-1)}$ defined in (6.3.10), satisfy closed superalgebras independently of choosing the scheme. On the other hand, we can use them to prove that the subsets $\mathcal{I}_{N,z}^{(2\theta-1)}$ given in (8.1.16) also produce closed superalgebras. Other useful relations are

$$
[\mathcal{G}_{-(N+l)}^{(1)},\mathcal{Q}_a^{N+k}]=\frac{1}{2}T_k(\eta)P_{n_+}(\eta+2k)\big|_{\eta=\mathcal{H}+\lambda_-}^{\eta=\mathcal{H}+\lambda_- +2(N-k)}(\mathcal{Q}_a^{l-k}+i\epsilon_{ab}\mathcal{Q}_b^{l-k})\,, \tag{C.3.7}
$$

where $l>k$ and $l-k\le n_+$. For $l<k$ we have

$$
[\mathcal{G}_{-(N+l)}^{(1)},\mathcal{Q}_a^{N+k}]=P_{n_-}(\mathcal{H}+\lambda_-)(\mathcal{Q}_a^{k-l}+i\epsilon_{ab}\mathcal{Q}_b^{k-l})\,, \tag{C.3.8}
$$

and also we can write $[\mathcal{G}_{\pm(N\pm k)}^{(1)},\mathcal{G}_{\pm(N\pm l)}^{(1)}]=0$ for any values of k and l.

C.4 List of polynomial functions for Sec. 6.5

<u>Eq. (6.5.7)</u>: $P_{\alpha,\beta}(\mathcal{H})=-P_{-\alpha,-\beta}(\mathcal{H}-2(\alpha+\beta))$, and

$P_{-1,1}=\mathcal{H}(6\mathcal{H}-20)-8\Pi_-(\mathcal{H}-3)\,,$	$P_{-1,-2}=-2\mathcal{H}+12(1-\Pi_-)\,,$	$P_{-1,+2}=10(\mathcal{H}-4)-12\Pi_-\,,$
$M_{-1,+2}=12\,,$	$P_{-1,-3}=P_{-2,-3}=-6\,,$	$P_{-1,+3}=-12\,,$
$M_{-1,+3}=24\,,$	$P_{-1,-4}=P_{-2,-4}=-8\,,$	$P_{-1,+4}=16(\mathcal{H}-5-\Pi_-)\,,$
$P_{-1,-5}=P_{-2,-5}=-10\,,$	$P_{-1,+5}=20(\mathcal{H}-6-\Pi_-)\,,$	

$$P_{-2,+2} = (\mathcal{H}-4)[8\mathcal{H}(2\mathcal{H}-7) + \Pi_-(4\mathcal{H}^2 - 44\mathcal{H} + 192)]\,, \qquad P_{-2,+3} = -18\mathcal{H} + 96 - 4(\mathcal{H}-30)\Pi_-\,,$$
$$M_{-2,+3} = M_{-2,+4} = -96\,, \qquad P_{-2,+4} = -2(11\mathcal{H} - \Pi_-) + 136\,,$$
$$P_{-2,+5} = 1104 - 340\mathcal{H} + 26\mathcal{H}^2 + \Pi_-(576 - 104\mathcal{H} + 10\mathcal{H}^2)\,, \qquad P_{-3,+4} = 24\mathcal{H} - 144 + \Pi_-(2\mathcal{H} - 76)\,,$$
$$M_{-3,+4} = 848\,, \qquad P_{-3,+5} = 30\mathcal{H} - 180 + \Pi_-(8\mathcal{H} - 180)\,,$$
$$M_{-3,+5} = 960\,,$$
$$P_{-4,+5} = 40(32 - 10\mathcal{H} + \mathcal{H}^2 - \Pi_-(7\mathcal{H} - 32))\,, \qquad M_{-4,+5} = -5760\,,$$

$$P_{-4,+4} = 4[7\mathcal{H}^3 - 56\mathcal{H}^2 + 116\mathcal{H} + 32 + \Pi_-(\mathcal{H}^3 - 64\mathcal{H}^2 + 572\mathcal{H} + 1472)]\,,$$
$$P_{-5,+5} = 2(320 - 2848\mathcal{H} + 1268\mathcal{H}^2 - 248\mathcal{H}^3 + 23\mathcal{H}^4) + 8\Pi_-(10000 - 6212\mathcal{H} + 1492\mathcal{H}^2 - 187\mathcal{H}^3 + 12\mathcal{H}^4)\,.$$

Eq. (6.5.8): $F_{\alpha,\beta}(\mathcal{H}) = -F_{-\alpha,-\beta}(\mathcal{H} - 2(\alpha+\beta))$, and

$$F_{+1,-1} = F_{-1,+3} = 0\,, \qquad N_{+1,-1} = -F_{-1,-2} = -N_{-2,+1} = 2\,, \qquad F_{-2,-1} = F_{-2,-2} = -4\,,$$
$$F_{-1,+2} = -N_{-1,+3} = 6\,, \qquad F_{-1,+4} = F_{-2,+1} = 8\,, \qquad F_{-1,+5} = 10\,,$$
$$N_{-1,+2} = F_{-2,+3} = -12\,, \qquad F_{-2,+4} = -16 \qquad N_{-2,+3} = N_{-2,+4} = 48\,,$$
$$F_{-1,+1} = 4(\mathcal{H}-3)\,, \qquad F_{-2,+5} = 24(\mathcal{H}-7)\,, \qquad F_{-2,+2} = 12(\mathcal{H}-4)^2\,,$$

while other elements are zero. Eq. (6.5.12): $\mathbb{C}_{\alpha,\beta} = \mathbb{C}_{\beta,\alpha}$, and

$$\mathbb{C}_{-2,-1} = \mathcal{G}^{(1)}_{+1}(\mathcal{H}-6) + 8\mathcal{G}^{(0)}_{+1}\,, \qquad \mathbb{C}_{-2,0} = \mathcal{G}^{(1)}_{-2} + 4\mathcal{G}^{(0)}_{-2}\,,$$
$$\mathbb{C}_{-2,1} = -(\mathcal{H} - 4\Pi_-)\mathcal{G}^{(1)}_{-3}\,, \qquad \mathbb{C}_{-2,2} = (\mathcal{H} + 4\Pi_-)\mathcal{G}^{(1)}_{-4}\,,$$
$$\mathbb{C}_{-2,5} = \mathcal{G}^{(1)}_{-7} = -\mathcal{G}^{(1)}_{-3}\mathcal{G}^{(1)}_{-4}\,, \qquad \mathbb{C}_{-1,0} = \mathcal{G}^{(1)}_{-1} + \mathcal{G}^{(0)}_{-1}\,,$$
$$\mathbb{C}_{-1,1} = -\mathcal{G}^{(1)}_{-2} + 2\mathcal{G}^{(0)}_{-2}\,, \qquad \mathbb{C}_{-1,4} = \mathcal{G}^{(1)}_{-5}\,,$$
$$\mathbb{C}_{-1,5} = \mathcal{G}^{(1)}_{-6} = -(\mathcal{G}^{(1)}_{-2})^2\,, \qquad \mathbb{C}_{1,2} = -(\mathcal{H}-2)(\mathcal{G}^{(1)}_{-1} + 6\mathcal{G}^{(0)}_{-1})\,,$$
$$\mathbb{C}_{1,3} = -\mathcal{G}^{(1)}_{-2} + 6\mathcal{G}^{(0)}_{-2}\,, \qquad \mathbb{C}_{1,4} = (\mathcal{H} - 4\sigma_3)\mathcal{G}^{(1)}_{-3}\,,$$
$$\mathbb{C}_{1,5} = (\mathcal{H} - 4 - 10\Pi_-)\mathcal{G}^{(1)}_{-4}\,, \qquad \mathbb{C}_{2,3} = (\mathcal{H}-2)\mathcal{G}^{(1)}_{-1} - 12(\mathcal{H}-4)\mathcal{G}^{(0)}_{-1}\,,$$
$$\mathbb{C}_{2,4} = (\mathcal{H}-2)\mathcal{G}^{(1)}_{-2} - 16(\mathcal{H}+3)\mathcal{G}^{(0)}_{-2}\,, \qquad \mathbb{C}_{2,5} = ((\mathcal{H}-2)(\mathcal{H}-4) - 8(\mathcal{H}-5)\Pi_-)\mathcal{G}^{(1)}_{-3}\,,$$
$$\mathbb{C}_{3,4} = (\mathcal{H}-2)\mathcal{G}^{(1)}_{-1} - 16(\mathcal{H}-5)\mathcal{G}^{(0)}_{-1}\,, \qquad \mathbb{C}_{3,5} = -(\mathcal{H}+2)\mathcal{G}^{(1)}_{-2} - 20(\mathcal{H}-4)\mathcal{G}^{(0)}_{-2}\,,$$
$$\mathbb{C}_{4,5} = [(\mathcal{H}-2)(\mathcal{H}+6) - 2\Pi_-(12\mathcal{H} - 117))\mathcal{G}^{(1)}_{-1} - 720\mathcal{G}^{(0)}_{-1}\,.$$

Eq. (6.5.13):

$$\begin{array}{llll}
\mathbb{Q}^1_{1,-2} = 2\,, & \mathbb{Q}^2_{1,-2} = 5\,, & \mathbb{Q}^1_{2,3} = 7(40-\mathcal{H})\,, & \mathbb{Q}^2_{2,3} = -3\,, \\
\mathbb{Q}^1_{1,-2} = 2\,, & \mathbb{Q}^2_{1,-2} = 5(\mathcal{H}-4)\,, & \mathbb{Q}^1_{2,4} = -10(\mathcal{H}-6)\,, & \mathbb{Q}^2_{2,4} = -4\,, \\
\mathbb{Q}^1_{1,-1} = -1\,, & \mathbb{Q}^2_{1,-1} = 3\mathcal{H}-10\,, & \mathbb{Q}^1_{2,5} = (336-118\mathcal{H}+11\mathcal{H}^2)\,, & \mathbb{Q}^2_{2,5} = 5\,, \\
\mathbb{Q}^1_{1,2} = 5(\mathcal{H}-4)\,, & \mathbb{Q}^2_{1,2} = \mathcal{H}\,, & \mathbb{Q}^1_{3,1} = 3\,, & \mathbb{Q}^2_{3,1} = 1\,, \\
\mathbb{Q}^1_{1,3} = -10\,, & \mathbb{Q}^2_{1,3} = -6\,, & \mathbb{Q}^1_{3,2} = -8(\mathcal{H}-3)\,, & \mathbb{Q}^2_{3,2} = 4\,, \\
\mathbb{Q}^1_{1,4} = 7(\mathcal{H}-36)\,, & \mathbb{Q}^2_{1,4} = 4\,, & \mathbb{Q}^1_{4,1} = 2\,, & \mathbb{Q}^2_{4,1} = -1\,, \\
\mathbb{Q}^1_{1,5} = 9(\mathcal{H}-56)\,, & \mathbb{Q}^2_{1,5} = 5\,, & \mathbb{Q}^1_{4,2} = 10(3-\mathcal{H})\,, & \mathbb{Q}^2_{4,2} = -5\,, \\
\mathbb{Q}^1_{2,-2} = -2\,, & \mathbb{Q}^2_{2,-2} = 4(\mathcal{H}-4)(2\mathcal{H}-7)\,, & \mathbb{Q}^1_{5,1} = 5\,, & \mathbb{Q}^2_{5,1} = 3\,, \\
\mathbb{Q}^1_{2,-1} = -1\,, & \mathbb{Q}^2_{2,-1} = 5(\mathcal{H}+2)\,, & \mathbb{Q}^1_{5,2} = 6(\mathcal{H}-1)\,, & \mathbb{Q}^2_{5,2} = 3\,.
\end{array}$$

Eq. (6.5.14):

$$\begin{array}{llll}
\mathbb{G}^1_{1,-2} = -1\,, & \mathbb{G}^2_{1,-2} = (8-\mathcal{H})\,, & \mathbb{G}^1_{2,-2} = -1\,, & \mathbb{G}^2_{2,-2} = (\mathcal{H}+8)(\mathcal{H}-6)\,, \\
\mathbb{G}^1_{1,-1} = 1\,, & \mathbb{G}^2_{1,-1} = (4-\mathcal{H})\,, & \mathbb{G}^1_{2,-1} = 1\,, & \mathbb{G}^2_{2,-1} = 4-\mathcal{H}\,, \\
\mathbb{G}^1_{1,0} = \mathbb{G}^2_{1,0} = 1\,, & \mathbb{G}^1_{2,0} = -\mathbb{G}^2_{2,0} = 1\,, & \mathbb{G}^1_{1,1} = \mathcal{H}-4\,, & \mathbb{G}^2_{1,1} = -1\,, \\
\mathbb{G}^1_{2,1} = \mathcal{H}-2\,, & \mathbb{G}^2_{2,1} = \mathcal{H}\,, & \mathbb{G}^1_{1,2} = \mathcal{H}-4\,, & \mathbb{G}^2_{1,2} = \mathcal{H}\,, \\
\mathbb{G}^1_{2,2} = -1\,, & \mathbb{G}^2_{2,2} = \mathcal{H}\,, & \mathbb{G}^1_{1,3} = \mathbb{G}^2_{1,3} = -1\,, & \mathbb{G}^1_{2,3} = 4-\mathcal{H}\,, \\
\mathbb{G}^2_{2,3} = -1\,, & \mathbb{G}^1_{1,4} = \mathcal{H}-4\,, & \mathbb{G}^2_{1,4} = -1\,, & \mathbb{G}^1_{2,4} = 2-\mathcal{H}\,, \\
\mathbb{G}^2_{2,4} = 1\,, & \mathbb{G}^1_{1,5} = \mathcal{H}-4\,, & \mathbb{G}^2_{1,5} = -1\,, & \mathbb{G}^1_{2,5} = (\mathcal{H}-2)(\mathcal{H}-4)\,, \\
\mathbb{G}^2_{2,5} = -1\,. & & &
\end{array}$$